Algebra 3, 2010

Compulsory exercises, second set
Andet sæt obligatoriske opgaver

23. February 2010

The students in the course have to submit a written solution to the following exercises in the mailbox of the instructor (Martin) tuesday 9. March at the latest at the start of the exercise classes. The written solutions (in Danish or English) should be carefully written with references to quoted results from the notes and the exercises.

Exercise C2.1: In this exercise results and examples from the final section 1.21 of the notes on group theory may be especially useful.

Let G be a finite group of order $55055 = 5 \cdot 7 \cdot 11^2 \cdot 13$. Let $P \in \mathrm{Syl}_{13}(G)$, $Q \in \mathrm{Syl}_{11}(G)$, $R \in \mathrm{Syl}_7(G)$, $S \in \mathrm{Syl}_5(G)$.

(1) Show that $m_{11}(G) = 1$ and that $Q \lhd G$.

(2) Use a result from the final section of Chapter 1 in the notes to show that the factor group G/Q is cyclic.

(3) Use the result of (2) to show that PQ, QR and QS and QRS are *normal subgroups* of G.

(4) Which of the subgroups PQ, QR, QS and QRS *must* be abelian?

(5) How is it possible to deduce from (4) that $P \lhd G$ and $R \lhd G$?
Hint: Use result(s) on characteristic subgroups.

Exercise C2.2: Let α be an algebraic number of odd degree with respect to $\mathbb{Q}$. Prove that α and α^2 have the same degree with respect to $\mathbb{Q}$.
Hint: Consider the inclusions $\mathbb{Q} \subseteq \mathbb{Q}(\alpha^2) \subseteq \mathbb{Q}(\alpha)$

Exercise C2.3: Let GF(5)= $\mathbb{Z}_5$ be the field with 5 elements.
(1) Show that the polynomial $p(x) = x^3 + x^2 + 1 \in \mathrm{GF}(5)[x]$ is irreducible over GF(5).

(2) Show that GF(125) is a splitting field for p over GF(5).

(3) Why is it true that $p(x)$ divides $x^{125} - x$ in $\mathrm{GF}(5)[x]$?