



Bachelor thesis in mathematics

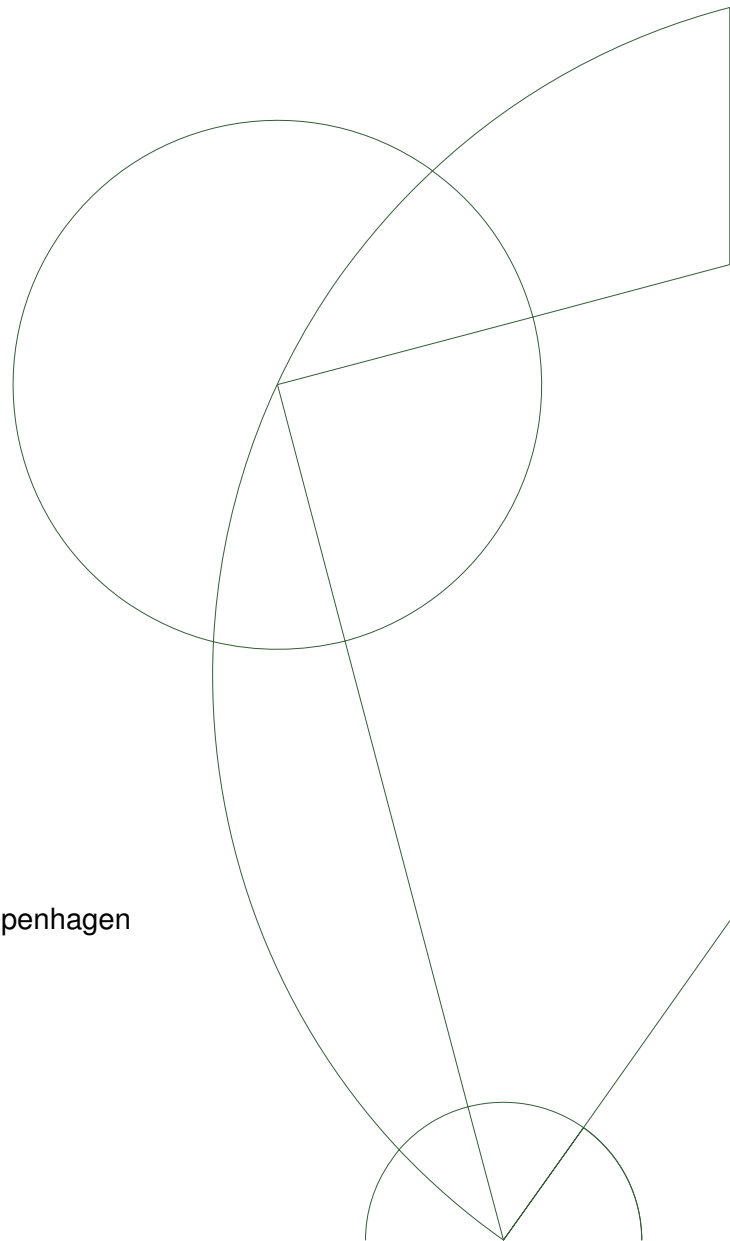
Rasmus Sylvester Bryder

Amenable Banach algebras

Department of Mathematical Sciences, University of Copenhagen

Advisor: Niels Grønbæk

January 14th, 2011



Abstract

This thesis deals with the concept of *amenability* of Banach algebras. A Banach algebra \mathcal{A} is said to be *amenable* if it holds for any Banach \mathcal{A} -bimodule \mathfrak{X} that all derivations of \mathcal{A} into the dual space \mathfrak{X}' of \mathfrak{X} are inner, or if the *first Hochschild cohomology group* of \mathcal{A} with coefficients in \mathfrak{X}' is trivial. In the thesis, some hereditary properties of amenable Banach algebras are proved. Furthermore, a connection to amenable groups is established, in the result that a group G is amenable if and only if the discrete group algebra $\ell^1(G)$ is amenable. This is then used to prove that Banach algebras such as the continuous complex-valued functions on compact Hausdorff spaces, the compact operators on arbitrary Hilbert spaces and the integrable functions on \mathbb{R} are amenable. The last chapter of the thesis deals with an examination of Hochschild cohomology groups of higher order, leading to the surprising result that they are trivial whenever the Banach algebra \mathcal{A} over which they are defined is amenable and they have coefficients in the dual space of a Banach \mathcal{A} -bimodule. The proof of this uses some interesting isomorphisms, as well as tensor products over Banach spaces.

Resumé

Dette projekt beskæftiger sig med begrebet *amenabilitet* for Banach-algebraer. En Banach-algebra \mathcal{A} siges at være *amenabel*, hvis der for ethvert Banach- \mathcal{A} -bimodul \mathfrak{X} gælder, at alle derivationer fra \mathcal{A} over i det duale rum \mathfrak{X}' til \mathfrak{X} er indre, svarende til at den såkaldte *første Hochschild-kohomologigruppe* over \mathcal{A} med koefficienter i \mathfrak{X}' er triviell. I projektet gennemgås en række arvelighedsegenskaber for amenable Banach-algebraer. Endvidere knyttes en forbindelse til amenable grupper i form af resultatet at en gruppe G er amenabel hvis og kun hvis den diskrete gruppealgebra $\ell^1(G)$ er amenabel. Dette benyttes til at vise, at Banach-algebraer som de kontinuerte, komplekse funktioner over kompakte Hausdorff-rum, de kompakte operatorer over vilkårlige Hilbert-rum og de integrable funktioner over \mathbb{R} er amenable. Sidst i projektet undersøges Hochschild-kohomologigrupper af højere orden nærmere, med henblik på at vise, at disse over amenable Banach-algebraer \mathcal{A} er trivielle, når de har koefficienter i det duale rum til et Banach- \mathcal{A} -bimodul. Beviset for dette benytter en række interessante isomorfier, samt tensorprodukter over Banach-rum.

CONTENTS

Preface	2
Preliminaries	3
1 Amenable Banach algebras	7
1.1 Motivation and definition	7
1.2 Hereditary properties of amenable Banach algebras	8
2 Examples of amenable Banach algebras	15
2.1 Amenable groups	15
2.2 The discrete group algebra	18
2.3 Hereditary properties of amenable groups	24
2.4 Compact operators on Hilbert spaces	25
2.5 The group algebra of \mathbb{R}	29
3 Hochschild cohomology in Banach algebras	33
3.1 Higher order Hochschild cohomology groups	33
3.2 Tensor products	36
A Multiplicative linear functionals	40
B Hilbert space results	42
C A condition implying group amenability	44
D Outside theorems	47
Bibliography	48

PREFACE

This bachelor thesis was written mostly during a dark and snowy Danish winter, under the influence of a lot of coffee, microwave food and Steely Dan. It was written in English, in order to perhaps increase the number of potential readers.

Perhaps unusual for a bachelor thesis is the section of preliminary theory; this is included to ease the possible newcomer, should he or she have had no experience with some of the core concepts of the actual thesis. If the reader has had at least some experience with Banach spaces and topology, the preliminaries section should, hopefully, only encourage the reader to read on to the thesis itself. To optimize understanding thoroughly, though, some knowledge of functional analysis, Banach algebras and modules is needed.

Whenever a Banach algebra is defined in the thesis, it won't be proved that it actually *is* a Banach algebra; likewise, we won't show that the multiplications on modules that we define throughout actually are proper, well-defined multiplications.

Chapter 1 deals with the concept of amenable Banach algebras and how hereditary properties are deduced from them. Chapter 2 deals with the concept of amenable groups and how we establish a connection to a specific Banach algebra from this concept, enabling deduction of amenability of some very central Banach algebras. Some of the proofs in the section take up a lot of space, and in order to increase readability of them, they have been split up into smaller parts, so that the reader can deduce the result in simpler, non-trivial steps. Chapter 3 finally deals with a generalization of one of the central concepts of Banach algebra amenability, namely the Hochschild cohomology group, and how a line is drawn between these and amenability.

Some of the proofs in the thesis have been included in three appendices, since they took focus from the central point of the section in which they were originally placed, and they will be referred to whenever needed. A list of important theorems used in the thesis is included at the end. It has overall been the desire to make this thesis as self-contained and focused as possible, without relying too much on outside theorems to make the points throughout.



January 2011

Acknowledgements

Niels, my advisor, has been a very helpful one and provided me with a lot of inspiration for what to include and what not to, as well as guidelines for some of the proofs, and for this I would like to thank him. Marcus, Ulrich and Kristian should also be thanked for their proof-reading and advising. Finally, I would like to thank my parents for housing and taking care of me during some of the hard times, and Emma for helping out whenever I needed it. Without any of these persons, this would not have been much of a thesis.

PRELIMINARIES

Here we will deal with introducing notation and structures to be used in the thesis, mainly via definitions, and the hope is that the newcomer and curious reader will be able to read the other chapters without studying extraordinarily much outside of the thesis.

Banach spaces and algebras

All vector spaces considered in this thesis are over \mathbb{C} . A *Banach space* is a normed vector space that is complete with respect to the metric induced by the norm. For any Banach space \mathfrak{X} , $(\mathfrak{X})_1$ denotes the closed unit ball of \mathfrak{X} :

$$(\mathfrak{X})_1 = \{x \in \mathfrak{X} \mid \|x\| \leq 1\}.$$

An *algebra* \mathcal{A} is a vector space \mathcal{A} equipped with a map $\mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$, $(x, y) \mapsto xy$ satisfying for all $\alpha \in \mathbb{C}$, $x, y, z \in \mathcal{A}$ that

$$(xy)z = x(yz), \quad x(y+z) = xy + xz, \quad (x+y)z = xz + yz, \quad (\alpha x)y = \alpha(xy) = x(\alpha y).$$

The map is, quite naturally (some would probably say conveniently), called the *product* or the *multiplication*. A *Banach algebra* \mathcal{A} is an algebra \mathcal{A} equipped with a norm that makes it into a Banach space and additionally satisfies the inequality $\|xy\| \leq \|x\|\|y\|$ for all $x, y \in \mathcal{A}$.

For algebras \mathcal{A} and \mathfrak{B} , an *algebra homomorphism* is a linear mapping $\varphi : \mathcal{A} \rightarrow \mathfrak{B}$ satisfying

$$\varphi(ab) = \varphi(a)\varphi(b), \quad a, b \in \mathcal{A},$$

so that the compositions are preserved by φ .

Example 0.1. \mathbb{C} with the usual addition, product and norm is a Banach algebra.

Example 0.2. Let X be a non-void set, and let $\ell^\infty(X)$ denote the set of bounded mappings $f : X \rightarrow \mathbb{C}$. With pointwise addition, product and scalar multiplication and equipped with the uniform norm $\|f\|_\infty = \sup_{x \in X} |f(x)|$, $\ell^\infty(X)$ becomes a Banach algebra.

Example 0.3. Let X be a non-void compact topological space, and let $C(X)$ denote the set of continuous mappings $f : X \rightarrow \mathbb{C}$. With pointwise addition, product and scalar multiplication and equipped with the uniform norm, $C(X)$ becomes a Banach algebra.

For any Banach space \mathfrak{X} , \mathfrak{X}' denotes the Banach dual of \mathfrak{X} , namely the Banach space of continuous linear functionals on \mathfrak{X} .

Subalgebras, ideals and quotients

Now for some small formalities regarding subspaces and quotient spaces. Let \mathcal{A} be a Banach algebra and let S be a subspace of \mathcal{A} . S is a *subalgebra* of \mathcal{A} if $s_1, s_2 \in S$ implies $s_1 s_2 \in S$. S is a *two-sided ideal* of \mathcal{A} if $sa \in S$ and $as \in S$ for all $s \in S$, $a \in \mathcal{A}$. By assuming that S is closed, we obtain a norm on the quotient space \mathcal{A}/S of \mathcal{A} modulo S by defining

$$\|s\| = \inf_{a \in \mathfrak{B}} \|a\|, \quad \mathfrak{B} \in \mathcal{A}/S,$$

called *the canonical norm*, making \mathcal{A}/S into a Banach space. By assuming S is a closed two-sided ideal, \mathcal{A}/S can be equipped with a well-defined product, making it a Banach algebra called the *quotient algebra*. Worth noting is that the canonical mapping $\pi : \mathcal{A} \rightarrow \mathcal{A}/S$ is a contractive algebra homomorphism, easily seen from the definition of the norm on \mathcal{A}/S . In the thesis, we will always consider quotient spaces and algebras by making them this way.

Let \mathfrak{X} be a Banach space with a closed subspace \mathcal{Y} , and define the quotient space \mathfrak{X}/\mathcal{Y} in the way mentioned above. We define the *dual mapping* $\pi^* : (\mathfrak{X}/\mathcal{Y})' \rightarrow \mathfrak{X}'$ by $\pi^*(\lambda) = \lambda \circ \pi$, and we obtain the following characterization of the bounded linear functionals on the Banach space \mathfrak{X}/\mathcal{Y} :

Lemma 0.4. *Let \mathfrak{X} and \mathcal{Y} be as above. Then $(\mathfrak{X}/\mathcal{Y})'$ is isometrically isomorphic to*

$$\mathcal{Y}^\perp := \{\varphi \in \mathfrak{X}' \mid \varphi(\mathcal{Y}) = \{0\}\}.$$

Proof. It is clear that for any $\lambda \in (\mathfrak{X}/\mathcal{Y})'$, $\pi^*(\lambda) = \lambda \circ \pi$ is linear and

$$|(\pi^*(\lambda))(x)| = |(\lambda \circ \pi)(x)| = |\lambda(x + \mathcal{Y})| \leq \|\lambda\| \|x + \mathcal{Y}\| \leq \|\lambda\| \|x\|$$

for any $x \in \mathfrak{X}$. Then $\|\pi^*(\lambda)\| \leq \|\lambda\|$, and $\pi^*(\lambda)(\mathcal{Y}) = \lambda(\pi(\mathcal{Y})) = 0$ because $\pi(\mathcal{Y}) = \mathcal{Y}$ is the zero vector on \mathfrak{X}/\mathcal{Y} , so $\pi^*((\mathfrak{X}/\mathcal{Y})') \subseteq \mathcal{Y}^\perp$. Now, for any $\varphi \in \mathcal{Y}^\perp$, define $\tilde{\varphi}(x + \mathcal{Y}) = \varphi(x)$ for $x \in \mathfrak{X}$. $\tilde{\varphi}$ is well-defined: assume $x_1 + \mathcal{Y} = x_2 + \mathcal{Y}$. Then $x_1 - x_2 \in \mathcal{Y}$, so $\varphi(x_1 - x_2) = 0$ and $\varphi(x_1) = \varphi(x_2)$. $\tilde{\varphi}$ is obviously linear. Finally, let $s \in \mathfrak{X}/\mathcal{Y}$. For any $a \in s$, $a + \mathcal{Y} = s$ and so

$$|\tilde{\varphi}(s)| = |\tilde{\varphi}(a + \mathcal{Y})| = |\varphi(a)| \leq \|\varphi\| \|a\|$$

for any $a \in s$. Taking infimum over all $a \in s$ thus gives $|\tilde{\varphi}(s)| \leq \|\varphi\| \|s\|$, so $\tilde{\varphi}$ is bounded. Define the mapping $\psi : \mathcal{Y}^\perp \rightarrow (\mathfrak{X}/\mathcal{Y})'$ by $\psi(\varphi) = \tilde{\varphi}$. It is clear that for any $\lambda \in (\mathfrak{X}/\mathcal{Y})'$, $x \in \mathfrak{X}$, we have

$$\lambda(x + \mathcal{Y}) = (\lambda \circ \pi)(x) = \pi^*(\lambda)(x) = \psi(\pi^*(\lambda))(x + \mathcal{Y}),$$

and for any $\varphi \in \mathcal{Y}^\perp$, $x \in \mathfrak{X}$, we have

$$\varphi(x) = \tilde{\varphi}(x + \mathcal{Y}) = \pi^*(\tilde{\varphi})(x) = \pi^*(\psi(\varphi))(x).$$

Hence π^* is a bijection $(\mathfrak{X}/\mathcal{Y})' \rightarrow \mathcal{Y}^\perp$ with inverse ψ . Because $\tilde{\varphi} = \lambda$ when $\varphi = \pi^*(\lambda)$, we have

$$\|\pi^*(\lambda)\| \leq \|\lambda\| = \|\tilde{\varphi}\| \leq \|\varphi\| = \|\pi^*(\lambda)\|,$$

so π^* is an isometry. □

Bimodules

We go on to defining bimodules over a Banach algebra \mathcal{A} .

Definition 0.5. Let \mathcal{A} be a Banach algebra. A *left \mathcal{A} -module* is a vector space \mathfrak{M} , equipped with a mapping $\mathcal{A} \times \mathfrak{M} \rightarrow \mathfrak{M}$, $(a, m) \mapsto am$, that is linear in both variables and satisfies

$$a(bm) = (ab)m, \quad a, b \in \mathcal{A}, m \in \mathfrak{M}.$$

A *right \mathcal{A} -module* is a vector space \mathfrak{M} equipped with a mapping $\mathfrak{M} \times \mathcal{A} \rightarrow \mathfrak{M}$, $(m, a) \mapsto ma$, linear in both variables and satisfying

$$(ma)b = m(ab), \quad a, b \in \mathcal{A}, m \in \mathfrak{M}.$$

The two mappings are called *the left* and *the right module multiplication* respectively. An *\mathcal{A} -bimodule* is then a vector space \mathfrak{M} that is both a left and a right \mathcal{A} -module and satisfies

$$a(mb) = (am)b, \quad a, b \in \mathcal{A}, m \in \mathfrak{M}.$$

A *Banach \mathcal{A} -bimodule* is an \mathcal{A} -bimodule \mathfrak{X} equipped with a norm such that \mathfrak{X} becomes a Banach space and there exists $S > 0$ such that

$$\|ax\| \leq S\|a\|\|x\|, \quad \|xa\| \leq S\|x\|\|a\|, \quad a \in \mathcal{A}, x \in \mathfrak{X}.$$

Left and *right Banach \mathcal{A} -bimodules* satisfy the first and second inequality respectively. Note that this requirement actually makes the module multiplication operations jointly continuous; namely, for $a_n \rightarrow a$ in \mathcal{A} and $x_n \rightarrow x$ in \mathfrak{X} , we have that $a_n x \rightarrow ax$ and $ax_n \rightarrow ax$ in \mathfrak{X} , and $xa_n \rightarrow xa$ and $x_n a \rightarrow xa$ as well.

Let \mathcal{A} be a Banach algebra and let \mathfrak{X} and \mathfrak{Y} be Banach \mathcal{A} -bimodules. A homomorphism $\varphi : \mathfrak{X} \rightarrow \mathfrak{Y}$ is said to be a *bimodule homomorphism* if φ preserves module multiplication, i.e.

$$\varphi(ax) = a\varphi(x), \quad \varphi(xa) = \varphi(x)a, \quad a \in \mathcal{A}, x \in \mathfrak{X}.$$

Left and *right module homomorphisms* are defined analogously.

The angle bracket notation for functional evaluation will be used throughout (to ease readability a little): that is, for any linear functional φ on \mathfrak{X} , we define

$$\langle x, \varphi \rangle := \varphi(x), \quad x \in \mathfrak{X}.$$

Let \mathcal{A} be a Banach algebra and \mathfrak{B} a Banach \mathcal{A} -bimodule. There is a common way to make the Banach dual \mathfrak{B}' of \mathfrak{B} into a Banach \mathcal{A} -bimodule; by defining

$$\langle x, a\varphi \rangle := \langle xa, \varphi \rangle, \quad \langle x, \varphi a \rangle := \langle ax, \varphi \rangle$$

for all $a \in \mathcal{A}$, $x \in \mathfrak{B}$ and $\varphi \in \mathfrak{B}'$, \mathfrak{B}' is made into a Banach \mathcal{A} -bimodule, which shouldn't be hard to check. In this thesis, whenever \mathfrak{B}' is made into a Banach \mathcal{A} -bimodule, it will be done *this way*, unless otherwise noted (and it *will* be noted).

Nets and weak-star topology

Finally, some topological concepts need to fall into place, and the first thing to do is to define nets.

Definition 0.6. A *directed set* is a partially ordered set (V, \leq) , such that for all $u, v \in V$ there exists $w \in V$ such that $w \leq u$ and $w \leq v$. Let \mathfrak{X} be a topological space. A *net* in \mathfrak{X} is a mapping $v \mapsto \alpha_v$ of a directed set V into \mathfrak{X} . We typically denote the net by $(\alpha_v)_{v \in V}$, as we do sequences (a net is a sort of generalized sequence).

A net $(\alpha_v)_{v \in V}$ in \mathfrak{X} is said to converge to $\alpha \in \mathfrak{X}$ if there for any neighbourhood U of α exists $v_U \in V$ such that $v \geq v_U$ implies $\alpha_v \in U$; we write $\alpha_v \rightarrow \alpha$ or $\alpha = \lim_{v \in V} \alpha_v$, and we say that α is a *limit point* for $(\alpha_v)_{v \in V}$.

Some important results from metric space theory pop up in net versions. For instance, if \mathfrak{X} is a topological space and $S \subseteq \mathfrak{X}$, then $x \in \overline{S}$ if and only if there is a net of points in S converging to x , and if \mathfrak{X} and \mathfrak{Y} are topological spaces, then a map $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ is continuous if and only if $f(\alpha_v) \rightarrow f(\alpha)$ for every net $(\alpha_v)_{v \in V}$ in \mathfrak{X} with $\alpha_v \rightarrow \alpha$. Finally, if $\alpha_v \rightarrow \alpha$ in a normed space and $\|\alpha_v\| \leq A$ for some $A \in \mathbb{R}$, then $\|\alpha\| \leq A$ as well. These results will not be referred to explicitly in the chapters to come.

Let \mathfrak{X} be a Banach space, and for any $x \in \mathfrak{X}$, define the function $\hat{x} : \mathfrak{X}' \rightarrow \mathbb{C}$ by

$$\langle \varphi, \hat{x} \rangle := \langle x, \varphi \rangle, \quad \varphi \in \mathfrak{X}'.$$

It is easy to check that all \hat{x} for $x \in \mathfrak{X}$ are contained in the *second dual* \mathfrak{X}'' , the Banach dual of \mathfrak{X}' , and that the mapping $\mathfrak{X} \rightarrow \mathfrak{X}''$, $x \mapsto \hat{x}$, is actually an injection.

The *weak-star topology* (or w^* topology, as it will be denoted in this thesis) is the weakest topology on \mathfrak{X}' that makes every function in the set $\{\hat{x} \mid x \in \mathfrak{X}\}$ continuous. In this topology, a net (φ_α) in \mathfrak{X}' will converge to φ if and only if $\langle \varphi_\alpha, \hat{x} \rangle$ converges to $\langle \varphi, \hat{x} \rangle$, or $\langle x, \varphi_\alpha \rangle \rightarrow \langle x, \varphi \rangle$, in \mathbb{C} , and thus the weak-star topology on \mathfrak{X}' is completely characterized by pointwise convergence of bounded linear functionals φ on \mathfrak{X} .

Amenable Banach algebras

In this chapter, we will motivate the definition of amenability of a Banach algebra \mathcal{A} and then go on to prove some very useful hereditary properties for amenable Banach algebras. Some of these incorporate new concepts that may seem extraneous to the core concept, but nonetheless prove incredibly useful in their construction.

1.1 Motivation and definition

Let us start off with a definition.

Definition 1.1. Let \mathcal{A} be a Banach algebra and \mathfrak{X} a Banach \mathcal{A} -bimodule. A *bounded \mathfrak{X} -derivation* is a bounded linear mapping D from \mathcal{A} into \mathfrak{X} satisfying

$$D(ab) = D(a)b + aD(b)$$

for all $a, b \in \mathcal{A}$. The set of all bounded \mathfrak{X} -derivations on \mathcal{A} is denoted by $\mathcal{Z}^1(\mathcal{A}, \mathfrak{X})$.

The above equality should be familiar to those accustomed to differentiation; it is indeed Leibniz' law, albeit in a very generalized context. A neophyte wouldn't immediately think that there could be anything peculiar about such mappings, but then again, thus start all great adventures.

Let \mathcal{A} be a Banach algebra and \mathfrak{X} a Banach \mathcal{A} -bimodule. For any $x \in \mathfrak{X}$, define $\delta_x : \mathcal{A} \rightarrow \mathfrak{X}$ by

$$\delta_x(a) = ax - xa, \quad a \in \mathcal{A}.$$

It is clear that δ_x is linear and bounded. Furthermore, because

$$\delta_x(ab) = (ab)x - x(ab) = a(bx) - a(xb) + (ax)b - (xa)b = a\delta_x(b) + \delta_x(a)b$$

for all $a, b \in \mathcal{A}$ since \mathfrak{X} is an \mathcal{A} -bimodule, we obtain $\delta_x \in \mathcal{Z}^1(\mathcal{A}, \mathfrak{X})$. Derivations of this kind are called *inner*, and we denote the set of all inner \mathfrak{X} -derivations on \mathcal{A} by $\mathcal{B}^1(\mathcal{A}, \mathfrak{X})$. For $x \in \mathfrak{X}$, the mapping δ_x is called the *inner derivation generated by x* , and we will denote any inner \mathfrak{X} -derivation by the above δ notation. $\mathcal{B}^1(\mathcal{A}, \mathfrak{X})$ is clearly a linear subspace of $\mathcal{Z}^1(\mathcal{A}, \mathfrak{X})$, thus allowing us to define the quotient space

$$\mathcal{H}^1(\mathcal{A}, \mathfrak{X}) = \mathcal{Z}^1(\mathcal{A}, \mathfrak{X}) / \mathcal{B}^1(\mathcal{A}, \mathfrak{X})$$

which is called the *the first Hochschild cohomology group of \mathcal{A} with coefficients in \mathfrak{X}* . It would indeed be quite wonderous if all bounded \mathfrak{X} -derivations on the Banach algebra \mathcal{A} were inner, and this in turn motivates the not quite as strong, but still incredibly handy property from which this thesis gets its title.

Definition 1.2. A Banach algebra \mathcal{A} is said to be *amenable* if $\mathcal{H}^1(\mathcal{A}, \mathfrak{X}') = \{0\}$ for every Banach \mathcal{A} -bimodule \mathfrak{X}' , i.e. if every bounded \mathfrak{X}' -derivation is inner.

In the above definition, \mathfrak{X}' is made into a Banach \mathcal{A} -bimodule the usual way.

The term *amenable* doesn't originate from the study of Banach algebras but rather from what we shall look closer on in the beginnings of chapter 2, but it still seems appropriate grounded in Banach algebra theory. In the most colloquial sense of the word, Banach algebras that have the amenability property do indeed seem amenable.

1.2 Hereditary properties of amenable Banach algebras

A question naturally arises; from the amenability of a Banach algebra, can we then determine whether another Banach algebra is amenable? Thankfully, the answer is that, yes, in some cases, we can.

Theorem 1.3. *Let \mathcal{A} and \mathfrak{B} be Banach algebras. If \mathcal{A} is amenable and there exists a continuous homomorphism $\varphi : \mathcal{A} \rightarrow \mathfrak{B}$ such that φ has dense range in \mathfrak{B} , then \mathfrak{B} is amenable.*

Proof. Let \mathfrak{X} be a Banach \mathfrak{B} -bimodule, and let $D : \mathfrak{B} \rightarrow \mathfrak{X}'$ be a bounded \mathfrak{X}' -derivation. \mathfrak{X} becomes an \mathcal{A} -bimodule by defining multiplications

$$ax := \varphi(a)x, \quad xa := x\varphi(a), \quad a \in \mathcal{A}, x \in \mathfrak{X},$$

and it is easily seen to be a Banach \mathcal{A} -bimodule by φ being continuous. For $a, b \in \mathcal{A}$, we have

$$D(\varphi(ab)) = D(\varphi(a))\varphi(b) + \varphi(a)D(\varphi(b)) = D(\varphi(a))b + aD(\varphi(b))$$

and $\|D(\varphi(a))\| \leq \|D\|\|\varphi(a)\| = \|D\|\|\varphi\|\|a\|$, so $D \circ \varphi : \mathcal{A} \rightarrow \mathfrak{X}'$ is a bounded \mathfrak{X}' -derivation. Hence by the assumption, $D \circ \varphi$ is inner, and thus exists $f \in \mathfrak{X}'$ such that

$$D(\varphi(a)) = af - fa = \varphi(a)f - f\varphi(a), \quad a \in \mathcal{A},$$

or $D(y) = yf - fy$ for all $y \in \varphi(\mathcal{A})$. For $b \in \mathfrak{B}$, $b = \lim b_n$ for some $(b_n) \in \varphi(\mathcal{A})$ and so

$$D(b) = \lim D(b_n) = \lim(b_n f - f b_n) = \lim(b_n f) - \lim(f b_n) = bf - fb,$$

by continuity of D and the module multiplications. □

An immediate consequence of this theorem is the following:

Corollary 1.4. *Let \mathcal{A} be an amenable Banach algebra, and let \mathcal{I} be a closed two-sided ideal in \mathcal{A} . Then the Banach algebra \mathcal{A}/\mathcal{I} is amenable.*

Proof. The canonical mapping $\mathcal{A} \rightarrow \mathcal{A}/\mathcal{I}$ is a contractive, surjective homomorphism and therefore continuous. □

We now turn our attention to subalgebras of a Banach algebra \mathcal{A} , in which case the preceding theorem won't do us much good. In general, it is not true that any closed subalgebra of an amenable Banach algebra is amenable; we can't even be sure that any closed two-sided ideal is amenable. As it turns out, we need only demand one very specific thing from such ideals, and for a proper description of this, we need the extraneous definitions mentioned in the introduction of this chapter.

Definition 1.5. Let \mathcal{A} be a Banach algebra.

A net $(\alpha_v)_{v \in V}$ in \mathcal{A} is *bounded* if there exists $S \geq 0$ such that $\|\alpha_v\| \leq S$ for all $v \in V$.

A net $(e_v)_{v \in V}$ in \mathcal{A} is called a *left approximate identity* for \mathcal{A} if $e_v a \rightarrow a$ for all $a \in \mathcal{A}$; and it is called a *right approximate identity* if $a e_v \rightarrow a$. If a net is both a left and a right approximate identity, it is called a *(two-sided) approximate identity*.

Let \mathfrak{X} be a Banach \mathcal{A} -bimodule. A net $(e_v)_{v \in V}$ in \mathcal{A} is called a *(left/right/two-sided) approximate identity for \mathfrak{X}* if it satisfies the above, when replacing all $a \in \mathcal{A}$ with all $x \in \mathfrak{X}$.

What does all this have to do with amenability? The answer is the following theorem.

Theorem 1.6. *Let \mathcal{A} be an amenable Banach algebra. Then \mathcal{A} has a bounded approximate identity.*

Proof. The proof makes use of a wide array of constructions. Let \mathfrak{A} be \mathcal{A} itself made into a Banach \mathcal{A} -bimodule upon which \mathcal{A} acts via the multiplications

$$a \cdot b := ab, \quad b \cdot a := 0, \quad a \in \mathcal{A}, b \in \mathfrak{A}.$$

We now make \mathfrak{A}' into a Banach \mathcal{A} -bimodule the usual way, thus implying

$$\langle b, a\varphi \rangle = \langle ba, \varphi \rangle = 0, \quad a \in \mathcal{A}, b \in \mathfrak{A}', \varphi \in \mathfrak{A}',$$

and then make \mathfrak{A}'' into a Banach \mathcal{A} -bimodule the same way by defining

$$\langle \varphi, aF \rangle := \langle \varphi a, F \rangle, \quad \langle \varphi, Fa \rangle := \langle a\varphi, F \rangle = 0, \quad a \in \mathcal{A}, \varphi \in \mathfrak{A}'', F \in \mathfrak{A}'',$$

so that $Fa = 0$ for all $a \in \mathcal{A}, F \in \mathfrak{A}''$.

Now we let $D : \mathcal{A} \rightarrow \mathfrak{A}''$ be the canonical embedding of \mathcal{A} into its second dual by defining $D(a) = \hat{a}$ for $a \in \mathcal{A}$ (see the preliminaries). Then for all $a, b \in \mathcal{A}$ and $\varphi \in \mathfrak{A}'$,

$$\langle \varphi, D(a)b + aD(b) \rangle = \langle \varphi, aD(b) \rangle = \langle \varphi a, \hat{b} \rangle = \langle b, \varphi a \rangle = \langle ab, \varphi \rangle = \langle \varphi, D(ab) \rangle,$$

proving that D is a \mathfrak{A}'' -derivation, making D inner to boot because of the assumption that \mathcal{A} is amenable. This of course means that there must be some $\Psi \in \mathfrak{A}''$ such that

$$\hat{a} = a\Psi - \Psi a = a\Psi, \quad a \in \mathcal{A}.$$

Goldstine's theorem (Theorem D.5) implies that there exists a bounded net $(\kappa_m)_{m \in M}$ in \mathcal{A} such that $(\widehat{\kappa_m})_{m \in M}$ converges to Ψ in the w^* topology on \mathfrak{A}'' . Therefore $\langle \widehat{\kappa_m}, \lambda \rangle \rightarrow \langle \Psi, \lambda \rangle$, or $\langle \lambda, \widehat{\kappa_m} \rangle \rightarrow \langle \lambda, \Psi \rangle$ for all $\lambda \in \mathfrak{A}'$. Then especially we have $\langle \lambda a, \widehat{\kappa_m} \rangle \rightarrow \langle \lambda a, \Psi \rangle$ for all $\lambda \in \mathfrak{A}'$ and $a \in \mathcal{A}$, thus implying

$$\langle a\kappa_m, \lambda \rangle = \langle \lambda a, \widehat{\kappa_m} \rangle \rightarrow \langle \lambda a, \Psi \rangle = \langle \lambda, a\Psi \rangle = \langle \lambda, \hat{a} \rangle = \langle a, \lambda \rangle$$

for all $\lambda \in \mathfrak{A}'$ and $a \in \mathcal{A}$. By Theorem D.1, \mathcal{A} has a bounded right approximate identity $(e_v)_{v \in V}$. By repeating the proof all over, but switching around the module multiplications on \mathfrak{A} such that the left \mathcal{A} -multiplication always gives 0, we analogously obtain a bounded left approximate identity $(f_w)_{w \in W}$ for \mathcal{A} .

We now turn these two approximate identities into one altogether bounded approximate identity for \mathcal{A} . We turn $V \times W$ into a directed set by defining the partial order relation $(v_1, w_1) \leq (v_2, w_2)$ if and only if $v_1 \leq v_2$ and $w_1 \leq w_2$ for $v_1, v_2 \in V, w_1, w_2 \in W$. Now define the net $(g_{v,w})_{(v,w) \in V \times W}$ by $g_{v,w} = e_v + f_w - e_v f_w$ for all $v \in V, w \in W$. $(g_{v,w})$ is obviously bounded. For $a \in \mathcal{A}$, we then have

$$\|ag_{v,w} - a\| \leq \|ae_v - a\| + \|af_w - ae_v f_w\| = \|ae_v - a\| + \|a - ae_v\| \|f_w\|,$$

so that $a = \lim_{v,w} ag_{v,w}$ because (f_w) is bounded. Similarly, because (e_v) is bounded,

$$\|g_{v,w}a - a\| \leq \|f_w a - a\| + \|e_v a - e_v f_w a\| = \|f_w a - a\| + \|e_v\| \|a - f_w a\|,$$

so that $a = \lim_{v,w} g_{v,w}a$. Thus, $(g_{v,w})$ is a bounded approximate identity for \mathcal{A} . \square

Let \mathcal{A} be a Banach algebra and let \mathfrak{X} be a Banach \mathcal{A} -bimodule. A *left \mathcal{A} -submodule* \mathfrak{M} of \mathfrak{X} is a subspace of \mathfrak{X} that satisfies $ax \in \mathfrak{M}$ for all $a \in \mathcal{A}, x \in \mathfrak{M}$; a *right \mathcal{A} -submodule* \mathfrak{M} satisfies $xa \in \mathfrak{M}$. Let \mathcal{Y} be a closed left and right submodule of \mathfrak{X} . We make \mathfrak{X}/\mathcal{Y} into a Banach \mathcal{A} -bimodule by defining

$$a(x + \mathcal{Y}) = ax + \mathcal{Y}, \quad (x + \mathcal{Y})a = xa + \mathcal{Y}, \quad a \in \mathcal{A}, x \in \mathfrak{X}.$$

The dual mapping $(\mathfrak{X}/\mathcal{Y})' \rightarrow \mathcal{Y}^\perp$ then becomes an \mathcal{A} -bimodule isomorphism, defining dual modules the usual way; this is easy to check, and it will be used thoroughly in the following.

A very important theorem becomes useful now.

Theorem 1.7 (Cohen’s factorization theorem). *Let \mathcal{A} be a Banach algebra, let \mathfrak{X} be a left/right Banach \mathcal{A} -module and let $z \in \mathfrak{X}$, $\delta > 0$. If \mathcal{A} contains a bounded left/right approximate identity for \mathfrak{X} , then there exists $a \in \mathcal{A}$, $y \in \mathfrak{X}$ such that $z = ay/z = ya$ and $\|z - y\| < \delta$.*

Proof. Omitted. Consult [1, Proposition 11.10]. \square

Let us now take a look at a lemma proving an important formality in the upcoming Theorem 1.10.

Lemma 1.8. *Let \mathcal{A} be a Banach algebra with a bounded approximate identity $(e_v)_{v \in V}$, and let \mathfrak{X} be a Banach \mathcal{A} -bimodule. Then $\mathfrak{X}_1 := \{xa \mid a \in \mathcal{A}, x \in \mathfrak{X}\}$ and $\mathfrak{X}_0 := \mathcal{A}\mathfrak{X}_1$ are respectively a closed right and a closed left and right submodule of \mathfrak{X} .*

Proof. Let \mathfrak{B} denote $\overline{\text{span } \mathfrak{X}_1}$. \mathfrak{B} is clearly a right submodule of \mathfrak{X} , and so is the closure, since closures of subspaces are subspaces, and if $x_n \rightarrow x$ for $x_n \in \mathfrak{B}$, then $x_n a \rightarrow xa$ for all $a \in \mathcal{A}$, so that $xa \in \mathfrak{B}$. Let $S > 0$ denote the constant bounding (e_v) . Since \mathfrak{B} is closed, it thus becomes a Banach right \mathcal{A} -module. For all $\xi = \sum_n x_n a_n \in \mathfrak{B}$, $\xi e_v = \sum_n x_n (a_n e_v) \rightarrow \xi$, and for any $\zeta \in \mathfrak{B}$ and any $\varepsilon > 0$, there exists $\eta \in \mathfrak{B}$ such that $\|\zeta - \eta\| < \varepsilon$, so that

$$\|\zeta e_v - \zeta\| \leq \|(\zeta - \eta)e_v\| + \|\eta e_v - \eta\| + \|\eta - \zeta\| < \varepsilon(S + 2),$$

for $v \geq w$, where $w \in V$ is chosen such that $\|\eta e_w - \eta\| < \varepsilon$. Hence (e_v) is a bounded right approximate identity for the right Banach \mathcal{A} -module \mathfrak{B} . By Cohen’s factorization theorem, $\mathfrak{B} = \mathfrak{B}\mathcal{A}$ and so $\mathfrak{X}_1 \subseteq \mathfrak{B} = \mathfrak{B}\mathcal{A} \subseteq \mathfrak{X}_1$, so $\mathfrak{X}_1 = \mathfrak{B}$.

If we now let $\mathcal{C} = \text{span } \mathfrak{X}_0$, \mathcal{C} is easily seen to be a left and right submodule of \mathfrak{X} ; the closure is as well, and thus $\overline{\mathcal{C}}$ becomes a Banach \mathcal{A} -bimodule. (e_v) is easily seen, by the same method as above, to be a bounded approximate identity of $\overline{\mathcal{C}}$, and by Cohen’s factorization theorem, $\overline{\mathcal{C}} = \overline{\mathcal{C}}\mathcal{A} = \mathcal{A}\overline{\mathcal{C}} = \mathcal{A}\overline{\mathcal{C}}\mathcal{A}$, so that $\mathfrak{X}_0 \subseteq \overline{\mathcal{C}} = \mathcal{A}\overline{\mathcal{C}}\mathcal{A} \subseteq \mathcal{A}\mathfrak{X}_1\mathcal{A} = \mathfrak{X}_0$. \square

This next definition will come in really handy in all cases, as we shall see:

Definition 1.9. Let \mathcal{A} be a Banach algebra. A Banach \mathcal{A} -bimodule \mathfrak{X} is called *neo-unital* if

$$\mathfrak{X} = \{axb \mid a, b \in \mathcal{A}, x \in \mathfrak{X}\}.$$

By the proof of Lemma 1.8, we have that if \mathcal{A} is a Banach algebra with a bounded approximate identity $(e_v)_{v \in V}$ and \mathfrak{X} is a neo-unital Banach \mathcal{A} -bimodule (that is, if $\mathfrak{X} = \mathcal{A}\mathfrak{X}\mathcal{A}$), then $(e_v)_{v \in V}$ is a bounded approximate identity for \mathfrak{X} .

Neo-unitality seems really neat at first glance, but it becomes even neater now: indeed, in a bowling analogy, this next theorem is a real strike.

Theorem 1.10. *Let \mathcal{A} be a Banach algebra with a bounded approximate identity $(e_v)_{v \in V}$. Then \mathcal{A} is amenable if and only if $\mathcal{H}^1(\mathcal{A}, \mathfrak{X}') = \{0\}$ for all neo-unital Banach \mathcal{A} -bimodules \mathfrak{X} .*

Proof. The “only if” part is clear. Let \mathfrak{X} be a Banach \mathcal{A} -bimodule, and let $D \in \mathcal{Z}^1(\mathcal{A}, \mathfrak{X}')$; we aim to show that D is inner and thus that there exists $x \in \mathfrak{X}'$ such that $D = \delta_x$. By letting $\mathfrak{X}_1 := \{xa \mid a \in \mathcal{A}, x \in \mathfrak{X}\}$ and $\mathfrak{X}_0 := \mathcal{A}\mathfrak{X}_1 \subseteq \mathfrak{X}_1$; \mathfrak{X}_1 and \mathfrak{X}_0 are closed submodules by the preceding lemma, and furthermore, \mathfrak{X}_0 is a Banach \mathcal{A} -bimodule that is neo-unital by Cohen’s factorization theorem; indeed, since (e_v) is a bounded approximate identity for \mathfrak{X}_0 , then for $x \in \mathfrak{X}_0$, $x = cy = c(zd)$ for some $c, d \in \mathcal{A}$, $y, z \in \mathfrak{X}_0$. The proof is quite meticulous and comes in two halves, one in which we prove that $\mathcal{H}^1(\mathcal{A}, \mathfrak{X}'_1) = \{0\}$ and one in which we then prove that $\mathcal{H}^1(\mathcal{A}, \mathfrak{X}') = \{0\}$ in much the same way we proved the first half.

Part 1. Make \mathfrak{X}' , \mathfrak{X}'_1 and \mathfrak{X}'_0 into Banach \mathcal{A} -bimodules the usual way. Let $\Delta \in \mathcal{Z}^1(\mathcal{A}, \mathfrak{X}'_1)$ and let $\pi_0 : \mathfrak{X}'_1 \rightarrow \mathfrak{X}'_0$ be the restriction map defined by $\pi_0(f) = f|_{\mathfrak{X}_0}$. π_0 is an \mathcal{A} -bimodule

homomorphism; indeed, for $f, g \in \mathfrak{X}'_1$, $a \in \mathcal{A}$, we have $\pi(f + g) = \pi(f) + \pi(g)$ and moreover for $x \in \mathfrak{X}_0$,

$$\langle x, af \rangle = \langle xa, f \rangle = \langle xa, f|_{\mathfrak{X}_0} \rangle = \langle x, af|_{\mathfrak{X}_0} \rangle \quad \text{and} \quad \langle x, fa \rangle = \langle ax, f \rangle = \langle ax, f|_{\mathfrak{X}_0} \rangle = \langle x, f|_{\mathfrak{X}_0} a \rangle.$$

Now $\pi_0 \circ \Delta$ belongs to $\mathcal{Z}^1(\mathcal{A}, \mathfrak{X}'_0)$, so because of neo-unitality of \mathfrak{X}_0 , $\pi_0 \circ \Delta$ is inner, and there exists $\Phi_0 \in \mathfrak{X}'_0$ such that $\pi_0 \circ \Delta = \delta_{\Phi_0}$. By Hahn-Banach, we can extend Φ_0 to a $\Phi \in \mathfrak{X}'_1$ such that $\Phi|_{\mathfrak{X}_0} = \Phi_0$.

We focus now on

$$\mathfrak{X}_0^\perp = \{\varphi \in \mathfrak{X}'_1 \mid \varphi(\mathfrak{X}_0) = \{0\}\}.$$

\mathfrak{X}_0^\perp is obviously closed in \mathfrak{X}'_1 and a closed left and right submodule of \mathfrak{X}'_1 , since $a\varphi$ and φa vanish on \mathfrak{X}_0 for all $\varphi \in \mathfrak{X}'_1 \cap \mathfrak{X}_0^\perp$ and $a \in \mathcal{A}$, and can thus be considered as an \mathcal{A} -bimodule. Then for $a \in \mathcal{A}$ and $x \in \mathfrak{X}_0$,

$$\langle x, \Delta(a) \rangle = \langle x, \pi_0 \circ \Delta(a) \rangle = \langle x, \delta_{\Phi_0}(a) \rangle = \langle x, a\Phi_0 - \Phi_0 a \rangle = \langle x, a\Phi - \Phi a \rangle = \langle x, \delta_\Phi(a) \rangle,$$

so that $(\Delta - \delta_\Phi)(a)$ vanishes on \mathfrak{X}_0 . Furthermore, $(\Delta - \delta_\Phi)(a) \in \mathfrak{X}'_1$, and so because the mapping $\Delta - \delta_\Phi : \mathcal{A} \rightarrow \mathfrak{X}'_1 \cap \mathfrak{X}_0^\perp$ can easily be checked to be a bounded and linear derivation, $\Delta - \delta_\Phi \in \mathcal{Z}^1(\mathcal{A}, \mathfrak{X}'_1 \cap \mathfrak{X}_0^\perp)$.

We now consider $(\mathfrak{X}_1/\mathfrak{X}_0)'$ and $\mathfrak{X}'_1 \cap \mathfrak{X}_0^\perp$ as isomorphic Banach \mathcal{A} -bimodules (by Lemma 0.4). Since $\mathcal{A}\mathfrak{X}_1 = \mathfrak{X}_0$, $\mathfrak{X}_1/\mathfrak{X}_0$ satisfies $\mathcal{A}(\mathfrak{X}_1/\mathfrak{X}_0) = \{0\}$, and so we have $(\mathfrak{X}_1/\mathfrak{X}_0)'\mathcal{A} = \{0\}$.

Let $\Lambda \in \mathcal{Z}^1(\mathcal{A}, (\mathfrak{X}_1/\mathfrak{X}_0)')$; then $\Lambda(ab) = a\Lambda(b)$ and specially $\Lambda(ae_v) = a\Lambda(e_v)$ for $a, b \in \mathcal{A}$, $v \in V$. Since all $\Lambda(e_v)$ are contained in a bounded ball in \mathfrak{X}' , Banach-Alaoglu's theorem (Theorem D.2) tells us that $(\Lambda(e_v))$ has a convergent subnet $(\Lambda(e_w))_{w \in W}$ in the w^* topology. Let $\Psi := \lim_w \Lambda(e_w)$; then

$$\Lambda(a) = \lim_{w \in W} \Lambda(ae_w) = \lim_{w \in W} a\Lambda(e_w) = a\Psi = a\Psi - \Psi a$$

for all $a \in \mathcal{A}$, so Λ is inner. Hence $\mathcal{H}^1(\mathcal{A}, \mathfrak{X}'_1 \cap \mathfrak{X}_0^\perp) = \{0\}$ by Lemma 0.4, and there exists a $\Phi_1 \in \mathfrak{X}'_1 \cap \mathfrak{X}_0^\perp$, so that $\Delta - \delta_{\Phi_0} = \delta_{\Phi_1}$ and $\Delta = \delta_{\Phi_0 + \Phi_1}$. Because Δ was arbitrary, we at last obtain $\mathcal{H}^1(\mathcal{A}, \mathfrak{X}'_1) = \{0\}$.

Part 2. The hard part of the proof is over; generalizing further involves a lot of recycling of the above arguments. Let $\Delta \in \mathcal{H}^1(\mathcal{A}, \mathfrak{X}')$ and let $\pi : \mathfrak{X}' \rightarrow \mathfrak{X}'_1$ be the restriction map defined by $\pi(f) = f|_{\mathfrak{X}_1}$. π is an \mathcal{A} -bimodule homomorphism by the same argument as above. Now $\pi \circ \Delta$ belongs to $\mathcal{Z}^1(\mathcal{A}, \mathfrak{X}'_1)$ and is thus inner, since $\mathcal{H}^1(\mathcal{A}, \mathfrak{X}'_1) = \{0\}$; let $\phi_0 \in \mathfrak{X}'_1$ such that $\pi \circ \Delta = \delta_{\phi_0}$. By Hahn-Banach, we can extend ϕ_0 to a $\phi \in \mathfrak{X}'$ such that $\phi|_{\mathfrak{X}_1} = \phi_0$.

We find in the same manner as above that $(\Delta - \delta_\phi)(a)$ vanishes on \mathfrak{X}_1 for every $a \in \mathcal{A}$, and thus $\Delta - \delta_\phi \in \mathcal{Z}^1(\mathcal{A}, \mathfrak{X}_1^\perp)$, with $\mathfrak{X}_1^\perp = \{\varphi \in \mathfrak{X}' \mid \varphi(\mathfrak{X}_1) = \{0\}\}$ considered as a Banach \mathcal{A} -bimodule.

For any $a \in \mathcal{A}$, $x \in \mathfrak{X}$, we have $xa \in \mathfrak{X}_1$, so that $(\mathfrak{X}/\mathfrak{X}_1)\mathcal{A} = \{0\}$ and $\mathcal{A}(\mathfrak{X}/\mathfrak{X}_1)' = \{0\}$. Let $\Lambda \in \mathcal{Z}^1(\mathcal{A}, (\mathfrak{X}/\mathfrak{X}_1)')$; then $\Delta(ab) = \Delta(a)b$ and specially $\Lambda(e_v a) = \Lambda(e_v)a$ for $a, b \in \mathcal{A}$, $v \in V$. If Ψ is once again the limit of a w^* -convergent subnet $(\Lambda(e_w))_{w \in W}$ of $(\Lambda(e_v))$, then

$$\Lambda(a) = \lim_{w \in W} \Lambda(e_w a) = \lim_{w \in W} \Lambda(e_w)a = \Psi a = a(-\Psi) - (-\Psi)a$$

for all $a \in \mathcal{A}$, so Λ is inner. Since $\mathfrak{X}_1^\perp \simeq (\mathfrak{X}/\mathfrak{X}_1)'$ as Banach \mathcal{A} -bimodules by Lemma 0.4, we have $\mathcal{H}^1(\mathcal{A}, \mathfrak{X}_1^\perp) = \{0\}$; thus exists a $\psi \in \mathfrak{X}_1^\perp$ such that $\Delta - \delta_\phi = \delta_\psi$, and so $\Delta = \delta_{\phi + \psi}$. \square

Why do we prove this theorem, may one ask: the answer would be that it *is* immensely useful that we only need consider neo-unital Banach bimodules over a Banach algebra in order to determine its amenability, as we shall see shortly. This next theorem also gets rid of a lot of unwanted obstacles. At least one, anyway.

Lemma 1.11. *Let \mathcal{A} be a Banach algebra, \mathcal{I} a closed two-sided ideal in \mathcal{A} with a bounded approximate identity $(e_v)_{v \in V}$, \mathfrak{X} a neo-unital Banach \mathcal{I} -bimodule and $D \in \mathcal{Z}^1(\mathcal{I}, \mathfrak{X}')$. Then \mathfrak{X} can be made into a Banach \mathcal{A} -bimodule canonically so that there exists an extension of D to \mathcal{A} , $\tilde{D} \in \mathcal{Z}^1(\mathcal{A}, \mathfrak{X}')$.*

Proof. For $x \in \mathfrak{X}$, let $a \in \mathcal{I}$ and $y \in \mathfrak{X}$ such that $x = ay$, possible because \mathfrak{X} is neo-unital. For $\alpha \in \mathcal{A}$, define $\alpha \cdot x = (\alpha a)y$. We need to prove that the two products are independent of the choices of a and y . Thus, let $a_1 \in \mathcal{I}$ and $y_1 \in \mathfrak{X}$ such that $x = a_1 y_1$. Then for all $v \in V$,

$$\alpha(e_v a)x = (\alpha e_v)ax = (\alpha e_v)a_1 x_1 = \alpha(e_v a_1)x_1,$$

and taking limits gives us $\alpha ax = \alpha a_1 x_1$. This way \mathfrak{X} is made into a left \mathcal{A} -module; the very same way, we can make \mathfrak{X} into a right \mathcal{A} -module and thus an \mathcal{A} -bimodule. Finally, \mathfrak{X} is a Banach \mathcal{A} -bimodule, since for $x = ay \in \mathfrak{X}$, where $a \in \mathcal{I}$, $y \in \mathfrak{X}$, and for $\alpha \in \mathcal{A}$, we have for the left multiplication that

$$\|\alpha x\| = \|\alpha(ay)\| \leq S\|\alpha\|\|ay\| = S\|\alpha\|\|x\|,$$

where $S > 0$ was the bounding constant of \mathfrak{X} , and the same for the right.

Now, to extend D to \mathcal{A} , we proceed as follows: let $\tilde{D} : \mathcal{A} \rightarrow \mathfrak{X}'$ be defined by

$$\langle x, \tilde{D}(a) \rangle = \lim_v \langle x, D(e_v a) - D(e_v) a \rangle, \quad x \in \mathfrak{X}.$$

Let $x \in \mathfrak{X}$ and let $b \in \mathcal{I}$, $z \in \mathfrak{X}$, such that $x = bz$. Now for $a \in \mathcal{A}$,

$$\begin{aligned} \langle x, D(e_v a) - D(e_v) a \rangle &= \langle z, D(e_v a)b - D(e_v) ab \rangle \\ &= \langle z, (D(e_v ab) - e_v a D(b)) - D(e_v ab) + e_v D(ab) \rangle \\ &= \langle z, e_v D(ab) \rangle - \langle z, e_v a D(b) \rangle \\ &= \langle z e_v, D(ab) \rangle - \langle z e_v a, D(b) \rangle \\ &\rightarrow \langle z, D(ab) \rangle - \langle z a, D(b) \rangle, \end{aligned}$$

because (e_v) is a bounded approximate identity for \mathfrak{X} , so the limit exists for all $x \in \mathfrak{X}$, making \tilde{D} well-defined. For $i \in \mathcal{I}$, we have

$$\langle x, \tilde{D}(i) \rangle = \lim_v \langle x, D(e_v i) - D(e_v) i \rangle = \lim_v \langle x, e_v D(i) \rangle = \lim_v \langle x e_v, D(i) \rangle = \langle x, D(i) \rangle,$$

because $D \in \mathcal{Z}^1(\mathcal{I}, \mathfrak{X}')$ and $D(i)$ is continuous, and hence $\tilde{D}|_{\mathcal{I}} = D$. \tilde{D} is linear by linearity of D , and \tilde{D} is bounded: for unit-normed $x \in \mathfrak{X}$, $a \in \mathcal{A}$ and for $v \in V$, we have

$$|\langle x, D(e_v a) - D(e_v) a \rangle| \leq 2\|D\|\|a\|\|e_v\|\|x\| \leq 2\|D\| \sup_v \|e_v\|,$$

so that $\|\tilde{D}\| \leq 2\|D\| \sup_v \|e_v\|$ as well. It then only remains to show that \tilde{D} actually is a derivation. Let $a, b \in \mathcal{A}$, $i, j \in \mathcal{I}$ and $x \in \mathfrak{X}$. Because D is a derivation,

$$\begin{aligned} \langle jxi, \tilde{D}(a) \rangle &= \lim_v \langle xi, D(e_v a)j - D(e_v) aj \rangle \\ &= \lim_v \langle xi, D(e_v aj) - e_v a D(j) - D(e_v aj) + e_v D(aj) \rangle \\ &= \langle xi, D(aj) \rangle - \langle xia, D(j) \rangle \\ &= \lim_v \langle xi, D(ae_v j) \rangle - \langle xia, D(j) \rangle \\ &= \lim_v \langle jxi, D(ae_v) \rangle + \lim_v \langle xia e_v, D(j) \rangle - \langle xia, D(j) \rangle \\ &= \lim_v \langle jxi, D(ae_v) \rangle, \end{aligned}$$

because $iae_v \rightarrow ia$, and because of neo-unitality of \mathfrak{X} , $\langle x, \tilde{D}(a) \rangle = \lim_v \langle x, D(ae_v) \rangle$ for all $x \in \mathfrak{X}$. Let $(f_w)_{w \in V} = (e_v)_{v \in V}$. By continuity of D , $D(a(be_v)) = \lim_w D((af_w)(e_v b))$ for all $v \in V$, and so

$$\begin{aligned}
 \langle x, \tilde{D}(ab) \rangle &= \lim_v \langle x, D(abe_v) \rangle \\
 &= \lim_v \lim_w \langle x, D((af_w)(be_v)) \rangle \\
 &= \lim_v \lim_w \langle be_v x, D(af_w) \rangle + \lim_v \lim_w \langle xaf_w, D(be_v) \rangle \\
 &= \lim_v \langle be_v x, \tilde{D}(a) \rangle + \lim_v \langle \lim_w xaf_w, D(be_v) \rangle \\
 &= \lim_v \langle be_v x, \tilde{D}(a) \rangle + \lim_v \langle xa, D(be_v) \rangle \\
 &= \langle x, \tilde{D}(a)b \rangle + \langle x, a\tilde{D}(b) \rangle,
 \end{aligned}$$

by continuity of all $D(i)$ for $i \in \mathcal{I}$. Hence $\tilde{D} \in \mathcal{Z}^1(\mathcal{A}, \mathfrak{X}')$. \square

Using the above two theorems, we now finally reap the harvest we have sown.

Theorem 1.12. *Let \mathcal{A} be an amenable Banach algebra, and let \mathcal{I} be a closed two-sided ideal in \mathcal{A} . Then \mathcal{I} is amenable if and only if \mathcal{I} contains a bounded approximate identity.*

Proof. The “only if” part follows from Theorem 1.6. Assume that \mathcal{I} contains a bounded approximate identity. By Lemma 1.10, it suffices to show that $\mathcal{H}^1(\mathcal{I}, \mathfrak{X}') = \{0\}$ for any neo-unital Banach \mathcal{I} -bimodule \mathfrak{X} . Let \mathfrak{X} be a neo-unital Banach \mathcal{I} -bimodule and $D \in \mathcal{Z}^1(\mathcal{I}, \mathfrak{X}')$. By Lemma 1.11, we make \mathfrak{X} into a Banach \mathcal{A} -bimodule canonically such that D has a unique extension $\tilde{D} \in \mathcal{Z}^1(\mathcal{A}, \mathfrak{X}')$, possible because \mathcal{I} has a bounded approximate identity. Since \mathcal{A} is amenable, $\tilde{D} = \delta_\phi$ for some $\phi \in \mathfrak{X}'$, and so by restriction to \mathcal{I} , $D = \delta_\phi|_{\mathcal{I}}$, making D inner. \square

Thus all amenable closed ideals of amenable Banach algebras are characterized by having a bounded approximate identity. Let us take it a step further:

Theorem 1.13. *Let \mathcal{A} be a Banach algebra and \mathcal{I} a closed two-sided ideal in \mathcal{A} containing a bounded approximate identity $(e_v)_{v \in V}$. Then \mathcal{A} is amenable if and only if \mathcal{I} and \mathcal{A}/\mathcal{I} are amenable.*

Proof. The “only if” part is clear from Theorem 1.12 and Corollary 1.4.

Suppose then that \mathcal{I} and \mathcal{A}/\mathcal{I} are amenable Banach algebras, and let \mathfrak{X} be a Banach \mathcal{A} -bimodule. Letting $D \in \mathcal{Z}^1(\mathcal{A}, \mathfrak{X}')$ gives $D|_{\mathcal{I}} \in \mathcal{Z}^1(\mathcal{I}, \mathfrak{X}')$, and restricting the \mathcal{A} multiplications on \mathfrak{X} to \mathcal{I} , making \mathfrak{X} a Banach \mathcal{I} -bimodule, gives us that there exists $\phi \in \mathfrak{X}'$ such that $D = \delta_\phi$ on \mathcal{I} . Therefore we now just need to get the case for \mathcal{A}/\mathcal{I} covered.

$D - \delta_\phi$ is a derivation on \mathcal{A} , and for all $a \in \mathcal{A}$, $b \in \mathcal{I}$, $x \in \mathfrak{X}$, we have

$$\begin{aligned}
 0 &= (D - \delta_\phi)(ab) = (D - \delta_\phi)(a)b + a(D - \delta_\phi)(b) = (D - \delta_\phi)(a)b \quad \text{and} \\
 0 &= (D - \delta_\phi)(ba) = (D - \delta_\phi)(b)a + b(D - \delta_\phi)(a) = b(D - \delta_\phi)(a),
 \end{aligned}$$

since $D = \delta_\phi$ on \mathcal{I} , so $\langle bx, (D - \delta_\phi)(a) \rangle = 0$ and $\langle xb, (D - \delta_\phi)(a) \rangle = 0$. Let

$$\mathfrak{X}_{\mathcal{I}} = \text{span}\{ax + yb \mid a, b \in \mathcal{I}, x, y \in \mathfrak{X}\}.$$

Then the above equations show that for all $a \in \mathcal{A}$, $\langle y, (D - \delta_\phi)(a) \rangle = 0$ for all $y \in \mathfrak{X}_{\mathcal{I}}$ and thus for all $y \in \mathcal{Y} := \overline{\mathfrak{X}_{\mathcal{I}}}$ by continuity of $(D - \delta_\phi)(a)$. Hence $D - \delta_\phi$ maps \mathcal{A} into \mathcal{Y}^\perp .

Because $\mathfrak{X}_{\mathcal{I}}$ is a subspace of \mathfrak{X} , then so is \mathcal{Y} , and \mathcal{Y} is easily seen to be a closed left and right \mathcal{A} -submodule of \mathfrak{X} . \mathfrak{X}/\mathcal{Y} is made into a Banach \mathcal{A}/\mathcal{I} -bimodule by defining

$$(a + \mathcal{I})(x + \mathcal{Y}) := ax + \mathcal{Y}, \quad (x + \mathcal{Y})(a + \mathcal{I}) := xa + \mathcal{Y}, \quad a \in \mathcal{A}, x \in \mathfrak{X}.$$

Since the dual mapping π^* is a module isomorphism $(\mathfrak{X}/\mathcal{Y})' \rightarrow \mathcal{Y}^\perp$, \mathcal{Y}^\perp can be made into a Banach \mathcal{A}/\mathcal{I} -bimodule by defining

$$\langle x, \rho\alpha \rangle := \langle x + \mathcal{Y}, \tilde{\rho}\alpha \rangle, \quad \langle x, \alpha\rho \rangle := \langle x + \mathcal{Y}, \alpha\tilde{\rho} \rangle, \quad \rho \in \mathcal{Y}^\perp, \alpha \in \mathcal{A}/\mathcal{I}, x \in \mathfrak{X},$$

where $\tilde{\rho}(x + \mathcal{Y}) = \rho(x)$ for all $x \in \mathfrak{X}$ (since $\pi^{*-1}(\rho) = \tilde{\rho}$ per the proof of Lemma 0.4).

Because $D - \delta_\phi = 0$ on \mathcal{I} , the mapping $\Delta : \mathcal{A}/\mathcal{I} \rightarrow \mathcal{Y}^\perp$ by $\Delta(a + \mathcal{I}) = (D - \delta_\phi)(a)$ is well-defined; if $a - b \in \mathcal{I}$, then $(D - \delta_\phi)(a) = (D - \delta_\phi)(b)$. It is easy to see that $\Delta \in \mathcal{Z}^1(\mathcal{A}/\mathcal{I}, \mathcal{Y}^\perp)$, and by the assumption, there exists $\varphi \in \mathcal{Y}^\perp$ such that $\Delta = \delta_\varphi$. Letting $\pi : \mathcal{A} \rightarrow \mathcal{A}/\mathcal{I}$ denote the canonical mapping, we have that $D - \delta_\phi = \Delta \circ \pi = \delta_\varphi \circ \pi$, and so

$$\begin{aligned} \langle x, (D - \delta_\phi)(a) \rangle &= \langle x, \pi(a)\varphi - \varphi\pi(a) \rangle \\ &= \langle x + \mathcal{Y}, \pi(a)\tilde{\varphi} \rangle - \langle x + \mathcal{Y}, \tilde{\varphi}\pi(a) \rangle \\ &= \langle xa + \mathcal{Y}, \tilde{\varphi} \rangle - \langle ax + \mathcal{Y}, \tilde{\varphi} \rangle \\ &= \langle xa, \varphi \rangle - \langle ax, \varphi \rangle \\ &= \langle x, a\varphi - \varphi a \rangle. \end{aligned}$$

Thus $D - \delta_\phi = \delta_\varphi$, and letting $\Psi = \phi + \varphi$, we finally obtain that $D = \delta_\Psi$. \square

We finally prove a theorem concerning *directed unions* (conditions 1 and 2 of the theorem) of subalgebras of a Banach algebra. We can actually conclude amenability of the Banach algebra in which they are contained, provided they satisfy certain premises.

Theorem 1.14. *Let \mathcal{A} be a Banach algebra, and let $\{\mathcal{A}_\lambda \mid \lambda \in \Lambda\}$ be a system of closed subalgebras of \mathcal{A} such that*

1. $\mathcal{A} = \overline{\bigcup_{\lambda \in \Lambda} \mathcal{A}_\lambda}$;
2. if $\lambda, \lambda' \in \Lambda$, there exists $\mu \in \Lambda$ with $\mathcal{A}_\mu \supseteq \mathcal{A}_\lambda \cup \mathcal{A}_{\lambda'}$;
3. if \mathfrak{X} is a Banach \mathcal{A}_λ -bimodule for all $\lambda \in \Lambda$, then there exists $K > 0$ such that for all $\lambda \in \Lambda$ and $D \in \mathcal{Z}^1(\mathcal{A}_\lambda, \mathfrak{X}')$, there is a $\rho \in \mathfrak{X}'$ such that $D = \delta_\rho$ and $\|\rho\| \leq K\|D\|$.

Then \mathcal{A} is amenable.

Proof. Let \mathfrak{X} be a Banach \mathcal{A} -bimodule and let $D \in \mathcal{Z}^1(\mathcal{A}, \mathfrak{X}')$. Defining $\lambda \leq \mu$ for $\mathcal{A}_\lambda \subseteq \mathcal{A}_\mu$ makes \mathcal{A} into a directed set. Now, for all $\lambda \in \Lambda$, \mathfrak{X} can be made naturally into a Banach \mathcal{A}_λ -bimodule and we naturally have $D|_{\mathcal{A}_\lambda} \in \mathcal{Z}^1(\mathcal{A}_\lambda, \mathfrak{X}')$; thus for all $\lambda \in \Lambda$, choose a $\rho_\lambda \in \mathfrak{X}'$ such that $D|_{\mathcal{A}_\lambda} = \delta_{\rho_\lambda}$ and $\|\rho_\lambda\| \leq K\|D|_{\mathcal{A}_\lambda}\| \leq K\|D\|$.

The net $(\rho_\lambda)_\lambda$ in \mathfrak{X}' is thus bounded; inducing the w^* topology on \mathfrak{X}' and applying Banach-Alaoglu's theorem, $(\rho_\lambda)_\lambda$ has a subnet $(\rho_\mu)_{\mu \in M}$, $M \subseteq \Lambda$, converging to a $\rho \in \mathfrak{X}'$.

If $a \in \bigcup_{\lambda \in \Lambda} \mathcal{A}_\lambda$, then there is $\lambda \in \Lambda$ such that $a \in \mathcal{A}_\lambda$. Choosing $\mu \in M$ such that $\lambda \leq \mu$ thus gives that $a \in \mathcal{A}_\lambda \subseteq \mathcal{A}_\mu \subseteq \mathcal{A}_\sigma$ for all $\sigma \in M$ such that $\mu \leq \sigma$. For all $\sigma \in M$ such that $\mu \leq \sigma$, we thus have that for $x \in \mathfrak{X}$ that

$$\langle x, D(a) \rangle = \langle x, D|_{\mathcal{A}_\sigma}(a) \rangle = \langle x, \delta_{\rho_\sigma}(a) \rangle = \langle xa, \rho_\sigma \rangle - \langle ax, \rho_\sigma \rangle,$$

converging by w^* convergence to $\langle xa, \rho \rangle - \langle ax, \rho \rangle = \langle x, \delta_\rho(a) \rangle$. Hence $D(a) = \delta_\rho(a)$ for all $a \in \bigcup_{\lambda \in \Lambda} \mathcal{A}_\lambda$, and by continuity of D and δ_ρ on \mathcal{A} , we obtain $D = \delta_\rho$ on \mathcal{A} . \square

All in all, the theorems in this chapter should provide most of the framework in which we shall now take various Banach algebras under examination, as well as familiarize the reader with the most relevant concepts. The title of the next chapter is then really quite obvious!

Examples of amenable Banach algebras

The term *amenable* actually originates in group theory, first conceived in connection with the well-known Banach-Tarski paradox by von Neumann. One might then wonder why the same name for a property occurs in Banach algebra theory; the first thing that comes to mind is that group theory *does* apply to (elements of) Banach algebra theory, but amenability itself does obviously not directly translate into group language. However, the discrete group algebra allows for an easy connection as we shall soon see, and with this connection, we are able to prove amenability of a variety of Banach algebras; as was written in the first draft of the thesis:

If this whole bachelor thesis could be compared to *The French Connection*, this would be the car chase. In other words, this is where everything really gets exciting, we take some chances and wind up with some great results. The way that we introduce and deal with it here [however] may come as a surprise...

2.1 Amenable groups

For any group G , $\ell^\infty(G)$ denotes the Banach space consisting of all bounded functions from G into \mathbb{C} with pointwise addition and scalar multiplication and with the uniform norm

$$\|f\|_\infty = \sup_{s \in G} |f(s)|.$$

Any $f \in \ell^\infty(G)$ is called *positive* if $f(g) \geq 0$ for all $g \in G$. For $H \subseteq G$, the *indicator function on H* is the bounded mapping $1_H : G \rightarrow \mathbb{C}$ that equals 1 on all elements of H and 0 everywhere else; we will use the symbol 1, in terms of $\ell^\infty(G)$, to denote the indicator function 1_G .

Definition 2.1. 1. Let G be a group and $g \in G$. The *left translation operator on $\ell^\infty(G)$ with respect to g* is the mapping $\tau_g : \ell^\infty(G) \rightarrow \ell^\infty(G)$ defined by $(\tau_g(f))(h) = f(g^{-1}h)$ for $f \in \ell^\infty(G)$, $h \in G$, i.e. each $f \in \ell^\infty(G)$ defines a mapping $h \mapsto f(g^{-1}h)$.

2. A linear functional $\mu : \ell^\infty(G) \rightarrow \mathbb{C}$ is called a *mean on $\ell^\infty(G)$* if $\mu(1) = 1$ and μ is positive, i.e. if $\mu(f) \geq 0$ for all positive $f \in \ell^\infty(G)$. The set of means on $\ell^\infty(G)$ is denoted \mathfrak{M}_G .

3. A mean on $\ell^\infty(G)$ is called *invariant* if it satisfies

$$\mu(\tau_g(f)) = \mu(f), \quad g \in G, f \in \ell^\infty(G),$$

i.e. if it is invariant under all left translations.

For any $g \in G$, τ_g is well-defined since $|\tau_g(f)(h)| = |f(gh^{-1})| \leq \|f\|_\infty$ for any $f \in \ell^\infty(G)$ and $h \in G$, so that $\tau_g(f) \in \ell^\infty(G)$ with $\|\tau_g(f)\|_\infty \leq \|f\|_\infty$; additionally, since for any $h \in G$ we have $|f(h)| = |f(g^{-1}(gh))| = |\tau_g(f)(gh)| \leq \|\tau_g(f)\|_\infty$, τ_g is actually an isometry. Also, τ_n is linear, and for positive $f \in \ell^\infty(\mathbb{Z})$, $\tau_n(f)(k) = f(k-n) \geq 0$, so τ_n preserves positivity; finally, $\tau_n(1) = 1$.

The previous definitions then culminate in this one:

Definition 2.2. A group G is *amenable* if there exists an invariant mean on $\ell^\infty(G)$.

It is not clear from the definition what an invariant mean on a group actually has to do with the term *mean*, or with taking means for the matter. The following theorem makes it clearer.

Theorem 2.3. *Any finite group G is amenable.*

Proof. The linear functional $\mu : \ell^\infty(G) \rightarrow \mathbb{C}$ defined by

$$\mu(f) = |G|^{-1} \sum_{g \in G} f(g)$$

is an invariant mean of $\ell^\infty(G)$, since $\mu(\tau_h)(f) = \mu(f)$ for any $h \in G$. □

In this light, an amenable group is then a group over which it is possible to take the mean of a bounded function's values over all elements in G , an operation it is not hard not to be familiar with. This mean-taking operation indeed makes such a group seem less abstract, and the term *amenable*, now for groups with invariant means and not Banach algebras, becomes quite appropriate too.

As invariant means are linear functionals on $\ell^\infty(G)$ for a group G , it would be interesting if the set \mathfrak{M} of means has some useful topological properties in $(\ell^\infty(G))'$. As the next theorem proves, it is indeed so.

Theorem 2.4. *\mathfrak{M}_G is a non-empty w^* -compact convex subset of the unit ball of $(\ell^\infty(G))'$, with all elements of \mathfrak{M}_G having norm 1.*

Proof. We proceed in five steps:

(1) *\mathfrak{M} is non-empty.* For any $g \in G$, define $\hat{g}(f) = f(g)$ for $f \in \ell_\infty(G)$. \hat{g} is clearly linear and furthermore, it's bounded, since $|\hat{g}(f)| = |f(g)| \leq \|f\|_\infty$, so $\hat{g} \in (\ell^\infty(G))'$. Since $\hat{g}(1) = 1$ and $\hat{g}(f) = f(g) \geq 0$ for any positive $f \in \ell_\infty(\mathbb{Z})$, \hat{g} is a mean, so \mathfrak{M}_G is non-empty.

(2) *\mathfrak{M}_G is convex.* First, let $\mu_1, \mu_2 \in \mathfrak{M}_G$, for any $\alpha \in [0, 1]$, define

$$q_\alpha(f) = \alpha\mu_1(f) + (1 - \alpha)\mu_2(f)$$

for all $f \in \ell_\infty(G)$. Then

$$|q_\alpha(f)| \leq \alpha|\mu_1(f)| + (1 - \alpha)|\mu_2(f)| \leq \alpha\|\mu_1\| + (1 - \alpha)\|\mu_2\|$$

for all $f \in \ell_\infty(G)$ with $\|f\|_\infty \leq 1$, so $q_\alpha \in (\ell_\infty(G))'$. Because μ_1 and μ_2 are means and $\alpha \in [0, 1]$, then if $f \in \ell_\infty(G)$ is positive, then

$$q_\alpha(f) = \alpha\mu_1(f) + (1 - \alpha)\mu_2(f) \geq \alpha \cdot 0 + (1 - \alpha) \cdot 0 = 0$$

and $q_\alpha(1) = \alpha\mu_1(1) + (1 - \alpha)\mu_2(1) = \alpha + (1 - \alpha) = 1$. Thus $q_\alpha \in \mathfrak{M}_G$, so \mathfrak{M}_G is convex.

(3) *\mathfrak{M} is weak*-closed.* Now, let μ be an element of the closure of \mathfrak{M}_G in $(\ell^\infty(G))'$ equipped with the w^* topology. Then there is a net of means $(\mu_\alpha)_{\alpha \in V}$ converging to μ in the w^* topology, implying that $\mu_\alpha(f) \rightarrow \mu(f)$ for all $f \in \ell_\infty(G)$. We want to show that $\mu(f) \geq 0$ for any positive $f \in \ell_\infty(G)$ and that $\mu(1) = 1$.

Let $f \in \ell_\infty(G)$ be positive. Assume that $\mu(f) < 0$ and let $B = \{z \in \mathbb{C} \mid |z - \mu(f)| < |\mu(f)|\}$ that is a neighbourhood of $\mu(f)$ in \mathbb{C} . Then for all $v \in V$,

$$|\mu_v(f) - \mu(f)| = \mu_v(f) - \mu(f) \geq -\mu(f) = |\mu(f)|,$$

so $\mu_v(f) \notin B$ for all $v \in V$. This implies that $\mu_v(f)$ doesn't converge to $\mu(f)$ since there is a neighbourhood B of $\mu(f)$ such that $(\mu_v(f))_{v \in V}$ isn't eventually in B , contradicting the fact that $\mu_v(f) \rightarrow \mu(f)$ for all $f \in \ell_\infty(G)$; hence $\mu(f) \geq 0$.

Finally, assuming that $\mu(1) \neq 1$, there exists $\varepsilon > 0$ such that $|\mu(1) - 1| > \varepsilon$. Then for all $v \in V$, $|\mu_v(1) - \mu(1)| = |1 - \mu(1)| > \varepsilon$, implying that $\mu_v(1) \notin \{z \in \mathbb{C} \mid |z - \mu(1)| < \varepsilon\}$, a neighbourhood of $\mu(1)$, for any $v \in V$. This implies that $\mu_v(1)$ does not converge to $\mu(1)$ which is again a contradiction. Thus $\mu(1) = 1$, and we conclude that $\mu \in \mathfrak{M}_G$, so that \mathfrak{M}_G is weak*-closed.

(4) \mathfrak{M}_G is contained in the unit ball of $(\ell^\infty(G))'$. Let $\mu \in \mathfrak{M}_G$. We will first prove that $|\mu(f)| \leq \|f\|_\infty$ for real-valued $f \in \ell^\infty(G)$ and then to go on to the general case. Therefore, let $f \in \ell^\infty(G)$ be real-valued. Because $|f(g)| \leq \|f\|_\infty$ for all $g \in G$, then the functions $\|f\|_\infty 1 \pm f$ are positive, so

$$\|f\|_\infty + \mu(f) = \mu(\|f\|_\infty 1 + f) \geq 0, \quad \|f\|_\infty - \mu(f) = \mu(\|f\|_\infty 1 - f) \geq 0,$$

because μ is linear. This yields $\pm\mu(f) \leq \|f\|_\infty$, so $|\mu(f)| \leq \|f\|_\infty$.

For complex-valued $f \in \ell^\infty(G)$, take $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ such that $|\mu(f)| = \lambda\mu(f) = \mu(\lambda f)$. Let $f_1 = \operatorname{Re}\lambda f$ and $f_2 = \operatorname{Im}\lambda f$. Now $\mu(f_1) + i\mu(f_2) = \mu(f_1 + if_2) = \mu(\lambda f) = |\mu(f)| \geq 0$. This implies that $\mu(f_1) + i\mu(f_2) \in \mathbb{R}$, so $\mu(f_2) = 0$. Finally, since $|f_1(g)| \leq |\lambda f(g)|$ for all $g \in G$, then

$$|\mu(f)| = \mu(\lambda f) = \mu(f_1) \leq \|f_1\|_\infty \leq \|\lambda f\|_\infty = \|f\|_\infty,$$

because f_1 is real-valued and $|\lambda| = 1$. Therefore, μ is bounded with norm ≤ 1 . Since $\mu(1) = 1$, we also have $\|\mu\| \geq 1$, so $\|\mu\| = 1$.

(5) \mathfrak{M}_G is w^* -compact. By Banach-Alaoglu's theorem, the unit ball of $(\ell^\infty(G))'$ is compact in $(\ell^\infty(G))'$ equipped with the w^* topology. Since \mathfrak{M}_G is a weak*-closed subset of the unit ball of $(\ell^\infty(G))'$ by (3) and (4), it is therefore weak*-compact. \square

Not all groups are amenable; an example is the free group \mathbb{F}_2 of two generators. However, a well-known large class of groups are amenable.

Theorem 2.5. *Let G be an abelian group. Then G is amenable.*

Proof. Let $\mathfrak{M} = \mathfrak{M}_G$. For $g \in G$ and $\varphi \in (\ell^\infty(G))'$, consider the map $f \mapsto \varphi \circ \tau_g(f)$, $f \in \ell^\infty(G)$. It is linear and furthermore, $|\varphi(\tau_g(f_1))| \leq \|\varphi\| \|\tau_g(f_1)\|_\infty = \|\varphi\| \|f_1\|_\infty$, so $\varphi \circ \tau_g$ is bounded.

For all $g \in G$, we now define maps $T_g : (\ell^\infty(G))' \rightarrow (\ell^\infty(G))'$ given by $T_g(\varphi) = \varphi \circ \tau_g$. For $g \in G$, $\mu \in \mathfrak{M}$ and $f \in \ell^\infty(G)$, note that $T_g(\mu)(1) = \mu(\tau_g(1)) = \mu(1) = 1$ and $T_g(\mu)(f) = \mu(\tau_g(f)) \geq 0$, since μ is a mean and $\tau_g(f)$ is positive. This implies that $T_g(\mu)$ is a mean itself, so $T_g(\mathfrak{M}) \subseteq \mathfrak{M}$. This implies that $F = \{T_g|_{\mathfrak{M}} \mid g \in G\}$ consists of maps $\mathfrak{M} \rightarrow \mathfrak{M}$.

We now prove in steps that all maps in F are (1) continuous, (2) affine and (3) commute.

(1) For any $g \in G$, let $\mu_1, \mu_2 \in \mathfrak{M}$. Then

$$\|T_g|_{\mathfrak{M}}(\mu_1) - T_g|_{\mathfrak{M}}(\mu_2)\| = \|\mu_1 \circ \tau_g - \mu_2 \circ \tau_g\| = \|(\mu_1 - \mu_2) \circ \tau_g\| \leq \|\mu_1 - \mu_2\|,$$

since we proved when defining T_g that $\|\varphi \circ \tau_g\| \leq \|\varphi\|$ for all $\varphi \in (\ell^\infty(G))'$. This implies that all $T_g|_{\mathfrak{M}}$, $g \in G$, are continuous linear operators $\mathfrak{M} \rightarrow \mathfrak{M}$.

(2) For $\alpha \in [0, 1]$ and $\mu_1, \mu_2 \in \mathfrak{M}$, we have that $\alpha\mu_1 + (1 - \alpha)\mu_2 \in \mathfrak{M}$ since \mathfrak{M} is convex. Thus

$$\begin{aligned} T_g|_{\mathfrak{M}}(\alpha\mu_1 + (1 - \alpha)\mu_2)(f) &= ((\alpha\mu_1 + (1 - \alpha)\mu_2) \circ \tau_g)(f) \\ &= (\alpha\mu_1)(\tau_g(f)) + ((1 - \alpha)\mu_2)(\tau_g(f)) \\ &= \alpha\mu_1(\tau_g(f)) + (1 - \alpha)\mu_2(\tau_g(f)) \\ &= \alpha(T_g|_{\mathfrak{M}}(\mu_1) + (1 - \alpha)T_g|_{\mathfrak{M}}(\mu_2))(f) \end{aligned}$$

for all $g \in G$ and $f \in \ell^\infty(G)$, so $T_g|_{\mathfrak{M}}$ is affine for any $g \in G$.

(g) For $g, h \in G$, $\mu \in \mathfrak{M}$ and $f \in \ell^\infty(G)$, we first have

$$\tau_g(\tau_h(f))(k) = \tau_g(f)(h^{-1}k) = f(g^{-1}h^{-1}k) = f(h^{-1}g^{-1}k) = \tau_h(f)(g^{-1}k) = \tau_h(\tau_g(f))(k)$$

for all $k \in G$, so $\tau_g(\tau_h(f)) = \tau_h(\tau_g(f))$. This in turn implies that

$$\begin{aligned} (T_g|_{\mathfrak{M}}T_h|_{\mathfrak{M}}(\mu)(f))(k) &= ((\mu \circ \tau_h) \circ \tau_g)(f)(k) \\ &= \mu(\tau_h(\tau_g(f)))(k) \\ &= \mu(\tau_g(\tau_h(f)))(k) \\ &= ((\mu \circ \tau_g) \circ \tau_h)(f)(k) \\ &= (T_h|_{\mathfrak{M}}T_g|_{\mathfrak{M}}(\mu)(f))(k), \end{aligned}$$

because $T_g(\mathfrak{M}), T_h(\mathfrak{M}) \subseteq \mathfrak{M}$, so all elements of F commute.

Applying Markov-Kakutani's fixed point theorem (Theorem D.6) to \mathfrak{M} (that indeed satisfies all the requirements, by Theorem 2.4) and the family F of functions $\mathfrak{M} \rightarrow \mathfrak{M}$, there exists a $\mu \in \mathfrak{M}$ which is fixed by all $T_g|_{\mathfrak{M}}$ for $g \in G$: thus this μ satisfies $\mu \circ \tau_g = T_g|_{\mathfrak{M}}(\mu) = \mu$ for all $g \in G$, so μ is invariant and G is therefore amenable. \square

This is a great result, and we will use it in abundance throughout the chapter. However, we are going to keep digging for just a little while.

2.2 The discrete group algebra

We now turn our heads back to Banach algebraic territory. For a group G , we define $\ell^1(G)$ to be the set of functions $f : G \rightarrow \mathbb{C}$ for which

$$\sum_{s \in G} |f(s)| < \infty.$$

With pointwise addition and scalar multiplication, and equipped with the norm

$$\|f\| = \sum_{s \in G} |f(s)|,$$

$\ell^1(G)$ becomes a Banach space. We make $\ell^1(G)$ into a Banach algebra, the *discrete group algebra on G* , by making *convolution* the product, that is

$$(f * g)(s) = \sum_{t \in G} f(t)g(t^{-1}s), \quad s \in G.$$

The reason why we define this Banach algebra will become apparent in a short while; it is quite wonderful. The first question one should ask concerning $\ell^1(G)$, however, isn't about reasons for defining it but rather about whether it is actually possible to sum over a perhaps uncountable set.

The answer is that, yes, it *is* possible. We can define the sum over G in terms of nets, as the limit of the net of sums over finite subsets of G . This is possibly the strictest way to define the sum; for this cause, we define $\mathcal{P}_f(G)$ to be the set of all finite subsets of G . However, since the series with which we operate have finite sum, we often have the possibility to take the easy way out and note that every function $f \in \ell^1(G)$ has countable support: indeed, if we let $A_k = \{s \in G \mid |f(s)| > 1/k\}$ for $k \in \mathbb{N}$, then

$$\frac{1}{k} \text{card}(A_k) < \sum_{s \in A_k} |f(s)| \leq \sum_{s \in G} |f(s)| < \infty,$$

so $\text{supp}(f) = \bigcup_{k \in \mathbb{N}} A_k$ is a countable union of finite sets and thus countable. Hence all our standard rules for manipulating with series over countable index sets also hold for series over uncountable ones, as long as we operate with the ones with finite sum.

Definition 2.6. Let \mathcal{A} be a Banach algebra. A non-zero linear functional $\sigma : \mathcal{A} \rightarrow \mathbb{C}$ is *multiplicative* if it satisfies the equation

$$\sigma(ab) = \sigma(a)\sigma(b), \quad a, b \in \mathcal{A}.$$

In Appendix A, it is proved that *any* multiplicative linear functional σ over *any* Banach algebra is bounded, with $\|\sigma\| \leq 1$.

Lemma 2.7. *Let \mathcal{A} be an amenable Banach algebra and let σ be a multiplicative linear functional on \mathcal{A} . Then there exists $F \in \mathcal{A}'' \setminus \{0\}$ such that $\langle \sigma, F \rangle = 1$ and*

$$\langle fa, F \rangle = \sigma(a)\langle f, F \rangle, \quad a \in \mathcal{A}, f \in \mathcal{A}'.$$

Proof. We make \mathcal{A}' into a Banach \mathcal{A} -bimodule by defining multiplications

$$\langle x, a\varphi \rangle := \sigma(a)\langle x, \varphi \rangle, \quad \langle x, \varphi a \rangle := \langle ax, \varphi \rangle, \quad x, a \in \mathcal{A}, \varphi \in \mathcal{A}'.$$

For $a \in \mathcal{A}$, then because $\sigma \in \mathcal{A}'$, we have $a\sigma = \sigma(a)\sigma$ and

$$\langle x, \sigma a \rangle = \langle ax, \sigma \rangle = \sigma(a)\langle x, \sigma \rangle = \langle x, \sigma(a)\sigma \rangle, \quad x \in \mathcal{A},$$

so $a\sigma = \sigma a = \sigma(a)\sigma$. Now consider $\mathbb{C}\sigma$. $\mathbb{C}\sigma$ is an \mathcal{A} -submodule of \mathcal{A}' : it is clear that $\mathbb{C}\sigma$ is a subspace of \mathcal{A}' and letting $a \in \mathcal{A}$, $\lambda\sigma \in \mathbb{C}\sigma$, we obtain that

$$a(\lambda\sigma) = \sigma(a)\lambda\sigma \in \mathbb{C}\sigma, \quad \langle x, (\lambda\sigma)a \rangle = \lambda\langle ax, \sigma \rangle = \langle x, \lambda\sigma(a)\sigma \rangle, \quad x \in \mathcal{A},$$

so $(\lambda\sigma)a = (\lambda\sigma(a))\sigma \in \mathbb{C}\sigma$. Finally, $\mathbb{C}\sigma$ is closed: let $(\lambda_n\sigma) \in \mathbb{C}\sigma$ converge to a $f \in \mathcal{A}'$. Now $|\lambda_m - \lambda_n| = \|\sigma\|^{-1}\|\lambda_m\sigma - \lambda_n\sigma\|$ for $m, n \in \mathbb{N}$, so (λ_n) is Cauchy and thus converges to a $\lambda \in \mathbb{C}$. It is now clear that for $n \rightarrow \infty$,

$$\|\lambda_n\sigma - \lambda\sigma\| = |\lambda_n - \lambda|\|\sigma\| \rightarrow 0.$$

We therefore define the Banach quotient space $\mathfrak{B} = \mathcal{A}'/\mathbb{C}\sigma$ and let π denote the canonical mapping $\mathcal{A}' \rightarrow \mathfrak{B}$. \mathfrak{B} is a Banach \mathcal{A} -bimodule, and the dual mapping $\pi^* : \mathfrak{B}' \rightarrow \mathcal{A}''$ defined by $\pi^*(\varphi) = \varphi \circ \pi$ is an \mathcal{A} -bimodule monomorphism (by Lemma 0.4). As $0 < \|\sigma\| \leq 1$, we can now choose $f \in \mathcal{A}''$ with $\langle \sigma, f \rangle = 1$ by Hahn-Banach (choose $g \in \mathcal{A}''$ with $\langle \sigma, g \rangle = \|\sigma\|$ and let $f = \|\sigma\|^{-1}g$), and we make \mathcal{A}'' into a Banach \mathcal{A} -bimodule the usual way, that is

$$\langle \varphi, aF \rangle := \langle \varphi a, F \rangle, \quad \langle \varphi, Fa \rangle := \langle a\varphi, F \rangle = 0, \quad a \in \mathcal{A}, \varphi \in \mathcal{A}', F \in \mathcal{A}''.$$

Let $\delta : \mathcal{A} \rightarrow \mathcal{A}''$ denote the inner \mathcal{A}'' -derivation given by $\delta(a) = af - fa$. Now for $a \in \mathcal{A}$, we have

$$\langle \sigma, \delta(a) \rangle = \langle \sigma, af - fa \rangle = \langle \sigma a, f \rangle - \langle a\sigma, f \rangle = \langle \sigma a, f \rangle - \langle \sigma a, f \rangle = 0,$$

by what we found earlier. Thus $\delta(a)$ is a linear functional on \mathcal{A}' that vanishes on $\mathbb{C}\sigma$ which means that $\delta(a) \in (\mathbb{C}\sigma)^\perp = \pi^*(\mathfrak{B}')$ by Lemma 0.4. Since π^* is injective, this means that for every $a \in \mathcal{A}$ there exists a unique $g_a \in \mathfrak{B}'$ such that $\pi^*(g_a) = \delta(a)$. Define $D : \mathcal{A} \rightarrow \mathfrak{B}'$ by $D(a) = g_a$.

D is a bounded \mathfrak{B}' -derivation; indeed $D(a) = g_a$ is bounded and linear for $a \in \mathcal{A}$, and for any $a, b \in \mathcal{A}$ and $\varphi \in \mathcal{A}'$ we have

$$\langle b\varphi, a f \rangle - \langle \varphi a, f b \rangle = \langle (b\varphi)a, f \rangle - \langle b(\varphi a), f \rangle = 0,$$

so because π^* is an \mathcal{A} -bimodule homomorphism, we obtain

$$\begin{aligned}
 \langle \varphi, \pi^*(D(ab)) \rangle &= \langle \varphi, \delta(ab) \rangle \\
 &= \langle \varphi, (ab)f \rangle - \langle \varphi, f(ab) \rangle \\
 &= \langle \varphi a, bf \rangle - \langle b\varphi, fa \rangle \\
 &= \langle \varphi a, bf \rangle - \langle \varphi a, fb \rangle + \langle b\varphi, af \rangle - \langle b\varphi, fa \rangle \\
 &= \langle b\varphi, \delta(a) \rangle + \langle \varphi a, \delta(b) \rangle \\
 &= \langle \varphi, \pi^*(D(a))b \rangle + \langle \varphi, a\pi^*(D(b)) \rangle \\
 &= \langle \varphi, \pi^*(D(a)b + aD(b)) \rangle.
 \end{aligned}$$

Thus $\pi^*(D(ab)) = \pi^*(D(a)b + aD(b))$, and $D(ab) = D(a)b + aD(b)$ because π^* is injective.

Because \mathcal{A} is amenable, D must be an inner \mathfrak{B}' -derivation and so there exists $h \in \mathfrak{B}'$ such that $D(a) = ah - ha$ for all $a \in \mathcal{A}$. This in turn means that

$$a(\pi^*(h)) - (\pi^*(h))a = \pi^*(ah - ha) = \pi^*(D(a)) = \delta(a) = af - fa, \quad a \in \mathcal{A}.$$

Now let $F = f - \pi^*(h)$; then $F \in \mathcal{A}''$ and $\langle \sigma, F \rangle = \langle \sigma, f \rangle - \langle \sigma, \pi^*(h) \rangle = 1 - \langle \pi(\sigma), h \rangle = 1$ (which of course means that $F \neq 0$), and furthermore the above equation tells us that $aF = Fa$, so that $\langle fa, F \rangle = \langle f, aF \rangle = \langle f, Fa \rangle = \langle af, F \rangle = \langle \sigma(a)f, F \rangle = \sigma(a)\langle f, F \rangle$. \square

Perhaps these next definitions arrive a little late; however, we have not needed them until now.

Definition 2.8. A *unit element* or *identity* in an algebra \mathcal{A} is a non-zero element $e \in \mathcal{A}$ such that $xe = ex = x$ for all $x \in \mathcal{A}$. An algebra with an identity is called *unital*.

Definition 2.9. Let \mathcal{A} be a unital algebra, and let $a \in \mathcal{A}$. A *left inverse* of a is an element $b \in \mathcal{A}$ such that $ba = 1$, and a *right inverse* of a is an element $c \in \mathcal{A}$ such that $ac = 1$. An *inverse* of a is an element of \mathcal{A} that is both a left inverse and a right inverse of a . We say that an element of \mathcal{A} is *invertible* if it has an inverse.

If a has a left inverse b and a right inverse c , then $b = b1 = bac = 1c = c$ and a is invertible. This also shows that all elements of unital algebras have at most one inverse. We denote the unique inverse of $a \in \mathcal{A}$ by a^{-1} and the set of invertible elements in \mathcal{A} by $\text{Inv}(\mathcal{A})$.

Before we go on to prove possibly the most important theorem of this thesis, we need to introduce some notation. Let G be a group, and let $\mathfrak{B} = \ell^1(G)$ in the following.

For any $g \in G$, let \mathbf{g} denote the function $G \rightarrow \mathbb{C}$ defined by $\mathbf{g} = 1_{\{g\}}$. It is clear that $\mathbf{g} \in \mathfrak{B}$ and $\|\mathbf{g}\| = 1$ for all $g \in G$. Furthermore, recalling that the algebra product in \mathfrak{B} is the convolution $(f * g)(h) = \sum_{t \in G} f(t)g(t^{-1}h)$ for $f, g \in \mathfrak{B}$, we have

$$(f * \mathbf{g})(h) = f(hg^{-1}), \quad (\mathbf{g} * f)(h) = f(g^{-1}h), \quad f \in \mathfrak{B}, g, h \in G.$$

These functions in \mathfrak{B} have some very useful properties. Letting e denote the neutral element in G , it follows from the above formulae that \mathbf{e} is the identity of \mathfrak{B} , and furthermore, the mapping $\psi : G \rightarrow \text{Inv}(\mathfrak{B})$ given by $\psi(g) = \mathbf{g}$ is a monomorphism. Indeed, any $\mathbf{g} \in \mathfrak{B}$ is invertible with inverse \mathbf{g}^{-1} , as

$$(\mathbf{g} * \mathbf{g}^{-1})(h) = 1_{\{g^{-1}\}}(g^{-1}h),$$

which equals 1 iff $h = e$ and 0 otherwise, so $\mathbf{g} * \mathbf{g}^{-1} = \mathbf{e}$; likewise $\mathbf{g}^{-1} * \mathbf{g} = \mathbf{e}$. Finally

$$(\psi(a) * \psi(b))(h) = 1_{\{b\}}(a^{-1}h)$$

which equals 1 iff $b = a^{-1}h$ or $h = ab$, and thus $\psi(a) * \psi(b) = \psi(ab)$. Because $\mathbf{g} = \mathbf{e}$ implies $g = e$, ψ is injective. Thus the subset $\mathbf{G} = \{\mathbf{g} \mid g \in G\}$ of \mathfrak{B} is a multiplicative group.

We claim that for $f \in \mathfrak{B}$, the series $\sum_{g \in G} f(g)\mathbf{g}$ converges to f in norm. Define $\kappa_f : G \rightarrow \mathbb{C}$ by $\kappa_f(g) = \sum_{s \in G} f(s)\mathbf{s}(g)$. It is clear that $\kappa_f \in \mathfrak{B}$ since $|\kappa_f(g)| = |f(g)\mathbf{g}(g)| = |f(g)|$ for all $g \in G$. Now for $F \in \mathcal{P}_f(G)$, we have that

$$\left\| f - \sum_{s \in F} f(s)\mathbf{s} \right\| = \sum_{s \in G} \left| f(s) - \sum_{g \in F} f(g)\mathbf{g}(s) \right| = \sum_{s \in G \setminus F} |f(s)|,$$

since $\sum_{g \in F} f(g)\mathbf{g}(h) = f(h)$ for $h \in F$ and $\sum_{g \in F} f(g)\mathbf{g}(h) = 0$ for $h \in G \setminus F$. Letting $\varepsilon > 0$ and choosing $F_0 \in \mathcal{P}_f(G)$ such that $\sum_{g \in G \setminus F_0} |f(g)| < \varepsilon$ (possible since $\sum_{s \in G} |f(s)| < \infty$) thus implies that $f = \sum_{g \in G} f(g)\mathbf{g}$ by definition of convergence of nets. Hence \mathfrak{B} has the dense subset

$$\text{span } \mathbf{G} = \left\{ \sum_{g \in F} f(g)\mathbf{g} \mid F \in \mathcal{P}_f(G), f \in \mathfrak{B} \right\} = \left\{ \sum_{g \in F} \lambda_g \mathbf{g} \mid F \in \mathcal{P}_f(G), \lambda_g \in \mathbb{C} \right\}.$$

From this also arises the following lemma.

Lemma 2.10. *Let \mathcal{A} be a unital Banach algebra containing a bounded group G with algebra product as the composition. There exists a mapping $\varphi : \ell^1(G) \rightarrow \mathcal{A}$ that is a continuous algebra homomorphism, satisfying $\varphi(\mathbf{g}) = g$ for all $g \in G$.*

Proof. Let $S > 0$ be the constant bounding G . Define $\varphi(\sum \lambda_g \mathbf{g}) = \sum \lambda_g g$ for all $\sum \lambda_g \mathbf{g} \in \ell^1(G)$, $g \in G$. This map is well-defined, since any $\ell^1(G)$ function has countable support, so we only need consider countable sums $\sum_{n=1}^{\infty} \lambda_n \mathbf{g}_n \in \ell^1(G)$, and since

$$\sum_{n=1}^{\infty} \|\lambda_n \mathbf{g}_n\| \leq S \sum |\lambda_n| < \infty$$

by the triangle equality, $\sum \lambda_n \mathbf{g}_n$ converges in \mathcal{A} . φ is obviously linear and continuous on $\ell^1(G)$ since by the above inequality, $\|\varphi(\sum \lambda_g \mathbf{g})\| \leq S \|\sum \lambda_g \mathbf{g}\|$, and furthermore it is an algebra homomorphism on all finite sums, i.e. elements of $\text{span } \mathbf{G}$. Consider the set

$$\{(a, b) \in \ell^1(G) \times \ell^1(G) \mid \varphi(ab) = \varphi(a)\varphi(b)\}.$$

It is clearly closed on $\ell^1(G) \times \ell^1(G)$ because of continuity of φ and product of \mathcal{A} , and it contains all elements of $\text{span } \mathbf{G} \times \text{span } \mathbf{G}$. Since $\text{span } \mathbf{G} \times \text{span } \mathbf{G}$ is dense in $\ell^1(G) \times \ell^1(G)$, the set contains $\ell^1(G) \times \ell^1(G)$, φ is a continuous algebra homomorphism on $\ell^1(G)$. \square

Now comes the absolute zenith of this chapter.

Theorem 2.11. *Let G be a group. Then the group G is amenable if and only if the discrete group algebra $\ell^1(G)$ is amenable.*

Proof. Let $\mathcal{A} = \ell^1(G)$. Define $\sigma(a) = \sum_{g \in G} a(g)$ for $a \in \ell^1(G)$. The proof of Lemma 2.10 can be modified to yield that σ is well-defined and a continuous algebra homomorphism $\mathcal{A} \rightarrow \mathbb{C}$, and thus a continuous, multiplicative linear functional on \mathcal{A} .

Now suppose that \mathcal{A} is amenable. By Lemma 2.7, there exists $F \in \mathcal{A}'' \setminus \{0\}$ with $\langle \sigma, F \rangle = 1$ and $\langle fa, F \rangle = \sigma(a)\langle f, F \rangle$ for $f \in \mathcal{A}'$, $a \in \mathcal{A}$. For any $m \in \ell^\infty(G)$, let m' denote the linear functional on \mathcal{A} defined by

$$\langle a, m' \rangle = \sum_{g \in G} a(g)m(g).$$

For any $m \in \ell^\infty(G)$, m' is bounded, since for any $a \in \mathcal{A}$, we have

$$|\langle a, m' \rangle| = \left| \sum_{g \in G} a(g)m(g) \right| \leq \sum_{g \in G} |a(g)m(g)| \leq \|m\|_\infty \|a\|.$$

The mapping $m \mapsto m'$ is linear as well; it is also an isometry because for $m \in \ell^\infty(G)$ and $g \in G$,

$$|m(g)| = \left| \sum_{h \in G} \mathbf{g}(h)m(h) \right| = |\langle \mathbf{g}, m' \rangle| \leq \|m'\|,$$

and so $\|m\|_\infty \leq \|m'\|$. Now, for $g, h \in G$ and $m \in \ell^\infty(G)$, we have $\tau_{g^{-1}}(m) \in \ell^\infty(G)$ and $(\tau_{g^{-1}}(m))(h) = m(gh)$, whereupon

$$\langle a, (\tau_{g^{-1}}(m))' \rangle = \sum_{s \in G} a(s)m(gs) = \sum_{t \in G} a(g^{-1}t)m(t) = \sum_{t \in G} (\mathbf{g} * a)(t)m(t) = \langle \mathbf{g} * a, m' \rangle$$

for any $a \in \mathcal{A}$. Make \mathcal{A}' into a Banach \mathcal{A} -bimodule the usual way. Then $(\tau_{g^{-1}}(m))' = m'\mathbf{g}$. Now define $\phi : \ell^\infty(G) \rightarrow \mathbb{C}$ by $\phi(m) = \langle m', F \rangle$. Then obviously $\phi \in (\ell^\infty(G))'$; because $\langle a, 1' \rangle = \sigma(a)$, then $\langle 1, \phi \rangle = \langle \sigma, F \rangle = 1$ and for any $g \in G$,

$$\langle \tau_{g^{-1}}(m), \phi \rangle = \langle (\tau_{g^{-1}}(m))', F \rangle = \langle m'\mathbf{g}, F \rangle = \sigma(\mathbf{g})\langle m', F \rangle = \langle m', F \rangle = \langle m, \phi \rangle.$$

Hence $\langle 1, \phi \rangle = 1$ and ϕ is invariant under left translations. It follows from Lemma C.1 that G is amenable.

Suppose now that G is amenable. Then $\ell^\infty(G)$ has an invariant mean μ . Let \mathfrak{X} be a Banach \mathcal{A} -bimodule and let D be a bounded \mathfrak{X}' -derivation. Furthermore, making \mathfrak{X}' into a Banach \mathcal{A} -bimodule the usual way, we let S be a constant such that

$$\|\varphi a\| \leq S\|\varphi\|\|a\|, \quad \|a\varphi\| \leq S\|a\|\|\varphi\|, \quad a \in \mathcal{A}, \varphi \in \mathfrak{X}'.$$

For any $g \in G$, $\mathbf{g}^{-1} \in \mathcal{A}$, and so $D(\mathbf{g}^{-1}) \in \mathfrak{X}'$. From left \mathcal{A} -module multiplication, we obtain the linear functional $\mathbf{g}D(\mathbf{g}^{-1}) \in \mathfrak{X}'$. Letting $x \in \mathfrak{X}$, we let \mathbf{x} be the complex-valued function on G defined by

$$\mathbf{x}(g) = \langle x, \mathbf{g}D(\mathbf{g}^{-1}) \rangle, \quad g \in G,$$

and so we have

$$\|\mathbf{x}(g)\| = \|(\mathbf{g}D(\mathbf{g}^{-1}))(x)\| \leq S\|\mathbf{g}\|\|D(\mathbf{g}^{-1})\|\|x\| \leq S\|D\|\|x\|,$$

implying that $\mathbf{x} \in \ell^\infty(G)$. Now define the mapping $f : \mathfrak{X} \rightarrow \mathbb{C}$ by $f(x) = \mu(\mathbf{x})$. Because μ is linear, then so is f , and thus $f \in \mathfrak{X}'$ (since $\|\mu\| = 1$ by Lemma 2.4). We claim that $D = \delta_f$.

Fix $x \in \mathfrak{X}$ and $g \in G$; let $z = x\mathbf{g} - \mathbf{g}x \in \mathfrak{X}$. For $h \in G$, then

$$\mathbf{z}(h) = \langle z, \mathbf{h}D(\mathbf{h}^{-1}) \rangle = \langle x\mathbf{g}, \mathbf{h}D(\mathbf{h}^{-1}) \rangle - \langle \mathbf{g}x, \mathbf{h}D(\mathbf{h}^{-1}) \rangle,$$

because $\mathbf{h}D(\mathbf{h}^{-1}) \in \mathfrak{X}'$. We wish to reduce this considerably and notice that because D is an \mathfrak{X}' -derivation, then $D(\mathbf{h}^{-1} * \mathbf{g}) = D(\mathbf{h}^{-1})\mathbf{g} + \mathbf{h}^{-1}D(\mathbf{g})$ for $h \in G$, and we thus have

$$\begin{aligned} -(\mathbf{h}D(\mathbf{h}^{-1}))\mathbf{g} &= -\mathbf{h}(D(\mathbf{h}^{-1})\mathbf{g}) \\ &= -\mathbf{h}[D(\mathbf{h}^{-1} * \mathbf{g}) - \mathbf{h}^{-1}D(\mathbf{g})] \\ &= D(\mathbf{g}) - \mathbf{h}D(\mathbf{h}^{-1} * \mathbf{g}) \\ &= D(\mathbf{g}) - \mathbf{g}(\mathbf{g}^{-1} * \mathbf{h})D((\mathbf{g}^{-1} * \mathbf{h})^{-1}), \end{aligned}$$

and

$$\begin{aligned} -\langle \mathbf{g}x, \mathbf{h}D(\mathbf{h}^{-1}) \rangle &= -\langle x, (\mathbf{h}D(\mathbf{h}^{-1}))\mathbf{g} \rangle \\ &= -\langle x, D(\mathbf{g}) \rangle + \langle x, \mathbf{g}(\mathbf{g}^{-1} * \mathbf{h})D((\mathbf{g}^{-1} * \mathbf{h})^{-1}) \rangle \\ &= -\langle x, D(\mathbf{g}) \rangle + \langle x\mathbf{g}, (\mathbf{g}^{-1} * \mathbf{h})D((\mathbf{g}^{-1} * \mathbf{h})^{-1}) \rangle \end{aligned}$$

for $h \in G$, by using the \mathcal{A} -module multiplications in \mathfrak{X}' . If we now let $y = x\mathbf{g} \in \mathfrak{X}$, we obtain

$$\begin{aligned} \mathbf{z}(h) &= \langle y, \mathbf{h}D(\mathbf{h}^{-1}) \rangle - \langle \mathbf{g}x, \mathbf{h}D(\mathbf{h}^{-1}) \rangle \\ &= \langle y, \mathbf{h}D(\mathbf{h}^{-1}) \rangle - \langle y, (\mathbf{g}^{-1} * \mathbf{h})D((\mathbf{g}^{-1} * \mathbf{h})^{-1}) \rangle + \langle x, D(\mathbf{g}) \rangle \\ &= \mathbf{y}(h) - \mathbf{y}(g^{-1}h) + \langle x, D(\mathbf{g}) \rangle \\ &= \mathbf{y}(h) - \tau_g(\mathbf{y})(h) + \langle x, D(\mathbf{g}) \rangle \end{aligned}$$

for $h \in G$. Thus $\mathbf{z} = \mathbf{y} - \tau_g(\mathbf{y}) + \langle x, D(\mathbf{g}) \rangle 1$, 1 being the identity of $\ell^\infty(G)$. Because μ is an invariant mean, we have

$$\langle z, f \rangle = \mu(\mathbf{z}) = \mu(\mathbf{y}) - \mu(\tau_g(\mathbf{y})) + \langle x, D(\mathbf{g}) \rangle \mu(1) = \langle x, D(\mathbf{g}) \rangle,$$

and so

$$\langle x, D(\mathbf{g}) \rangle = \langle x\mathbf{g} - \mathbf{g}x, f \rangle = \langle x\mathbf{g}, f \rangle - \langle \mathbf{g}x, f \rangle = \langle x, \mathbf{g}f - f\mathbf{g} \rangle, \quad x \in \mathfrak{X}, g \in G.$$

Finally, because $a = \sum_{g \in G} a(g)\mathbf{g}$ for all $a \in \mathcal{A}$, then by continuity of D and module multiplications, $\langle x, D(a) \rangle = \langle x, af - fa \rangle = \langle x, \delta_f \rangle$ for all $a \in \mathcal{A}$. Hence \mathcal{A} is amenable. \square

Our first well-known example of a Banach algebra that is actually amenable now follows.

Corollary 2.12. *\mathbb{C} is an amenable Banach algebra with the usual product and norm.*

Proof. Let \mathcal{A} be the amenable Banach algebra $\ell^1(0)$, 0 denoting the trivial group. Define a map $\varphi : \mathcal{A} \rightarrow \mathbb{C}$ by $\varphi(f) = f(0)$; it is clearly an isomorphism and furthermore, it is an isometry, since $|\varphi(f)| = |f(0)| = \|f\|_1$ for all $f \in \mathcal{A}$. Apply Theorem 1.3. \square

Another useful theorem is quickly derived from Theorem 2.11.

Theorem 2.13. *Let \mathcal{A} be a unital Banach algebra containing a bounded, amenable group G (with the algebra product as the composition) with $\text{span } G$ dense in \mathcal{A} . Then \mathcal{A} is amenable.*

Proof. Apply Theorem 2.11, Lemma 2.10 and Theorem 1.3. \square

For any compact Hausdorff space \mathfrak{X} , let $C(\mathfrak{X})$ denote the Banach space of all continuous functions from \mathfrak{X} into \mathbb{C} with pointwise addition and scalar multiplication, equipped with the uniform norm. $C(\mathfrak{X})$ is made into a Banach algebra with the pointwise product

$$(fg)(x) = f(x)g(x), \quad f, g \in C(\mathfrak{X}), x \in \mathfrak{X}.$$

Theorem 2.14. *Let \mathfrak{X} be a compact Hausdorff space. Then $C(\mathfrak{X})$ is amenable.*

Proof. Let $C(\mathfrak{X}, \mathbb{R})$ denote the space of all continuous functions $\mathfrak{X} \rightarrow \mathbb{R}$. Let G denote the bounded group $\{e^{ih} \mid h \in C(\mathfrak{X}, \mathbb{R})\}$ (all elements have norm 1) in $C(\mathfrak{X})$ with pointwise multiplication as composition, i.e. the space of functions $x \mapsto e^{ih(x)}$ for $h \in C(\mathfrak{X}, \mathbb{R})$. G is obviously abelian, so G is amenable by Theorem 2.5 and the Banach algebra $\ell^1(G)$ is amenable. Let $\varphi : \ell^1(G) \rightarrow C(\mathfrak{X})$ be the continuous, contractive algebra homomorphism defined in Lemma 2.10:

$$\varphi(f)(t) = \sum_{g \in G} f(g)g(t), \quad f \in \ell^1(G), t \in \mathfrak{X}.$$

Now let $\mathfrak{B} = \overline{\varphi(\ell^1(G))}$; we're aiming at showing that $\mathfrak{B} = C(\mathfrak{X})$ by using the Stone-Weierstrass theorem (Theorem D.4). \mathfrak{B} is a subalgebra of $C(\mathfrak{X})$, since φ is a homomorphism and closures of subalgebras in Banach algebras are subalgebras (this is checked routinely by continuity of the algebra product).

If we let $h = 0$ and $b = e^{ih}$, then $1 = e^{ih} = \varphi(\mathbf{b}) \in \mathfrak{B}$, where \mathbf{b} is the embedding of $b \in G$ in $\ell^1(G)$. Finally, all $g \in G$ are equal to the complex conjugate of exactly one $\alpha \in G$, so that $g = \bar{\alpha} = g^{-1}$. Hence for $f \in \ell^1(G)$,

$$\overline{\varphi(f)(t)} = \overline{\sum_{g \in G} f(g)g(t)} = \sum_{\alpha \in G} \overline{f(\alpha)\bar{\alpha}(t)} = \sum_{\alpha \in G} \overline{f(\bar{\alpha})\alpha(t)} = \varphi(g)(t),$$

by continuity of complex conjugation, defining $g(\alpha) = \overline{f(\bar{\alpha})}$ for $\alpha \in G$. Obviously $g \in \ell^1(G)$, since

$$\sum_{\alpha \in G} |g(\alpha)| = \sum_{\alpha \in G} |f(\bar{\alpha})| = \sum_{g \in G} |f(g)| < \infty.$$

Hence $\overline{\varphi(f)}$ is contained in $\varphi(\ell^1(G))$ for any $f \in \varphi(\ell^1(G))$. For $f \in \mathfrak{B}$, let the sequence $(f_n) \in \varphi(\ell^1(G))$ converge uniformly to f ; we find that $|\overline{f(t)} - \overline{f_n(t)}| = |f(t) - f_n(t)|$ for all $t \in \mathfrak{X}$, and therefore $(\overline{f_n})$ converges uniformly to \overline{f} for $n \rightarrow \infty$. Thus \mathfrak{B} is self-adjoint.

Finally, if $x, y \in \mathfrak{X}$ and $x \neq y$, there exists a function $h \in C(\mathfrak{X}, \mathbb{R})$ such that $h(x) = 0$ and $h(y) = 1$ by the Urysohn lemma (Theorem D.7); letting $g = e^{ih}$ gives us $\varphi(\mathbf{g})(x) \neq \varphi(\mathbf{g})(y)$, so \mathfrak{B} separates points. Thus by the Stone-Weierstrass theorem, $\mathfrak{B} = C(\mathfrak{X})$, and by Theorem 1.3, $C(\mathfrak{X})$ is amenable. \square

Already, this dribbling back and forth between group properties and algebra properties seems to do us an immense amount of good, and it is a good time now to widen our perspective a little.

2.3 Hereditary properties of amenable groups

We turn our attention back to groups in order to uncover some of the more complicated examples of amenable Banach algebras. A thing to ask would be if one could conclude amenability of a group from a normal subgroup and the corresponding factor group. And, why, sure.

Theorem 2.15. *Let G be a group containing a normal subgroup N such that N and the factor group G/N are amenable. Then G is amenable.*

Proof. Let $\mu_H : \ell^\infty(H) \rightarrow \mathbb{C}$ be an invariant mean on $\ell^\infty(H)$, let $x \in \ell^\infty(G)$ and consider the map $\hat{x} : G/H \rightarrow \mathbb{C}$ defined by

$$\langle \bar{g}, \hat{x} \rangle = \langle \tau_{g^{-1}}(x)|_H, \mu_H \rangle,$$

where $\bar{g} = gH = Hg$. The map \hat{x} is well-defined: if $g_1, g_2 \in G$ satisfy $\bar{g}_1 = \bar{g}_2$, then $g_2 = g_1 h$ for some $h \in H$, and so

$$\langle a, \tau_{g_2^{-1}}(x) \rangle = \langle g_2 a, x \rangle = \langle g_1 h a, x \rangle = \langle h a, \tau_{g_1^{-1}}(x) \rangle = \langle a, \tau_{h^{-1}}(\tau_{g_1^{-1}}(x)) \rangle, \quad a \in G;$$

on top of this,

$$\langle a, \tau_{h^{-1}}(\tau_{g_1^{-1}}(x)|_H) \rangle = \langle h a, \tau_{g_1^{-1}}(x)|_H \rangle = \langle h a, \tau_{g_1^{-1}}(x) \rangle = \langle a, \tau_{h^{-1}}(\tau_{g_1^{-1}}(x)) \rangle, \quad a \in H,$$

and so that

$$\tau_{h^{-1}}(\tau_{g_1^{-1}}(x)|_H) = \tau_{h^{-1}}(\tau_{g_1^{-1}}(x))|_H,$$

so that, finally,

$$\mu_H(\tau_{g_2^{-1}}(x)|_H) = \mu_H(\tau_{h^{-1}}(\tau_{g_1^{-1}}(x))|_H) = \mu_H(\tau_{g_1^{-1}}(x)|_H)$$

because of left translation invariance of μ_H . Actually, $\hat{x} \in \ell^\infty(G/H)$, since by Lemma 2.4,

$$|\hat{x}(\bar{g})| = |\mu_H(\tau_{g^{-1}}(x)|_H)| = \sup_{h \in H} |\tau_{g^{-1}}(x)(h)| \leq \|x\|_\infty.$$

Because G/H is amenable, there exists an invariant mean $\mu_1 : \ell^\infty(G/H) \rightarrow \mathbb{C}$ on $\ell^\infty(G/H)$. Define $\mu(x) = \mu_1(\hat{x})$ for all $x \in \ell^\infty(G)$; this is the desired invariant mean on $\ell^\infty(G)$. Clearly μ is linear, positive because if $x \in \ell^\infty(G)$ is positive, then so are \hat{x} and $\mu_1(\hat{x})$, and $\mu(1) = 1$. In order to show that μ is an invariant mean of $\ell^\infty(G)$, we just need to show that it is left translation invariant. Therefore, let $h \in G$ and $x \in \ell^\infty(G)$; if $T_{\bar{h}}$ denotes the translation operator on $\ell^\infty(G/H)$ with respect to \bar{h} , then

$$\langle \bar{g}, \widehat{\tau_{h^{-1}}(x)} \rangle = \langle \tau_{g^{-1}}(\tau_{h^{-1}}(x))|_H, \mu_H \rangle = \langle \tau_{(hg)^{-1}}(x)|_H, \mu_H \rangle = \langle \bar{hg}, \hat{x} \rangle = \langle \bar{g}, T_{\bar{h}^{-1}}(\hat{x}) \rangle$$

for all $g \in G$, so that $\langle \tau_{h^{-1}}(x), \mu \rangle = \langle \widehat{\tau_{h^{-1}}(x)}, \mu_1 \rangle = \langle T_{\bar{h}^{-1}}(\hat{x}), \mu_1 \rangle = \langle \hat{x}, \mu_1 \rangle = \langle x, \mu \rangle$. \square

This next theorem was originally proved in [5] and becomes quite essential in the next section because of the corollary after it.

Theorem 2.16. *Let G be a group. If G is a directed union of a system of amenable groups H_α , in the sense that $G = \bigcup_\alpha H_\alpha$ and for any H_α, H_β , there exists $H_\gamma \supseteq H_\alpha \cup H_\beta$, then G is amenable.*

Proof. For each α , let μ_α be an invariant mean on $\ell^\infty(H_\alpha)$. Define $\bar{\mu}_\alpha(x) = \mu_\alpha(x|_{H_\alpha})$ for each α ; these are clearly means on $\ell^\infty(G)$, and moreover, every $\bar{\mu}_\alpha$ is left translation invariant by elements of H_α .

The set Λ_α of means on $\ell^\infty(G)$ that are left translation invariant under H_α is w^* -compact in $(\ell^\infty(G))'$ by Lemma 2.4: if the Λ_α sequence $(\lambda_{\alpha,n})$ converges to λ , then it converges pointwise, implying that Λ_α is closed in the set of means on $\ell^\infty(G)$. Additionally, the collection of all Λ_α has the finite intersection property: for any finite subcollection of Λ_α , $\{\Lambda_{\alpha_1}, \dots, \Lambda_{\alpha_n}\}$, there exists $H_\beta \supseteq \bigcup_{i=1}^n H_{\alpha_i}$, and $\bar{\mu}_\beta$ is a mean on $\ell^\infty(G)$ that is left translation invariant under all H_{α_i} , so $\bigcap_{i=1}^n \Lambda_{\alpha_i} \neq \emptyset$.

Hence $\Lambda = \bigcap_\alpha \Lambda_\alpha$ is non-empty, since the set of means on $\ell^\infty(G)$ is w^* -compact (whereupon any collection of closed subsets with the finite intersection property is non-empty). Take $\mu \in \Lambda$; μ is a mean on $\ell^\infty(G)$, and for all $h \in G$ and $x \in \ell^\infty(G)$, there exists H_α containing h and so $\mu \in \Lambda_\alpha$, thereby implying that $\mu(\tau_h(x)) = \mu(x)$ so that μ is invariant. \square

Corollary 2.17. *Let G be a group. If every finitely generated subgroup of G is amenable, then G is amenable.*

Proof. The system of all finitely generated subgroups of G make up a directed union of G . \square

2.4 Compact operators on Hilbert spaces

As mentioned earlier, the results of the previous section could be useful, and they become that right here. We recall that a *Hilbert space* is a vector space equipped with an inner product such that it becomes complete with respect to the norm induced by the inner product. In this section, the inner product on a Hilbert space \mathcal{H} will be denoted by (\cdot, \cdot) and the Banach space of bounded linear operators $\mathcal{H} \rightarrow \mathcal{H}$ is denoted by $\mathcal{L}(\mathcal{H})$. Some theorems for Hilbert spaces are assumed in the section to be known to the reader.

Definition 2.18. An operator $A \in \mathcal{L}(\mathcal{H})$ on a Hilbert space \mathcal{H} is *compact* if the closure of $A((\mathcal{H})_1)$ is compact in \mathcal{H} . The set of all compact operators on the Hilbert space \mathcal{H} is denoted by $K(\mathcal{H})$.

An operator $F \in \mathcal{L}(\mathcal{H})$ on a Hilbert space \mathcal{H} is *finite rank* if the image of F is finite-dimensional in \mathcal{H} . The *rank* of F , $\text{rk } F$, is defined to be the dimension of the image.

All finite rank operators are obviously compact. It won't be proved here that compact operators on Hilbert spaces are limits of finite rank operators (for a thorough proof of this, consult [4, Theorem 5.9]) and thus have separable image. The identity operator is only compact on finite-dimensional spaces. We refer to Appendix B for a proof that the adjoint operators of compact operators are compact; the adjoint of a linear operator A is denoted A^* .

Let \mathcal{H} be a separable Hilbert space with orthonormal basis $(e_i)_{i \in \mathbb{N}}$. We define the *projection operator* $P_n : \mathcal{H} \rightarrow \mathcal{H}$ for $n \in \mathbb{N}$ by $P_n(x) = \sum_{i=1}^n (x, e_i) e_i$; it is linear and bounded, with $\|P_n\| = 1$ for all $n \in \mathbb{N}$. Projection operators are defined analogously in the finite-dimensional case. It can easily be seen that P_n has rank n .

The set of compact operators $K(\mathcal{H})$ on \mathcal{H} is a closed two-sided ideal in the Banach algebra $\mathcal{L}(\mathcal{H})$ with composition as product, and thus a Banach algebra itself.

Lemma 2.19. *For a separable Hilbert space \mathcal{H} , (P_n) is a bounded approximate identity for the Banach algebra $K(\mathcal{H})$.*

Proof. Let $A \in K(\mathcal{H})$ and $\varepsilon > 0$. $\overline{A((\mathcal{H})_1)}$ is compact in \mathcal{H} , and so there exists a finite number n of open $\frac{\varepsilon}{3}$ -balls with centres x_1, \dots, x_n covering $\overline{A((\mathcal{H})_1)}$. Since $P_n \rightarrow I$ in the strong operator norm topology (pointwise) on the Hilbert space \mathcal{H} , then for all $j = 1, \dots, n$ there exists $N_j \in \mathbb{N}$ such that $n \geq N_j$ implies $\|P_n x_j - x_j\| < \frac{\varepsilon}{3}$; let $N = \max\{N_j\}_{j=1}^n$.

For any $x \in \overline{A((\mathcal{H})_1)}$, there exists x_j with $1 \leq j \leq n$ such that $\|x - x_j\| < \frac{\varepsilon}{3}$, and so

$$\|P_n x - x\| \leq \|P_n(x - x_j) + P_n x_j - x_j + x_j - x\| < \varepsilon$$

for $n \geq N$. Thus for all $x \in (\mathcal{H})_1$, $\|P_n A(x) - A(x)\| < \varepsilon$ for $n \geq N$, and so $\|P_n A - A\| \leq \varepsilon$ for $n \geq N$. Thus $P_n A \rightarrow A$ in norm. Because $A^* \in K(\mathcal{H})$ and $(P_n A - A)^* = A^* P_n - A^*$, we have $B^* P_n \rightarrow B^*$ in $K(\mathcal{H})$ for all $B \in K(\mathcal{H})$. Since the map $B \mapsto B^*$ is bijective, we obtain $A P_n \rightarrow A$ for all $A \in K(\mathcal{H})$ in norm, so P_n is a bounded approximate identity. \square

For $i, j \in \mathbb{N}$, we define the *elementary operator* $E_{ij} : \mathcal{H} \rightarrow \mathcal{H}$ by $E_{ij}x = (x, e_j)e_i$; it is obviously linear and bounded, and moreover, of rank 1.

Let A be a compact operator on \mathcal{H} and let $\lambda_{ij} = (A(e_j), e_i)$ for all $i, j \in \mathbb{N}$. Then

$$P_n A P_n(x) = \sum_{i=1}^n (A P_n(x), e_i) e_i = \sum_{i=1}^n \sum_{j=1}^n (x, e_j) (A(e_j), e_i) e_i = \sum_{i=1}^n \sum_{j=1}^n \lambda_{ij} E_{ij}(x)$$

for $x \in \mathcal{H}$, and so $P_n A P_n \in \text{span}\{E_{ij} \mid i, j \in \mathbb{N}\}$ for all $n \in \mathbb{N}$. It follows from Lemma 2.19 that

$$\|P_n A P_n - A\| = \|P_n(A P_n - A) + P_n A - A\| \leq \|A P_n - A\| + \|P_n A - A\| \rightarrow 0$$

for $n \rightarrow \infty$, so A is contained in the closure of the linear span of elementary operators.

We now consider two important groups. Extract all permutations $\pi : \mathbb{N} \rightarrow \mathbb{N}$ from the permutation group on \mathbb{N} , $S_{\mathbb{N}}$, that satisfy $\pi(n) = n$ except for finitely many $n \in \mathbb{N}$. The set of these, which we denote by $S_{\mathbb{N}}^F$, is obviously a subgroup of $S_{\mathbb{N}}$. The set χ_2 of all mappings $\mathbb{N} \rightarrow \{-1, 1\}$ is a group with pointwise product as the composition with the constant function 1 as neutral element, and extracting the mappings $\xi : \mathbb{N} \rightarrow \{-1, 1\}$ such that $\xi(n) = 1$ except for finitely many $n \in \mathbb{N}$, obviously gives us a subgroup of χ_2 , which we will denote by χ_2^F .

For a Hilbert space \mathcal{H} , consider the subalgebra $K(\mathcal{H}) \oplus \mathbb{C}I$; it is closed in $\mathcal{L}(\mathcal{H})$ by Lemma B.3, and thus a Banach algebra itself.

Theorem 2.20. *Let \mathcal{H} be a separable Hilbert space. Then $K(\mathcal{H}) \oplus \mathbb{C}I$ is amenable.*

Proof. We assume first that \mathcal{H} is infinite-dimensional. Let $\mathcal{A} = K(\mathcal{H}) \oplus \mathbb{C}I$. Since \mathcal{H} is separable, there exists an orthonormal basis $(e_n)_{n \in \mathbb{N}}$ for \mathcal{H} . Now, given $\xi \in \chi_2^F$ and $\pi \in \mathbb{S}_{\mathbb{N}}^F$, define $T_{\xi, \pi} : \mathcal{H} \rightarrow \mathcal{H}$ to be the bounded linear operator determined by

$$T_{\xi, \pi} \left(\sum_{n=1}^{\infty} \lambda_n e_n \right) = \sum_{n=1}^{\infty} \lambda_n \xi(n) e_{\pi(n)},$$

specially satisfying the equations $T_{\xi, \pi}(e_n) = \xi(n) e_{\pi(n)}$ for all $n \in \mathbb{N}$; boundedness is valid because for $x = \sum_{n=1}^{\infty} \lambda_n e_n$ in \mathcal{H} , such that $\sum_{n=1}^{\infty} |\lambda_n|^2 < \infty$, we have

$$\|T_{\xi, \pi}(x)\|^2 = \left\| \sum_{n=1}^{\infty} \lambda_n \xi(n) e_{\pi(n)} \right\|^2 = \sum_{n=1}^{\infty} |(\lambda_n \xi(n) e_{\pi(n)}, e_{\pi(n)})|^2 = \sum_{n=1}^{\infty} |\lambda_n \xi(n)|^2 = \|x\|^2. \quad (2.1)$$

Because $T_{\xi, \pi} - I$ is a finite rank operator and thus compact, $T_{\xi, \pi}$ is contained in \mathcal{A} .

Consider now the subset $G = \{T_{\xi, \pi} \mid \xi \in \chi_2^F, \pi \in \mathbb{S}_{\mathbb{N}}^F\}$ of \mathcal{A} , and let $\xi \in \chi_2^F$ and $\pi \in \mathbb{S}_{\mathbb{N}}^F$. Since

$$T_{\xi, \pi^{-1}}(T_{\xi, \pi}(e_n)) = T_{\xi, \pi}(T_{\xi, \pi^{-1}}(e_n)) = e_n$$

for all $n \in \mathbb{N}$, $T_{\xi, \pi}$ is invertible. If we additionally let $\xi' \in \chi_2^F$ and $\pi' \in \mathbb{S}_{\mathbb{N}}^F$, we see that

$$T_{\xi, \pi}(T_{\xi', \pi'}(e_n)) = (\xi' \xi)(n) e_{\pi(\pi'(n))} = T_{\xi' \xi, \pi \circ \pi'}(e_n), \quad (2.2)$$

and thus obtain that G is a group contained in \mathcal{A} . Considering the subset $J = \{T_{\xi, e} \mid \xi \in \chi_2^F\}$ of G , where e is the identity of $\mathbb{S}_{\mathbb{N}}^F$, it is seen from (2.2) that J is an abelian subgroup of G . J is normal; indeed, letting $\xi, \xi' \in \chi_2^F$ and $\pi \in \mathbb{S}_{\mathbb{N}}^F$, we see that

$$T_{\xi, \pi}(T_{\xi', e}(T_{\xi, \pi^{-1}}(e_n))) = \xi(n) \xi'(n) \xi(n) e_{\pi(\pi^{-1}(n))} = T_{\xi' \cdot \xi, e}(e_n).$$

This allows us to define the factor group G/J . Consider the surjective map $\varphi : G \rightarrow \mathbb{S}_{\mathbb{N}}^F$ defined by $\varphi(T_{\xi, \pi}) = \pi$; it is well-defined because for $T_{\xi, \pi} = T_{\xi', \pi'}$, then for all $n \in \mathbb{N}$, we have $\xi(n) e_{\pi(n)} = \xi'(n) e_{\pi'(n)}$ and $\pi(n) = \pi'(n)$ because (e_n) is a basis. It is also a homomorphism because

$$\varphi(T_{\xi, \pi} T_{\xi', \pi'}) = \varphi(T_{\xi' \xi, \pi \pi'}) = \pi \pi' = \varphi(T_{\xi, \pi}) \varphi(T_{\xi', \pi'}),$$

with kernel $\{T_{\xi, e} \mid \xi \in \chi_2^F\} = J$; hence by the first isomorphism theorem, $G/J \simeq \mathbb{S}_{\mathbb{N}}^F$. Since every finitely generated subgroup of $\mathbb{S}_{\mathbb{N}}^F$ is finite and thus amenable, $\mathbb{S}_{\mathbb{N}}^F$ is amenable by Corollary 2.17. Thus G is an amenable group by Theorem 2.15.

Now let $\mathfrak{B} = \text{span } G$. It is obvious from (2.1) that $\|T\| = 1$ for any $T \in G$, and thus by Theorem 2.13 it suffices to prove that \mathfrak{B} is dense in \mathcal{A} . Since it is obvious that $I \in G$ and thus $\mathbb{C}I \subseteq \mathfrak{B}$, then it suffices to show that all elementary operators E_{ij} are contained in \mathfrak{B} ; if so, then for all $f \in K(\mathcal{H})$ and for any $\varepsilon > 0$, there exists $F \in \text{span}\{E_{ij}\} \subseteq \mathfrak{B}$ such that $\|f - F\| < \varepsilon$, so that $f + \lambda I$ is approximated by $F + \lambda I$ for all $\lambda \in \mathbb{C}$.

Therefore, let $i, j \in \mathbb{N}$. Let π_0 be the permutation transposing i and j . Let $\xi_1(n) = 1$ for all $n \in \mathbb{N}$, let $\xi_2(n) = 1$ for $n \neq i$, $\xi_2(i) = -1$, and let $T_1 = T_{\xi_1, \pi_0}$ and $T_2 = T_{\xi_2, \pi_0}$. Then for $x \in \mathcal{H}$,

$$T_1(x) - T_2(x) = \sum_{n=1}^{\infty} (x, e_n) (1 - \xi_2(n)) e_{\pi_0(n)} = (x, e_i) (1 - \xi_2(i)) e_{\pi_0(i)} = 2(x, e_i) e_j = 2E_{ij}(x),$$

proving that $E_{ij} = \frac{1}{2}(T_1 - T_2) \in \mathfrak{B}$.

If \mathcal{H} is finite-dimensional, then G as written above is finite and thus amenable. Since $\mathcal{A} = \mathcal{H}$, as all operators $\mathcal{H} \rightarrow \mathcal{H}$ are finite-rank by the rank-nullity theorem and thus compact, $\text{span } G$ is dense in \mathcal{A} , using the same method as above. \square

The intricate interplay between abstract group theory and Banach algebra theory that amenability offers is very much visible in the above proof. It also provides the following corollary:

Corollary 2.21. *Let \mathcal{H} be a separable Hilbert space. Then $K(\mathcal{H})$ is amenable.*

Proof. $K(\mathcal{H})$ is a closed two-sided ideal in $\mathcal{L}(\mathcal{H})$ and thus $K(\mathcal{H}) \oplus \mathbb{C}I$ containing a bounded approximate identity by Lemma 2.19. Apply Theorem 2.20 and Theorem 1.12. \square

Corollary 2.22. *The Banach algebra $\mathbb{M}_n(\mathbb{C})$ of linear maps $\mathbb{C}^n \rightarrow \mathbb{C}^n$ (with composition as the product) is amenable.*

Proof. $K(\mathbb{C}^n)$ is isometrically isomorphic to $\mathbb{M}_n(\mathbb{C})$. \square

We are now ready to take the biggest step of all in this section.

Theorem 2.23. *Let \mathcal{H} be a Hilbert space. Then $K(\mathcal{H})$ is amenable.*

In order to prove this, recall that orthogonal projections on Hilbert spaces are self-adjoint and continuous; this proves useful in the next lemma.

Lemma 2.24. *Let $\mathfrak{X} \subseteq \mathcal{Y}$ be closed subspaces of a Hilbert space \mathcal{H} . The orthogonal projections on \mathfrak{X} and \mathcal{Y} , denoted $P_{\mathfrak{X}}$ and $P_{\mathcal{Y}}$ respectively, satisfy $P_{\mathfrak{X}}P_{\mathcal{Y}} = P_{\mathfrak{X}} = P_{\mathcal{Y}}P_{\mathfrak{X}}$.*

Proof. It is clear that $P_{\mathcal{Y}}P_{\mathfrak{X}} = P_{\mathfrak{X}}$, since $P_{\mathfrak{X}}(x) \in \mathfrak{X} \subseteq \mathcal{Y}$ for any $x \in \mathcal{H}$. The other equality is proven by adjoints: $P_{\mathfrak{X}} = P_{\mathfrak{X}}^* = (P_{\mathcal{Y}}P_{\mathfrak{X}})^* = P_{\mathfrak{X}}^*P_{\mathcal{Y}}^* = P_{\mathfrak{X}}P_{\mathcal{Y}}$. \square

Lemma 2.25. *Let \mathcal{A} be a separable subspace of a normed space \mathfrak{X} . Then $\overline{\mathcal{A}}$ is separable.*

Proof. \mathcal{A} contains a countable dense subset \mathfrak{B} , and this subset is dense in the closure of \mathcal{A} as well. Indeed, for any $x \in \overline{\mathcal{A}}$ and $\varepsilon > 0$, choose $a \in \mathcal{A}$ such that $\|x - a\| < \frac{\varepsilon}{2}$ and $b \in \mathfrak{B}$ such that $\|a - b\| < \frac{\varepsilon}{2}$; then $\|x - b\| < \varepsilon$. \square

Proof of Theorem 2.23. We have already proved this for separable Hilbert spaces, so assume that \mathcal{H} has dimension strictly larger than \aleph_0 . The proof comes in four parts. Let $\{\mathcal{H}_\lambda \mid \lambda \in \Lambda\}$ be the system of all closed, infinite-dimensional separable subspaces of \mathcal{H} , and for $\lambda \in \Lambda$, let P_λ denote the orthogonal projection on \mathcal{H}_λ and let \mathcal{A}_λ denote the subalgebra of $K(\mathcal{H})$ given by

$$\mathcal{A}_\lambda = \{P_\lambda B P_\lambda \mid B \in K(\mathcal{H})\}.$$

This groundwork gives a hint that Theorem 1.14 is the one to use here.

Part 1. We first prove that $\mathcal{A}_\lambda \cong K(\mathcal{H}_\lambda)$ isometrically. Let $\lambda \in \Lambda$, and define $\varphi_\lambda(F) = F'$ for $F \in \mathcal{A}_\lambda$, where $F'(x) = F|_{\mathcal{H}_\lambda}(x)$ for $x \in \mathcal{H}_\lambda$. φ_λ is clearly linear, and moreover maps into $K(\mathcal{H}_\lambda)$, since $\overline{F'((\mathcal{H}_\lambda)_1)} = \overline{F((\mathcal{H}_\lambda)_1)}$ is compact for $F \in \mathcal{A}_\lambda$. Indeed, $F((\mathcal{H}_\lambda)_1) \subseteq F((\mathcal{H})_1)$ which has compact closure in \mathcal{H} , so $F((\mathcal{H}_\lambda)_1)$ has compact closure in \mathcal{H} and thus \mathcal{H}_λ as well.

Actually, $\varphi_\lambda : \mathcal{A}_\lambda \rightarrow K(\mathcal{H}_\lambda)$ is an isometric algebra isomorphism. It is easily seen that φ_λ preserves products. Letting $G \in K(\mathcal{H}_\lambda)$ and $G^*(x_\lambda + x_\lambda^\perp) = G(x_\lambda)$ for $x_\lambda \in \mathcal{H}_\lambda$, $x_\lambda^\perp \in \mathcal{H}_\lambda^\perp$ defines a linear operator G^* on \mathcal{H} with norm $\|G\|$; it is compact, since the closure of the image $G^*((\mathcal{H})_1) = G((\mathcal{H}_\lambda)_1)$ is compact in \mathcal{H}_λ and thus in \mathcal{H} . It is easily seen that $\varphi_\lambda(P_\lambda G^* P_\lambda) = G$, so φ_λ is surjective. Finally, it is clear that $\|\varphi_\lambda(F)\| \leq \|F\|$; the opposite inequality is obtained by seeing that for any $F = P_\lambda A P_\lambda \in \mathcal{A}_\lambda$,

$$\|F(x)\| = \|P_\lambda A P_\lambda(x)\| = \|P_\lambda A P_\lambda P_\lambda(x)\| = \|F P_\lambda(x)\| = \|\varphi_\lambda(F)(P_\lambda(x))\| \leq \|\varphi_\lambda(F)\| \|x\|,$$

all projections having norm 1.

Part 2. Let $\alpha, \beta \in \Lambda$, and let \mathfrak{M} be the closure of $\mathcal{H}_\alpha + \mathcal{H}_\beta$ by Lemma 2.25. \mathfrak{M} is separable by Lemma 2.25, and so there exists $\lambda \in \Lambda$ with $\mathfrak{M} = \mathcal{H}_\lambda$. Let $A \in \mathcal{A}_\alpha$ and $B \in \mathcal{A}_\beta$; we aim to show that $A + B \in \mathcal{A}_\lambda$. There exists $F, G \in K(\mathcal{H})$ such that $A = P_\alpha F P_\alpha$ and $B = P_\beta G P_\beta$.

Let $H = A + B \in K(\mathcal{H})$, $K(\mathcal{H})$ being an ideal in $\mathcal{L}(\mathcal{H})$. Then by Lemma 2.24,

$$A + B = P_\alpha F P_\alpha + P_\beta G P_\beta = P_\lambda P_\alpha F P_\alpha P_\lambda + P_\lambda P_\beta G P_\beta P_\lambda = P_\lambda H P_\lambda \in \mathcal{A}_\lambda,$$

proving that for any $\alpha, \beta \in \Lambda$, there exists $\lambda \in \Lambda$ such that $\mathcal{A}_\alpha \cup \mathcal{A}_\beta \subseteq \mathcal{A}_\lambda$.

Part 3. Let $A \in K(\mathcal{H})$. $A(\mathcal{H})$ is separable and there exists an $\alpha \in \Lambda$ such that $\mathcal{H}_\alpha = \overline{A(\mathcal{H})}$. Likewise, $A^*(\mathcal{H})$ is separable by Theorem B.2, with a $\beta \in \Lambda$ such that $\mathcal{H}_\beta = \overline{A^*(\mathcal{H})}$. Hence $P_\alpha A = A$ and $P_\beta A^* = A^*$. Now, choose $\lambda \in \Lambda$ such that $\mathcal{A}_\alpha \cup \mathcal{A}_\beta \subseteq \mathcal{A}_\lambda$. Since

$$A = (A^*)^* = (P_\beta A^*)^* = A P_\beta^* = A P_\beta,$$

then $A = P_\alpha A P_\beta = P_\lambda P_\alpha A P_\beta P_\lambda = P_\lambda A P_\lambda \in \mathcal{A}_\lambda$, and $K(\mathcal{H}) = \bigcup_{\lambda \in \Lambda} \mathcal{A}_\lambda$.

Part 4. This last part is a smorgasbord of previous proofs. For all $\lambda \in \Lambda$, \mathcal{H}_λ is isometrically isomorphic to the Hilbert space of square-summable sequences in \mathbb{C} , ℓ_2 . In order to use Theorem 1.14, we will have to prove the last and third condition, which can then be modified by Part 1 to the statement that for all Banach $K(\ell_2)$ -bimodules \mathfrak{X} , there exists a $K > 0$ such that for all $D \in \mathcal{Z}^1(K(\ell_2), \mathfrak{X}')$ there is a $\rho \in \mathfrak{X}'$ such that $D = \delta_\rho$ and $\|\rho\| \leq K\|D\|$.

Let \mathfrak{X} be a neo-unital Banach $K(\ell_2)$ -bimodule \mathfrak{X} (Theorem 1.10) and $D \in \mathcal{Z}^1(K(\ell_2), \mathfrak{X}')$. What to do now? Let $\mathcal{A} = K(\ell_2) \oplus \mathbb{C}I$, and notice that by the proof of Theorem 2.20 that there exists an amenable group G with dense span in \mathcal{A} .

By Theorems 2.10 and 2.13, let $\varphi : \ell^1(G) \rightarrow \mathcal{A}$ be a dense, continuous homomorphism with norm 1, norm 1 arising from the fact that obviously, $\|\varphi(f)\| \leq \|f\|$ and assuming that $\|\varphi\| < 1$ contradicts that $\|\varphi(\mathbf{g})\| = 1 = \|\mathbf{g}\|$ for all $g \in G$. By neo-unitality of \mathfrak{X} , then since $K(\ell_2)$ is a closed two-sided ideal of \mathcal{A} containing a bounded approximate identity (Lemma 2.19), we make \mathfrak{X} into a Banach \mathcal{A} -bimodule with D extended to $\tilde{D} : \mathcal{A} \rightarrow \mathfrak{X}'$, by Theorem 1.11.

It is clear that \tilde{D} is inner when restricted to $\varphi(\ell^1(G))$, by continuity of φ and \tilde{D} . Let μ be the invariant mean on $\ell^\infty(G)$. If we take a look at how we construct the generator of \tilde{D} on $\varphi(\ell^1(G))$ using the proof of Theorem 2.11, remember that we for $x \in \mathfrak{X}$ define

$$\mathbf{x}(g) = \langle x, \mathbf{g} \tilde{D}(\mathbf{g}^{-1}) \rangle, \quad g \in G.$$

Defining $\rho(x) = \mu(\mathbf{x})$, we found that $\tilde{D} = \delta_\rho$ on $\varphi(\ell^1(G))$. It follows from the definitions that $|\rho(x)| \leq \|\mu\| \|\mathbf{x}\| \leq S \|\mu\| \|D\| \|x\|$ for $x \in \mathfrak{X}$, where $S > 0$ is the bounding constant of the Banach \mathcal{A} -bimodule \mathfrak{X} , so $\|\rho\| \leq S \|D\|_{\varphi(\ell^1(G))} \leq S \|\tilde{D}\|$.

Since $\varphi(\ell^1(G))$ is dense in \mathcal{A} , then by continuity of the extension \tilde{D} , \tilde{D} is inner on \mathcal{A} with generator ρ and thus, by restriction, on $K(\ell_2)$ with $\|\rho\| \leq S \|D\|$. Theorem 1.14 now applies, and $K(\mathcal{H})$ is amenable. \square

2.5 The group algebra of \mathbb{R}

Let $L^1(\mathbb{R})$ denote the vector space of equivalence classes (under equality almost everywhere) of Lebesgue-measurable functions that are integrable over \mathbb{R} with pointwise addition and scalar multiplication, to wit the measurable functions $f : \mathbb{R} \rightarrow \mathbb{C}$ for which

$$\int_{\mathbb{R}} |f(s)| ds < \infty.$$

$L^1(\mathbb{R})$ is a Banach space with the norm $\|f\|_1 := \int_{\mathbb{R}} |f(s)| ds$ and is made into a Banach algebra by using convolution as product, that is

$$(f * g)(s) = \int_{\mathbb{R}} f(t)g(s-t) dt, \quad f, g \in L^1(\mathbb{R}), s \in \mathbb{R}.$$

Convolution is commutative, making $L^1(\mathbb{R})$ a *commutative Banach algebra*.

$L^\infty(\mathbb{R})$ denotes the Banach space of equivalence classes (again, under equality almost everywhere) of *essentially bounded* Lebesgue-measurable functions with pointwise addition and scalar multiplication, i.e. the measurable functions $f : \mathbb{R} \rightarrow \mathbb{C}$ for which there exists an $S \geq 0$ such that $\{x \in \mathbb{R} \mid |f(x)| > S\}$ is a null set.

$L^1(\mathbb{R})$ is called *the group algebra of \mathbb{R}* . The reason for this name is that $L^1(\mathbb{R})$ is really just a special case of a general Banach algebra, $L^1(G)$, where G is a locally compact topological group, called the *group algebra*. Using the so-called left invariant Haar measure μ on G , $L^1(G)$ denotes the Banach space of functions on G integrable under this measure, with convolution as product given by

$$(f * g)(s) = \int_G f(t)g(t^{-1}s)d\mu(t), \quad s \in G,$$

whereupon $L^1(G)$ becomes a Banach algebra [1, Example 1.20]. By defining a locally compact group G to be amenable if there exists a left invariant mean on the space of bounded, right uniformly continuous functions $G \rightarrow \mathbb{C}$, B. E. Johnson proved a beautiful theorem [7, Theorem 2.5] stating that $L^1(G)$ is an amenable Banach algebra if and only if G is amenable (a definition of right uniformly continuous functions is given in [5, p. 21]).

Theorem 2.27 will prove a specific case, namely that $L^1(\mathbb{R})$ is an amenable Banach algebra, using that $(\mathbb{R}, +)$ is an abelian and thus amenable group by Theorem 2.5. We won't dwell on the more general case, and we'll just mention that the proof *can* be generalized to all locally compact groups G for which there exists a countable bounded approximate identity, making possible use of Lebesgue's dominated convergence theorem, essential for this proof.

It can be shown that $(L^1(\mathbb{R}))'$ is isometrically isomorphic to $L^\infty(\mathbb{R})$ by the mapping $\psi \mapsto \bar{\psi}$, $L^\infty(\mathbb{R}) \rightarrow (L^1(\mathbb{R}))'$, where

$$\langle f, \bar{\psi} \rangle = \int f\psi dm, \quad f \in L^1(\mathbb{R}). \quad (2.3)$$

We won't prove this fact; consult [4, Theorem 1.46] for a proof.

Lemma 2.26. *$L^1(\mathbb{R})$ has a countable bounded approximate identity.*

Proof. For $n \in \mathbb{N}$, $f_n : \mathbb{R} \rightarrow \mathbb{C}$ defined by $f_n = n1_{[0, n^{-1}]}$ does it; we won't prove this, and we do not even need a specific approximate identity in order to prove the next theorem. \square

The next theorem then ends the chapter in grand fashion.

Theorem 2.27. *$L^1(\mathbb{R})$ is amenable.*

Proof. Let $\mathcal{A} = L^1(\mathbb{R})$, and let \mathfrak{A} be the vector space $\ell^1(\mathbb{R}) \oplus \mathcal{A}$ with coordinatewise addition and scalar multiplication. To make \mathfrak{A} into an algebra, we already have the usual convolutions for the two spaces that make up \mathfrak{A} , and we need only to define a multiplication of elements of $\ell^1(\mathbb{R})$ with elements of \mathcal{A} . For any $\varphi \in \ell^1(\mathbb{R})$ and $f \in \mathcal{A}$, we know that $\varphi(t) = \sum_{s \in \mathbb{R}} \varphi(s)\mathbf{s}(t)$ and this motivates the definition

$$(\varphi * f)(t) = (f * \varphi)(t) := \sum_{s \in \mathbb{R}} \varphi(s)f(t-s), \quad \varphi \in \ell^1(\mathbb{R}), f \in \mathcal{A}.$$

Both of these functions are integrable over \mathbb{R} , since they have countable support and thus in the case of the first function,

$$\int |\varphi * f| dm \leq \int \sum_{s \in \mathbb{R}} |\varphi(s)f(t-s)| dt \leq \sum_{s \in \mathbb{R}} |\varphi(s)| \int |f(t-s)| dt \leq \|f\|_1 \|\varphi\|.$$

We make \mathfrak{A} into an algebra by defining

$$(\varphi_1 + f_1) * (\varphi_2 + f_2) := \varphi_1 * \varphi_2 + (\varphi_1 * f_2 + f_1 * \varphi_2 + f_1 * f_2), \quad \varphi_1, \varphi_2 \in \ell^1(\mathbb{R}), \quad f_1, f_2 \in \mathcal{A},$$

and finally into a Banach algebra by defining the norm $\|\varphi + f\| = \|\varphi\|_1 + \|f\|_1$ for $\varphi \in \ell^1(\mathbb{R})$ and $f \in \mathcal{A}$. Since $\varphi * f$ and $f * \varphi$ are contained in \mathcal{A} for $\varphi \in \ell^1(\mathbb{R})$ and $f \in \mathcal{A}$, \mathcal{A} is a closed ideal in \mathfrak{A} . Why this construction is needed becomes apparent in four steps, and we shall use the notation used in (2.3) for this.

Part 1. For any $f_1, f_2 \in \mathcal{A}$, $\psi \in L^\infty(\mathbb{R})$,

$$\langle f_1 * f_2, \bar{\psi} \rangle = \int_{\mathbb{R}} \psi(t) (f_1 * f_2)(t) dt = \int_{\mathbb{R}} \int_{\mathbb{R}} \psi(t) f_1(s) f_2(t-s) ds dt.$$

Because of absolute integrability, since by Tonelli's theorem,

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |\psi(t) f_1(s) f_2(t-s)| ds dt \leq \|\psi\|_\infty \int_{\mathbb{R}} |f_1(s)| \int_{\mathbb{R}} |f_2(t-s)| dt ds \leq \|\psi\|_\infty \|f_1\|_1 \|f_2\|_1,$$

we can then change the order of integration by Fubini's theorem, such that

$$\langle f_1 * f_2, \bar{\psi} \rangle = \int_{\mathbb{R}} f_1(s) \int_{\mathbb{R}} \psi(t) f_2(t-s) dt ds = \int_{\mathbb{R}} f_1(s) \langle \mathbf{s} * f_2, \bar{\psi} \rangle ds.$$

Part 2. Let \mathfrak{X} be a neo-unital Banach \mathcal{A} -bimodule. \mathfrak{X} can then be made into a Banach \mathfrak{A} -bimodule canonically by Lemma 1.11. Fix $x \in \mathfrak{X}$ and $F \in \mathfrak{X}'$. The function $\mathcal{A} \rightarrow \mathbb{C}$ defined by $f \mapsto \langle fx, F \rangle$ is linear and bounded; thus there exists $\bar{\Psi} \in L^\infty(\mathbb{R})$ such that $\langle f, \bar{\Psi} \rangle = \langle fx, F \rangle$ for all $f \in \mathcal{A}$. Let $f = f_1 * f_2$ for $f_1, f_2 \in \mathcal{A}$; then

$$\begin{aligned} F(f_1(f_2x)) &= F((f_1 * f_2)x) = \langle f_1 * f_2, \bar{\Psi} \rangle = \int_{\mathbb{R}} f_1(s) \langle \mathbf{s} * f_2, \bar{\Psi} \rangle ds \\ &= \int_{\mathbb{R}} f_1(s) \langle \mathbf{s}(f_2x), F \rangle ds, \end{aligned}$$

and since x and F were arbitrary, it holds for all $x \in \mathfrak{X}$ and $F \in \mathfrak{X}'$. Because \mathfrak{X} is neo-unital, all $y \in \mathfrak{X}$ can be written $y = fx$ for some $f \in \mathcal{A}$, $x \in \mathfrak{X}$, and so

$$\langle fy, F \rangle = \int_{\mathbb{R}} f(s) \langle \mathbf{s}y, F \rangle ds, \quad f \in \mathcal{A}, \quad y \in \mathfrak{X}, \quad F \in \mathfrak{X}'.$$

We show analogously that

$$\langle yf, F \rangle = \int_{\mathbb{R}} f(s) \langle y\mathbf{s}, F \rangle ds, \quad f \in \mathcal{A}, \quad y \in \mathfrak{X}, \quad F \in \mathfrak{X}'.$$

Part 3. Let $D \in \mathcal{Z}^1(\mathcal{A}, \mathfrak{X}')$, \mathfrak{X} a neo-unital Banach \mathcal{A} -bimodule, and extend D to \mathfrak{A} by Lemma 1.11 again. Using the notation of the embedding of an $x \in \mathfrak{X}$ in the second dual \mathfrak{X}'' , we have

$$\langle x, D(f) \rangle = \langle D(f), \hat{x} \rangle, \quad x \in \mathfrak{X}, \quad f \in \mathcal{A}.$$

Defining $D^*(y)(f) := y(D(f))$ for $y \in \mathfrak{X}''$ and $f \in \mathcal{A}$ defines the dual mapping $D^* : \mathfrak{X}'' \rightarrow \mathcal{A}'$, so that for a fixed $x \in \mathfrak{X}$ and arbitrary $f_1, f_2 \in \mathcal{A}$, there exists a $\bar{\Psi} \in L^\infty(\mathbb{R})$ such that

$$\langle f_1 * f_2, D^*(\hat{x}) \rangle = \langle f_1 * f_2, \bar{\Psi} \rangle = \int_{\mathbb{R}} f_1(s) \langle \mathbf{s} * f_2, \bar{\Psi} \rangle ds = \int_{\mathbb{R}} f_1(s) \langle \mathbf{s} * f_2, D^*(\hat{x}) \rangle ds,$$

and therefore

$$\langle x, D(f_1 * f_2) \rangle = \langle f_1 * f_2, D^*(\hat{x}) \rangle = \int_{\mathbb{R}} f_1(s) \langle \mathbf{s} * f_2, D^*(\hat{x}) \rangle ds = \int_{\mathbb{R}} f_1(s) \langle x, D(\mathbf{s} * f_2) \rangle ds$$

for all $x \in \mathfrak{X}$, $f_1, f_2 \in \mathcal{A}$. Now, let $(e_n)_{n \in \mathbb{N}}$ be a bounded approximate identity for \mathcal{A} . Then

$$\langle x, D(f) \rangle = \lim_{n \rightarrow \infty} \langle x, D(f * e_n) \rangle = \lim_{n \rightarrow \infty} \int_{\mathbb{R}} f(s) \langle x, D(\mathbf{s} * e_n) \rangle ds$$

for all $n \in \mathbb{N}$ and $f \in \mathcal{A}$, by continuity of D . We want to evaluate the above limit, and this we do by using Lebesgue's dominated convergence theorem. Fix $x \in \mathfrak{X}$, and let $\Psi \in L^\infty(\mathbb{R})$ such that $\langle f, D^*(\hat{x}) \rangle = \langle f, \bar{\Psi} \rangle$ for all $f \in \mathcal{A}$. For $n \in \mathbb{N}$ and fixed $f \in \mathcal{A}$, consider the function

$$s \mapsto f(s) \langle x, D(\mathbf{s} * e_n) \rangle = f(s) \langle \mathbf{s} * e_n, D^*(\hat{x}) \rangle = \int_{\mathbb{R}} f(s) \Psi(t) e_n(t - s) dt.$$

Every section $t \mapsto f(s) \Psi(t) e_n(t - s)$ for $s \in \mathbb{R}$ is integrable and $(s, t) \mapsto f(s) \Psi(t) e_n(t - s)$ is measurable because of measurability of projections and f , Ψ and e_n , so the above function is measurable, and moreover, bounded.

By the extension of the derivation, we have by the proof of Lemma 1.11 that

$$\langle x, D(\alpha) \rangle = \lim_{n \rightarrow \infty} \langle x, D(\alpha * e_n) \rangle, \quad \alpha \in \mathfrak{A}, x \in \mathfrak{X}.$$

Hence $f(s) \langle x, D(\mathbf{s}) \rangle = \lim_{n \rightarrow \infty} f(s) \langle x, D(\mathbf{s} * e_n) \rangle$. Finally, since $\int_{\mathbb{R}} |e_n(t - s)| dt = \int_{\mathbb{R}} |e_n| dm \leq S$ for all $n \in \mathbb{N}$ and some $S > 0$ by boundedness of (e_n) , the mappings $s \mapsto |f(s) \langle x, D(\mathbf{s} * e_n) \rangle|$ have the integrable upper bound $S \|\Psi\|_\infty f$ for all $n \in \mathbb{N}$, so

$$\langle x, D(f) \rangle = \lim_{n \rightarrow \infty} \int_{\mathbb{R}} f(s) \langle x, D(\mathbf{s} * e_n) \rangle ds = \int_{\mathbb{R}} f(s) \langle x, D(\mathbf{s}) \rangle ds$$

by Lebesgue's dominated convergence theorem.

Part 4. Let \mathfrak{X} be a neo-unital Banach \mathcal{A} -bimodule, and let $D \in \mathcal{Z}^1(\mathcal{A}, \mathfrak{X}')$, extended canonically to \mathfrak{A} . \mathfrak{X} can be treated as a Banach $\ell^1(\mathbb{R})$ -bimodule, and so the restriction of D to $\ell^1(\mathbb{R})$ is a bounded \mathfrak{X}' -derivation on $\ell^1(\mathbb{R})$ and thus an inner derivation (since \mathbb{R} is an abelian and thus amenable group). Thus there exists $\mathfrak{g} \in \mathfrak{X}'$ so that $D(\mathbf{s}) = \mathbf{s}\mathfrak{g} - \mathfrak{g}\mathbf{s}$ for all $s \in \mathbb{R}$. For any $f \in \mathcal{A}$, then by Parts 3 and 2 we have

$$\begin{aligned} \langle x, D(f) \rangle &= \int_{\mathbb{R}} f(s) \langle x, \mathbf{s}\mathfrak{g} - \mathfrak{g}\mathbf{s} \rangle ds \\ &= \int_{\mathbb{R}} f(s) (\langle x\mathbf{s}, \mathfrak{g} \rangle - \langle \mathbf{s}x, \mathfrak{g} \rangle) ds \\ &= \int_{\mathbb{R}} f(s) \langle x\mathbf{s}, \mathfrak{g} \rangle ds - \int_{\mathbb{R}} f(s) \langle \mathbf{s}x, \mathfrak{g} \rangle ds \\ &= \langle xf, \mathfrak{g} \rangle - \langle fx, \mathfrak{g} \rangle \\ &= \langle x, f\mathfrak{g} - \mathfrak{g}f \rangle. \end{aligned}$$

Thus D is inner as a derivation $\mathfrak{A} \rightarrow \mathfrak{X}'$, and by restriction, inner as a derivation on \mathcal{A} . By Theorem 1.10, \mathcal{A} is amenable. \square

Hochschild cohomology in Banach algebras

This last chapter turns the attention to Hochschild cohomology groups of a Banach algebra \mathcal{A} of higher order than 1 and draws a surprising connection to the first cohomology group. In order to prove it, we introduce tensor products over duals of Banach spaces, laying the groundwork for a construction that eases the proof of the connection considerably.

3.1 Higher order Hochschild cohomology groups

Let us take it easy and go back to the basics for a while.

Definition 3.1. For $n \in \mathbb{N}$ and Banach spaces $\mathcal{A}_1, \dots, \mathcal{A}_n, \mathfrak{X}$, a *bounded n -linear mapping* is a mapping $\Phi : \mathcal{A}_1 \times \dots \times \mathcal{A}_n \rightarrow \mathfrak{X}$ linear in all n variables, satisfying for some $S > 0$ that

$$\|\Phi(a_1, \dots, a_n)\| \leq S\|a_1\| \cdots \|a_n\|, \quad a_i \in \mathcal{A}_i, i = 1, \dots, n, .$$

$\mathcal{L}(\mathcal{A}_1, \dots, \mathcal{A}_n; \mathfrak{X})$ denotes the linear space of all bounded n -linear mappings $\mathcal{A}_1 \times \dots \times \mathcal{A}_n \rightarrow \mathfrak{X}$ made into a Banach space with the norm

$$\|\varphi\| = \sup \{ \|\varphi(a_1, \dots, a_n)\| \mid a_i \in (\mathcal{A}_i)_1, i = 1, \dots, n \}, \quad \varphi \in \mathcal{L}(\mathcal{A}_1, \dots, \mathcal{A}_n, \mathfrak{X}).$$

If $\mathcal{A}_1 = \dots = \mathcal{A}_n = \mathcal{A}$, we define $\mathcal{L}^n(\mathcal{A}; \mathfrak{X}) := \mathcal{L}(\mathcal{A}_1, \dots, \mathcal{A}_n; \mathfrak{X})$ and additionally define $\mathcal{L}^0(\mathcal{A}; \mathfrak{X}) := \mathfrak{X}$.

We will now make swift progress towards defining a higher order Hochschild cohomology group.

Definition 3.2. Let \mathcal{A} be a Banach algebra and \mathfrak{X} a Banach \mathcal{A} -bimodule. For $n \in \mathbb{N}$, we define the *n -coboundary operator* $\delta^n : \mathcal{L}^{n-1}(\mathcal{A}, \mathfrak{X}) \rightarrow \mathcal{L}^n(\mathcal{A}, \mathfrak{X})$ by

$$\delta^1(\Phi)(a) = a\Phi - \Phi a, \quad \Phi \in \mathfrak{X}, \quad a \in \mathcal{A}$$

for $n = 1$, and

$$\begin{aligned} \delta^n(\Phi)(a_1, \dots, a_n) &= a_1\Phi(a_2, \dots, a_n) + \sum_{i=1}^{n-1} (-1)^i \Phi(a_1, \dots, a_i a_{i+1}, \dots, a_n) \\ &\quad + (-1)^n \Phi(a_1, \dots, a_{n-1})a_n, \end{aligned}$$

for $n \geq 2$, where $\varphi \in \mathcal{L}^{n-1}(\mathcal{A}, \mathfrak{X})$, $a_1, \dots, a_n \in \mathcal{A}$.

These operators look quite complicated, manifested in this important lemma.

Lemma 3.3. *The coboundary operators satisfy $\delta^{n+1} \circ \delta^n = 0$ for all $n \in \mathbb{N}$.*

Proof. The result comes from a long, tedious bunch of calculating and cancelling out terms equal to zero in the resulting sum. For $n = 1$, then assuming $\Phi \in \mathfrak{X}$, $a_1, a_2 \in \mathcal{A}$, we have

$$\begin{aligned} \delta^2(\delta^1(\Phi))(a_1, a_2) &= a_1\delta^1(\Phi)(a_2) - \delta^1(\Phi)(a_1 a_2) - \delta^1(\Phi)(a_1)a_2 \\ &= -a_1 a_2 \Phi + \Phi a_1 a_2 + a_1(a_2 \Phi - \Phi a_2) - (a_1 \Phi - \Phi a_1)a_2 \\ &= 0. \end{aligned}$$

The process in obtaining the result for $n \geq 2$ can be described as follows. Let $a_1, \dots, a_{n+1} \in \mathcal{A}$ and $\Phi \in \mathcal{L}^{n-1}(\mathcal{A}, \mathfrak{X})$. Then $\delta^{n+1}(\delta^n(\Phi))(a_1, \dots, a_{n+1})$, once we write it out, has terms by which a_1 is multiplied from the left that cancel out and some by which a_{n+1} is multiplied from the right that cancel out as well. Left are the terms that contain no \mathcal{A} -module multiplications except for inside the $(n-1)$ -tuples of which the image of Φ is taken. Some of these $(n-1)$ -tuples have coordinates in which three successive a_i 's are multiplied, and these cancel each other out as well, which is easily seen by expanding $\delta^{n+1}(\delta^n(\Phi))(a_1, \dots, a_{n+1})$ one term at a time.

Now left are only the terms involving $(n-1)$ -tuples with two coordinates equal to $a_i a_{i+1}$ and $a_j a_{j+1}$ for some i, j with distance 2 at least. It is clear that for all $i, j \in \{1, \dots, n\}$ satisfying this, the image $\Phi(a_1, \dots, a_i a_{i+1}, \dots, a_j a_{j+1})$ occurs only twice in the sum overall, since we can switch i and j around and only do that to order to obtain this very term; the question is whether the signatures of these images are different. They are: fix $i = 1, \dots, n$ and let $j = 1, \dots, n$ with $|i - j| \geq 2$. Then the signature of $\Phi(a_1, \dots, a_i a_{i+1}, \dots, a_j a_{j+1})$ is $(-1)^i (-1)^{j-1}$ if $j > i$ and $(-1)^i (-1)^j$ if $j < i$. Symmetry ends the proof. \square

The previous proof may then be tedious, but our reward comes right now. Let \mathcal{A} be a Banach algebra and \mathfrak{X} a Banach \mathcal{A} -bimodule. It follows from the previous lemma that for any $n \in \mathbb{N}$, any $\varphi \in \text{ran } \delta^n$ satisfies $\delta^{n+1}(\varphi) = 0$, and therefore $\text{ran } \delta^n$ is a linear subspace of $\ker \delta^{n+1}$.

Now we note that for $\varphi \in \mathcal{L}^1(\mathcal{A}, \mathfrak{X})$ and $a_1, a_2 \in \mathcal{A}$, we have

$$\delta^2(\varphi)(a_1, a_2) = a_1 \varphi(a_2) - \varphi(a_1 a_2) + \varphi(a_1) a_2.$$

Thus if $\varphi \in \ker \delta^2$, then $\varphi(a_1 a_2) = a_1 \varphi(a_2) + \varphi(a_1) a_2$ for all $a_1, a_2 \in \mathcal{A}$, and therefore $\ker \delta^2$ consists of all bounded \mathfrak{X} -derivations and is equal to $\mathcal{Z}^1(\mathcal{A}, \mathfrak{X})$. It is also clear from the definition of δ^1 that $\text{ran } \delta^1 = \mathcal{B}^1(\mathcal{A}, \mathfrak{X})$. This inspires us to define

$$\mathcal{Z}^n(\mathcal{A}, \mathfrak{X}) := \ker \delta^{n+1}, \quad \mathcal{B}^n(\mathcal{A}, \mathfrak{X}) := \text{ran } \delta^n, \quad n \in \mathbb{N},$$

thus providing a well-defined extension of the cohomology group concept: staying true to the definition of the first Hochschild cohomology group, we define

$$\mathcal{H}^n(\mathcal{A}, \mathfrak{X}) = \mathcal{Z}^n(\mathcal{A}, \mathfrak{X}) / \mathcal{B}^n(\mathcal{A}, \mathfrak{X})$$

which is called the n th Hochschild cohomology group of \mathcal{A} with coefficients in \mathfrak{X} .

One wouldn't believe this theorem at first sight.

Theorem 3.4. *Let \mathcal{A} be an amenable Banach algebra. Then $\mathcal{H}^n(\mathcal{A}, \mathfrak{X}') = \{0\}$ for all Banach \mathcal{A} -bimodules \mathfrak{X} and all $n \in \mathbb{N}$.*

It is indeed quite remarkable that it is so, even though the idea behind the proof is very simple. Assuming the Banach algebra \mathcal{A} is amenable, it would be natural to look for a Banach \mathcal{A} -bimodule \mathcal{Y} such that the n th Hochschild cohomology group with coefficients in a dual Banach \mathcal{A} -bimodule \mathfrak{X}' would be isomorphic to $\mathcal{H}^1(\mathcal{A}, \mathcal{Y}')$, equal to $\{0\}$.

Let us therefore introduce the following mapping right away, to remove some obstacles immediately. Let \mathcal{A} be a Banach algebra and \mathfrak{X} a Banach \mathcal{A} -bimodule. For $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$, we define the mapping $\sigma_k^n : \mathcal{L}^{n+k}(\mathcal{A}; \mathfrak{X}) \rightarrow \mathcal{L}^n(\mathcal{A}; \mathcal{L}^k(\mathcal{A}; \mathfrak{X}))$ by

$$(\sigma_k^n(\varphi)(a_1, \dots, a_n))(b_1, \dots, b_k) = \varphi(a_1, \dots, a_n, b_1, \dots, b_k), \quad \varphi \in \mathcal{L}^{n+k}(\mathcal{A}; \mathfrak{X}), \quad a_i, b_j \in \mathcal{A}.$$

Hence for $\varphi \in \mathcal{L}^{n+k}(\mathcal{A}; \mathfrak{X})$ and $a \in \mathcal{A}^n$, we have $\sigma_k^n(\varphi)(a) \in \mathcal{L}^k(\mathcal{A}; \mathfrak{X})$.

It will come in handy to make $\mathcal{L}^k(\mathcal{A}; \mathfrak{X})$ into a Banach \mathcal{A} -bimodule. For $k \in \mathbb{N}$, we shall from here onward make $\mathcal{L}^k(\mathcal{A}; \mathfrak{X})$ into a Banach \mathcal{A} -bimodule under the multiplications

$$\begin{aligned} (a\varphi)(a_1, \dots, a_k) &:= a\varphi(a_1, \dots, a_k), \\ (\varphi a)(a_1, \dots, a_k) &:= \varphi(aa_1, \dots, a_k) + \sum_{j=1}^{k-1} (-1)^j \varphi(a, a_1, \dots, a_j a_{j+1}, \dots, a_k) \\ &\quad + (-1)^k \varphi(a, a_1, \dots, a_{k-1}) a_k, \end{aligned}$$

for $\varphi \in \mathcal{L}^k(\mathcal{A}; \mathfrak{X})$, $a, a_1, \dots, a_k \in \mathcal{A}$, whereupon for $n \in \mathbb{N}$, we can legitimately define Δ_k^n to be the n -coboundary operators δ^n with \mathfrak{X} replaced by $\mathcal{L}^k(\mathcal{A}; \mathfrak{X})$; hence Δ_k^n is a mapping $\mathcal{L}^{n-1}(\mathcal{A}; \mathcal{L}^k(\mathcal{A}; \mathfrak{X})) \rightarrow \mathcal{L}^n(\mathcal{A}; \mathcal{L}^k(\mathcal{A}; \mathfrak{X}))$, defined as in Definition 3.2.

Lemma 3.5. $\sigma_k^n : \mathcal{L}^{n+k}(\mathcal{A}; \mathfrak{X}) \rightarrow \mathcal{L}^n(\mathcal{A}; \mathcal{L}^k(\mathcal{A}; \mathfrak{X}))$ is an isometric \mathcal{A} -bimodule isomorphism.

Proof. σ_k^n is easily checked to be an \mathcal{A} -bimodule homomorphism, and for $\varphi \in \mathcal{L}^{n+k}(\mathcal{A}; \mathfrak{X})$ and $a_i, b_j \in (\mathcal{A})_1$,

$$\|(\sigma_k^n(\varphi)(a_1, \dots, a_n))(b_1, \dots, b_k)\| \leq \|\varphi\|,$$

so that $\|\sigma_k^n(\varphi)(a_1, \dots, a_n)\| \leq \|\varphi\|$ by remembering that $\sigma_k^n(\varphi)(a_1, \dots, a_n) \in \mathcal{L}^k(\mathcal{A}; \mathfrak{X})$, thus giving $\|\sigma_k^n(\varphi)\| \leq \|\varphi\|$. Conversely,

$$\|\varphi(a_1, \dots, a_n, b_1, \dots, b_k)\| \leq \|\sigma_k^n(\varphi)(a_1, \dots, a_n)\| \leq \|\sigma_k^n(\varphi)\|,$$

so σ_k^n is an isometry and thus injective. For $\Phi \in \mathcal{L}^n(\mathcal{A}; \mathcal{L}^k(\mathcal{A}; \mathfrak{X}))$, $\sigma_k^n(\varphi) = \Phi$ where

$$\varphi(a_1, \dots, a_{n+k}) = (\Phi(a_1, \dots, a_n))(a_{n+1}, \dots, a_{n+k}), \quad a_i \in \mathcal{A}.$$

φ is easily seen to be linear in all $n+k$ variables and bounded by $\|\Phi\|$. Hence σ_k^n is surjective. \square

Lemma 3.6. For $n \in \mathbb{N}$, $\Delta_k^n \circ \sigma_k^{n-1} = \sigma_k^n \circ \delta^{n+k}$.

Proof. We just calculate elementarily: for $\varphi \in \mathcal{L}^{n+k-1}(\mathcal{A}; \mathfrak{X})$ and $a_1, \dots, a_{n+k} \in \mathcal{A}$, we obtain by letting $b_i = a_{n+i}$ for $i = 1, \dots, k$ and $\Phi = \sigma_k^{n-1}(\varphi)(a_1, \dots, a_{n-1})$ that

$$\begin{aligned} &\delta^{n+k}(\varphi)(a_1, \dots, a_{n+k}) - (\Delta_k^n(\sigma_k^{n-1}(\varphi))(a_1, \dots, a_n))(a_{n+1}, \dots, a_{n+k}) \\ &= \sum_{i=n}^{n+k-1} (-1)^i \Phi(a_n, \dots, a_i a_{i+1}, \dots, a_{n+k}) + (-1)^{n+k} \Phi(a_n, \dots, a_{n+k-1}) a_{n+k} \\ &\quad - ((-1)^n \Phi a_n)(a_{n+1}, \dots, a_{n+k}) \\ &= (-1)^n \left(\Phi(a_n b_1, \dots, b_k) + \sum_{i=1}^{k-1} (-1)^i \Phi(a_n, \dots, b_i b_{i+1}, \dots, b_k) + (-1)^k \Phi(a_n, \dots, b_{k-1}) b_k \right) \\ &\quad - (-1)^n (\Phi a_n)(b_1, \dots, b_k) \\ &= 0, \end{aligned}$$

from the definition of the right \mathcal{A} -module multiplication on $\mathcal{L}^k(\mathcal{A}; \mathfrak{X})$. \square

Theorem 3.7. Let \mathcal{A} be a Banach algebra and \mathfrak{X} a Banach \mathcal{A} -bimodule. For $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$, $\mathcal{H}^{n+k}(\mathcal{A}, \mathfrak{X})$ and $\mathcal{H}^n(\mathcal{A}, \mathcal{L}^k(\mathcal{A}; \mathfrak{X}))$ are isomorphic as linear spaces.

Proof. For $k = 0$, the two spaces are equal, and we therefore assume $k \in \mathbb{N}$. We're aiming at proving that the mapping $\mathcal{H}^{n+k}(\mathcal{A}, \mathfrak{X}) \rightarrow \mathcal{H}^n(\mathcal{A}, \mathcal{L}^k(\mathcal{A}; \mathfrak{X}))$ given by

$$\varphi + \mathcal{B}^{n+k}(\mathcal{A}, \mathfrak{X}) \mapsto \sigma_k^n(\varphi) + \mathcal{B}^n(\mathcal{A}, \mathcal{L}^k(\mathcal{A}, \mathfrak{X})), \quad \varphi \in \mathcal{Z}^{n+k}(\mathcal{A}, \mathfrak{X}), \quad (3.1)$$

is an isomorphism.

For any $\varphi \in \mathcal{Z}^{n+k}(\mathcal{A}, \mathfrak{X})$, then by definition $\varphi \in \ker \delta^{n+k+1}$ and $\delta^{n+k+1}(\varphi) = 0$. Obviously, this occurs iff $(\sigma_k^{n+1} \circ \delta^{n+k+1})(\varphi) = 0$, since σ_k^{n+1} is injective, and so by Lemma 3.6, we have $(\Delta_k^{n+1} \circ \sigma_k^n)(\varphi) = 0$ and $\sigma_k^n(\varphi) \in \mathcal{Z}^n(\mathcal{A}, \mathcal{L}^k(\mathcal{A}; \mathfrak{X}))$. Thus we have proved

$$\varphi \in \mathcal{Z}^{n+k}(\mathcal{A}, \mathfrak{X}) \Leftrightarrow \sigma_k^n(\varphi) \in \mathcal{Z}^n(\mathcal{A}, \mathcal{L}^k(\mathcal{A}; \mathfrak{X})). \quad (3.2)$$

Now let $\varphi \in \mathcal{B}^{n+k}(\mathcal{A}, \mathfrak{X}) = \text{ran } \delta^{n+k}$. There exists $\Phi \in \mathcal{L}^{n+k-1}(\mathcal{A}; \mathfrak{X})$ such that $\delta^{n+k}(\Phi) = \varphi$, and so

$$\sigma_k^n(\varphi) = (\sigma_k^n \circ \delta^{n+k})(\Phi) = \Delta_k^n(\sigma_k^{n-1}(\Phi)) \in \text{ran } \Delta_k^n = \mathcal{B}^n(\mathcal{A}, \mathcal{L}^k(\mathcal{A}; \mathfrak{X})).$$

If $\varphi \in \mathcal{L}^{n+k}(\mathcal{A}; \mathfrak{X})$ such that $\sigma_k^n(\varphi) \in \mathcal{B}^n(\mathcal{A}, \mathcal{L}^k(\mathcal{A}; \mathfrak{X}))$, there exists $\Phi \in \mathcal{L}^{n-1}(\mathcal{A}; \mathcal{L}^k(\mathcal{A}; \mathfrak{X}))$ such that $\Delta_k^n(\Phi) = \sigma_k^n(\varphi)$. Since σ_k^n is an isomorphism, there exists $\Psi \in \mathcal{L}^{(n-1)+k}(\mathcal{A}; \mathfrak{X})$ such that $\sigma_k^{n-1}(\Psi) = \Phi$. Therefore

$$\sigma_k^n(\varphi) = (\Delta_k^n \circ \sigma_k^{n-1})(\Psi) = \sigma_k^n(\delta^{n+k}(\Psi)),$$

and since σ_k^n is injective, $\varphi = \delta^{n+k}(\Psi) \in \text{ran } \delta^{n+k} = \mathcal{B}^{n+k}(\mathcal{A}, \mathfrak{X})$. Hence

$$\varphi \in \mathcal{B}^{n+k}(\mathcal{A}, \mathfrak{X}) \Leftrightarrow \sigma_k^n(\varphi) \in \mathcal{B}^n(\mathcal{A}, \mathcal{L}^k(\mathcal{A}; \mathfrak{X})). \quad (3.3)$$

From (3.2) and (3.3), we observe that (3.1) specifies a well-defined homomorphism; it is surjective because of (3.2) and injective because of (3.3), and we are done. \square

Our problem is thus reduced to considering the first Hochschild cohomology group of a Banach algebra with coefficients in \mathcal{L}^k , and with the next section, we come much closer.

3.2 Tensor products

Let us go right to the definition.

Definition 3.8. For $n \in \mathbb{N}$, let $\mathfrak{X}_1, \dots, \mathfrak{X}_n$ be normed spaces with Banach duals $\mathfrak{X}'_1, \dots, \mathfrak{X}'_n$. Given $x_i \in \mathfrak{X}_i$, $i = 1, \dots, n$, we let $x_1 \otimes \dots \otimes x_n$ denote the bounded n -linear functional $\mathfrak{X}'_1 \times \dots \times \mathfrak{X}'_n \rightarrow \mathbb{C}$ defined by

$$(x_1 \otimes \dots \otimes x_n)(\varphi_1, \dots, \varphi_n) = \varphi_1(x_1) \cdots \varphi_n(x_n), \quad \varphi_i \in \mathfrak{X}'_i, i = 1, \dots, n.$$

The *algebraic tensor product* of $\mathfrak{X}_1, \dots, \mathfrak{X}_n$, denoted $\mathfrak{X}_1 \otimes \dots \otimes \mathfrak{X}_n$, is defined to be the linear span of all *elementary tensors* $x_1 \otimes \dots \otimes x_n$, $x_i \in \mathfrak{X}_i$, $i = 1, \dots, n$, in $\mathcal{L}(\mathfrak{X}_1, \dots, \mathfrak{X}_n; \mathbb{C})$; that is, all *finite* linear combinations of tensor products, called *tensors*.

As a subspace of $\mathcal{L}(\mathfrak{X}_1, \dots, \mathfrak{X}_n; \mathbb{C})$, $\mathfrak{X}_1 \otimes \dots \otimes \mathfrak{X}_n$ inherits the norm $\|\cdot\|$ from it, and it is clear from the definition that $\|x_1 \otimes \dots \otimes x_n\| \leq \|x_1\| \cdots \|x_n\|$ for all $x_i \in \mathfrak{X}_i$, $i = 1, \dots, n$. Let $x_1 \otimes \dots \otimes x_n \in \mathfrak{X}_1 \otimes \dots \otimes \mathfrak{X}_n$; as a result of Hahn-Banach, there exists an $\varphi_i \in \mathfrak{X}'_i$ with $\|\varphi_i\| = 1$ and $\langle x_i, \varphi_i \rangle = \|x_i\|$ for $i = 1, \dots, n$, so that

$$\|x_1 \otimes \dots \otimes x_n\| \geq |(x_1 \otimes \dots \otimes x_n)(\varphi_1, \dots, \varphi_n)| = |\langle x_1, \varphi_1 \rangle \cdots \langle x_n, \varphi_n \rangle| = \|x_1\| \cdots \|x_n\|.$$

Hence $\|x_1 \otimes \dots \otimes x_n\| = \|x_1\| \cdots \|x_n\|$ for all $x_i \in \mathfrak{X}_i$, $i = 1, \dots, n$; this property makes the inherited norm a so-called *cross norm* on $\mathfrak{X}_1 \otimes \dots \otimes \mathfrak{X}_n$.

A key property of the algebraic tensor product is this:

Theorem 3.9. *Let the mapping $\varphi : \mathfrak{X}_1 \times \dots \times \mathfrak{X}_n \rightarrow \mathcal{Y}$ be n -linear. There exists a unique linear mapping $\sigma : \mathfrak{X}_1 \otimes \dots \otimes \mathfrak{X}_n \rightarrow \mathcal{Y}$ such that $\sigma(x_1 \otimes \dots \otimes x_n) = \varphi(x_1, \dots, x_n)$ for all $x_i \in \mathfrak{X}_i$, $i = 1, \dots, n$.*

Proof. Omitted. Consult [1, Theorem 42.6]. \square

Definition 3.10. Let $n \in \mathbb{N}$ and $\mathfrak{X}_1, \dots, \mathfrak{X}_n$ be normed spaces. The *projective tensor norm* ρ on $\mathfrak{X}_1 \otimes \dots \otimes \mathfrak{X}_n$ is defined for $a \in \mathfrak{X}_1 \otimes \dots \otimes \mathfrak{X}_n$ by

$$\rho(a) = \inf \left\{ \sum_i \|x_1^i\| \cdots \|x_n^i\| \mid a = \sum_i x_1^i \otimes \cdots \otimes x_n^i, x_m^i \in \mathfrak{X}_m, m = 1, \dots, n \right\};$$

the above infimum is therefore taken over all (finite) representations of a in $\mathfrak{X}_1 \otimes \dots \otimes \mathfrak{X}_n$.

We do not actually know yet that ρ is a norm; let us check this. If we decompose $a = \sum_i x_1^i \otimes \cdots \otimes x_n^i$, then $\lambda a = \sum_i (\lambda x_1^i) \otimes \cdots \otimes x_n^i$ and $\rho(\lambda a) \leq |\lambda| \rho(a)$ for $\lambda \in \mathbb{C}$, since the decomposition of a was arbitrary. For $\lambda \neq 0$, $\rho(a) = \rho(\lambda^{-1} \lambda a) \leq |\lambda|^{-1} \rho(\lambda a)$, so $\rho(\lambda a) = |\lambda| \rho(a)$ follows clearly.

For $a, b \in \mathfrak{X}_1 \otimes \dots \otimes \mathfrak{X}_n$, write $a = \sum_{i=1}^j x_1^i \otimes \cdots \otimes x_n^i$ and $b = \sum_{i=1}^k y_1^i \otimes \cdots \otimes y_n^i$ for $x_m^i, y_m^i \in \mathfrak{X}_m, k = 1, \dots, n$; then $a + b = \sum_{i=1}^{j+k} z_1^i \otimes \cdots \otimes z_n^i$ where for $i = 1, \dots, n$, $z_m^i = x_m^i$ for $m = 1, \dots, j$, $z_m^i = y_m^{i-j}$ for $m = 1, \dots, k$. Thus

$$\rho(a + b) \leq \sum_{i=1}^{j+k} \|z_1^i\| \cdots \|z_n^i\| = \sum_{i=1}^j \|x_1^i\| \cdots \|x_n^i\| + \sum_{i=1}^k \|y_1^i\| \cdots \|y_n^i\|,$$

and therefore $\rho(a + b) \leq \rho(a) + \rho(b)$. If $a \in \mathfrak{X}_1 \otimes \dots \otimes \mathfrak{X}_n$ and $a = \sum_i x_1^i \otimes \cdots \otimes x_n^i$, then

$$|a(\varphi_1, \dots, \varphi_n)| = \left| \sum_i \varphi_1(x_1^i) \cdots \varphi_n(x_n^i) \right| \leq \|\varphi_1\| \cdots \|\varphi_n\| \sum_i \|x_1^i\| \cdots \|x_n^i\|, \quad \varphi_i \in \mathfrak{X}'_i,$$

and so $\|a\| \leq \rho(a)$. Thus if $\rho(a) = 0$, then $\|a\| = 0$ and $a = 0$. From this, we obtain the following.

Lemma 3.11. *The projective tensor norm ρ is a norm on $\mathfrak{X}_1 \otimes \dots \otimes \mathfrak{X}_n$ satisfying $\|a\| \leq \rho(a)$ for all $a \in \mathfrak{X}_1 \otimes \dots \otimes \mathfrak{X}_n$, and especially $\|x_1\| \cdots \|x_n\| = \rho(x_1 \otimes \cdots \otimes x_n)$ for all $x_i \in \mathfrak{X}_i, i = 1, \dots, n$.*

From this, we naturally form a Banach space. We remind the reader that a *completion* of a normed space \mathfrak{X} is a Banach space to which there exists a dense isometric homomorphism from \mathfrak{X} ; every normed space has a unique completion up to isometric isomorphism.

Definition 3.12. The completion of the normed space $\mathfrak{X}_1 \otimes \dots \otimes \mathfrak{X}_n$ with the projective tensor norm ρ is called *the projective tensor product of $\mathfrak{X}_1, \dots, \mathfrak{X}_n$* and is denoted $\mathfrak{X}_1 \hat{\otimes} \cdots \hat{\otimes} \mathfrak{X}_n$.

It isn't exactly clear how elements of $\mathfrak{X}_1 \hat{\otimes} \cdots \hat{\otimes} \mathfrak{X}_n$ then look. The next theorem clears that up.

Theorem 3.13. *$\mathfrak{X}_1 \hat{\otimes} \cdots \hat{\otimes} \mathfrak{X}_n$ can be represented as the linear subspace of $\mathcal{L}(\mathfrak{X}_1, \dots, \mathfrak{X}_n; \mathbb{C})$ consisting of all elements of the form $a = \sum_{i=1}^{\infty} x_1^i \otimes \cdots \otimes x_n^i$ where $\sum_{i=1}^{\infty} \|x_1^i\| \cdots \|x_n^i\| < \infty$. Furthermore, $\rho(a)$ is the infimum of the sums $\sum_{i=1}^{\infty} \|x_1^i\| \cdots \|x_n^i\|$ over all such representations of a .*

Proof. Let $a \in \mathfrak{X}_1 \hat{\otimes} \cdots \hat{\otimes} \mathfrak{X}_n$ and $\varepsilon > 0$. Since $\mathfrak{X}_1 \otimes \dots \otimes \mathfrak{X}_n$ is dense in $\mathfrak{X}_1 \hat{\otimes} \cdots \hat{\otimes} \mathfrak{X}_n$, then $a = \lim \lambda_m$ for a sequence (λ_m) in $\mathfrak{X}_1 \otimes \dots \otimes \mathfrak{X}_n$. Pick out a subsequence (a_m) of (λ_m) such that $\rho(a - a_m) \leq \frac{\varepsilon}{2^{m+3}}$ for $m \in \mathbb{N}$.

We want to write a as a sum of elementary tensors. Choose $x_j^i \in \mathfrak{X}_j, j = 1, \dots, n, i = 1, \dots, n_1$, such that $a_1 = \sum_{i=1}^{n_1} x_1^i \otimes \cdots \otimes x_n^i$ and

$$\rho(a_1) \leq \sum_{i=1}^{n_1} \|x_1^i\| \cdots \|x_n^i\| \leq \rho(a_1) + \frac{\varepsilon}{2^4} \leq \rho(a) + \rho(a - a_1) + \frac{\varepsilon}{2^4} \leq \rho(a) + \frac{\varepsilon}{2^3}.$$

Since $\rho(a_2 - a_1) \leq \rho(a - a_2) + \rho(a - a_1) \leq \frac{\varepsilon}{2^5} + \frac{\varepsilon}{2^4} < \frac{\varepsilon}{2^3}$, we can choose $x_j^i \in \mathfrak{X}_m$, $j = 1, \dots, n$, $i = n_1 + 1, \dots, n_2$, such that $a_2 - a_1 = \sum_{i=n_1+1}^{n_2} x_1^i \otimes \cdots \otimes x_n^i$ and

$$\rho(a_2 - a_1) \leq \sum_{i=n_1+1}^{n_2} \|x_1^i\| \cdots \|x_n^i\| \leq \rho(a_2 - a_1) + \left(\frac{\varepsilon}{2^3} - \rho(a_2 - a_1) \right) = \frac{\varepsilon}{2^3}.$$

Since $\rho(a_3 - a_2) \leq \rho(a - a_3) + \rho(a - a_2) \leq \frac{\varepsilon}{2^6} + \frac{\varepsilon}{2^5} < \frac{\varepsilon}{2^4}$, we can then choose $x_j^i \in \mathfrak{X}_j$, $j = 1, \dots, n$, $i = n_2 + 1, \dots, n_3$, such that $a_3 - a_2 = \sum_{i=n_2+1}^{n_3} x_1^i \otimes \cdots \otimes x_n^i$ and

$$\rho(a_3 - a_2) \leq \sum_{i=n_2+1}^{n_3} \|x_1^i\| \cdots \|x_n^i\| \leq \rho(a_3 - a_2) + \left(\frac{\varepsilon}{2^4} - \rho(a_3 - a_2) \right) = \frac{\varepsilon}{2^4}.$$

Continue expanding, choosing $x_m^i \in \mathfrak{X}_m$, $m = 1, \dots, n$ and estimating differences for $n \geq 4$. Now, let $b_1 = a_1$, $b_{m+1} = a_{m+1} - a_m$ for $m \in \mathbb{N}$, so that $a = \sum_{m=1}^{\infty} b_m = \sum_{i=1}^{\infty} x_1^i \otimes \cdots \otimes x_n^i$, and

$$\sum_{i=1}^{\infty} \|x_1^i\| \cdots \|x_n^i\| \leq \rho(a) + \frac{\varepsilon}{2} < \infty. \quad (3.4)$$

Hence $\sum_{i=1}^{\infty} x_1^i \otimes \cdots \otimes x_n^i$ converges absolutely. Let $\pi(a)$ denote the aforementioned infimum over all representations of a ; it follows from (3.4) that $\pi(a) \leq \rho(a)$. But

$$\rho(a) \leq \lim_{N \rightarrow \infty} \sum_{i=1}^N \rho(x_1^i \otimes \cdots \otimes x_n^i) = \sum_{i=1}^{\infty} \|x_1^i\| \cdots \|x_n^i\|,$$

so equation holds. Finally, if $a = \sum_{i=1}^{\infty} x_1^i \otimes \cdots \otimes x_n^i$ where $\sum_{i=1}^{\infty} \|x_1^i\| \cdots \|x_n^i\| < \infty$, then the sum converges in $\mathcal{L}(\mathfrak{X}_1, \dots, \mathfrak{X}_n; \mathbb{C})$ (a Banach space) and letting $a_N = \sum_{i=1}^N x_1^i \otimes \cdots \otimes x_n^i$, it is obvious that $\rho(a - a_N) \rightarrow 0$ for $N \rightarrow \infty$. \square

Therefore we only need to define properties for elementary tensors in $\mathfrak{X}_1 \otimes \cdots \otimes \mathfrak{X}_n$ in order to define them for the entire projective tensor product. For every $\varphi \in (\mathfrak{X}_1 \hat{\otimes} \cdots \hat{\otimes} \mathfrak{X}_n)'$, we define the n -linear mapping Λ_φ on $\mathfrak{X}_1 \times \cdots \times \mathfrak{X}_n$ by

$$\Lambda_\varphi(x_1, \dots, x_n) = \langle x_1 \otimes \cdots \otimes x_n, \varphi \rangle, \quad x_i \in \mathfrak{X}_i, i = 1, \dots, n.$$

Since $|\Lambda_\varphi(x_1, \dots, x_n)| \leq \|\varphi\| \|x_1\| \cdots \|x_n\|$, clearly $\Lambda_\varphi \in \mathcal{L}(\mathfrak{X}_1, \dots, \mathfrak{X}_n; \mathbb{C})$ and $\|\Lambda_\varphi\| \leq \|\varphi\|$.

Lemma 3.14. *The mapping $(\mathfrak{X}_1 \hat{\otimes} \cdots \hat{\otimes} \mathfrak{X}_n)' \rightarrow \mathcal{L}(\mathfrak{X}_1, \dots, \mathfrak{X}_n; \mathbb{C})$, $\varphi \mapsto \Lambda_\varphi$ is an isometric isomorphism.*

Proof. The mapping is clearly linear and isometric, since $\|x_1\| \cdots \|x_n\| = \|x_1 \otimes \cdots \otimes x_n\|$. For any mapping $\Lambda \in \mathcal{L}(\mathfrak{X}_1, \dots, \mathfrak{X}_n; \mathbb{C})$, then by Theorem 3.9, there exists a unique linear functional $\Phi : \mathfrak{X}_1 \otimes \cdots \otimes \mathfrak{X}_n \rightarrow \mathbb{C}$ such that $\Phi(x_1 \otimes \cdots \otimes x_n) = \Lambda(x_1, \dots, x_n)$ for all $x_i \in \mathfrak{X}_i$, $i = 1, \dots, n$. Φ is bounded; indeed for fixed $a \in \mathfrak{X}_1 \otimes \cdots \otimes \mathfrak{X}_n$,

$$|\Phi(a)| = \left| \Phi \left(\sum_i x_1^i \otimes \cdots \otimes x_n^i \right) \right| = \left| \sum_i \Lambda(x_1^i, \dots, x_n^i) \right| \leq \|\Lambda\| \sum_i \|x_1^i\| \cdots \|x_n^i\|, \quad x_i \in \mathfrak{X}_i,$$

for $a = \sum_i x_1^i \otimes \cdots \otimes x_n^i$, so that $|\Phi(a)| \leq \|\Lambda\| \rho(a)$, making Φ a bounded linear functional on the normed space $(\mathfrak{X}_1 \otimes \cdots \otimes \mathfrak{X}_n, \rho)$. By Hahn-Banach, we extend Φ to a bounded linear functional $\Phi_1 \in (\mathfrak{X}_1 \hat{\otimes} \cdots \hat{\otimes} \mathfrak{X}_n)'$ such that $\|\Phi_1\| = \|\Phi\| \leq \|\Lambda\|$ and

$$\Lambda(x_1, \dots, x_n) = \Phi(x_1 \otimes \cdots \otimes x_n) = \Phi_1(x_1 \otimes \cdots \otimes x_n) = \Lambda_{\Phi_1}(x_1, \dots, x_n),$$

so that $\varphi \mapsto \Lambda_\varphi$ is surjective, and we are done. \square

We prove just one more lemma before the real fun begins.

Lemma 3.15. *The mapping $\mathcal{L}(\mathfrak{X}_1, \dots, \mathfrak{X}_n; \mathbb{C}) \rightarrow \mathcal{L}(\mathfrak{X}_1, \dots, \mathfrak{X}_{n-1}; \mathfrak{X}'_n)$, $\varphi \mapsto T_\varphi$, where*

$$\langle x_n, T_\varphi(x_1 \otimes \cdots \otimes x_{n-1}) \rangle = \varphi(x_1, \dots, x_n), \quad x_i \in \mathfrak{X}_i, i = 1, \dots, n,$$

is an isometric isomorphism.

Proof. For all $\varphi \in \mathcal{L}(\mathfrak{X}_1, \dots, \mathfrak{X}_n; \mathbb{C})$, $x_i \in \mathfrak{X}_i$ and $i = 1, \dots, n-1$, $T_\varphi(x_1 \otimes \cdots \otimes x_{n-1})$ is linear, because of n -linearity of φ , and bounded, since $\|T_\varphi(x_1 \otimes \cdots \otimes x_{n-1})\| \leq \|\varphi\| \|x_1\| \cdots \|x_{n-1}\|$. Therefore $T_\varphi \in \mathcal{L}(\mathfrak{X}_1, \dots, \mathfrak{X}_{n-1}; \mathfrak{X}'_n)$. T_φ is clearly linear and isometric from the definition. If we let $\Phi \in \mathcal{L}(\mathfrak{X}_1, \dots, \mathfrak{X}_{n-1}; \mathfrak{X}'_n)$, then define $\varphi : \mathfrak{X}_1 \times \cdots \times \mathfrak{X}_n \rightarrow \mathbb{C}$ by

$$\varphi(x_1, \dots, x_n) = \langle x_n, \Phi(x_1 \otimes \cdots \otimes x_{n-1}) \rangle;$$

φ is clearly linear and bounded, and $T_\varphi = \Phi$, so the mapping is surjective. \square

Let \mathcal{A} be a Banach algebra and \mathfrak{X} a Banach \mathcal{A} -bimodule. For $n \in \mathbb{N}$, define

$$\mathcal{Y} = \underbrace{\mathcal{A} \hat{\otimes} \cdots \hat{\otimes} \mathcal{A}}_{n \text{ times}} \hat{\otimes} \mathfrak{X}. \quad (3.5)$$

We make \mathcal{Y} into a Banach \mathcal{A} -bimodule by the module multiplications

$$\begin{aligned} (a_1 \otimes \cdots \otimes a_n \otimes x)a &:= a_1 \otimes \cdots \otimes a_n \otimes (xa), \\ a(a_1 \otimes \cdots \otimes a_n \otimes x) &:= (aa_1) \otimes \cdots \otimes a_n \otimes x + \sum_{i=1}^{n-1} (-1)^i a \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \otimes x \\ &\quad + (-1)^n a \otimes a_1 \otimes \cdots \otimes (a_n x). \end{aligned}$$

By Lemmas 3.14 and 3.15, we obtain the following result.

Corollary 3.16. *For Banach spaces $\mathfrak{X}, \mathcal{Y}$, the dual of the $(n+1)$ -fold projective tensor product $\mathfrak{X} \hat{\otimes} \cdots \hat{\otimes} \mathfrak{X} \hat{\otimes} \mathcal{Y}$ and the Banach space $\mathcal{L}^n(\mathfrak{X}; \mathcal{Y}')$ are isometrically isomorphic.*

For a Banach algebra \mathcal{A} and \mathfrak{X} a Banach \mathcal{A} -bimodule, then Lemma 3.16 yields an isometric isomorphism $F : (\mathcal{A} \hat{\otimes} \cdots \hat{\otimes} \mathcal{A} \otimes \mathfrak{X})' \rightarrow \mathcal{L}^n(\mathcal{A}; \mathfrak{X}')$ given by

$$\langle x, F(\varphi)(a_1, \dots, a_n) \rangle = \langle a_1 \otimes \cdots \otimes a_n \otimes x, \varphi \rangle, \quad a_i \in \mathcal{A}, i = 1, \dots, n, x \in \mathfrak{X}.$$

F is actually an \mathcal{A} -bimodule isomorphism, if we induce the usual module multiplications on the dual on $\mathcal{A} \hat{\otimes} \cdots \hat{\otimes} \mathcal{A} \otimes \mathfrak{X}$; indeed, letting $b = (a_1, \dots, a_n)$,

$$\langle x, F(\alpha\varphi)(b) \rangle = \langle (a_1 \otimes \cdots \otimes a_n \otimes x)\alpha, \varphi \rangle = \langle a_1 \otimes \cdots \otimes a_n \otimes (x\alpha), \varphi \rangle = \langle x, (\alpha F)(\varphi)(b) \rangle,$$

for $\alpha, a \in \mathcal{A}$ and $x \in \mathfrak{X}$, as a result of how we defined module multiplications on $\mathcal{L}^k(\mathcal{A}, \mathfrak{X})$. It is not hard to check that F preserves right \mathcal{A} -module multiplications too. Hence F makes \mathcal{Y}' and $\mathcal{L}^n(\mathcal{A}; \mathfrak{X}')$ isometrically isomorphic as \mathcal{A} -bimodules. This is not just fantastic; it helps us prove Theorem 3.4.

Proof of Theorem 3.4. Let \mathcal{Y} be the projective tensor product defined in (3.5). Then since \mathcal{Y} is a Banach \mathcal{A} -bimodule, hence by Theorem 3.7,

$$\mathcal{H}^{n+1}(\mathcal{A}, \mathfrak{X}') \cong \mathcal{H}^1(\mathcal{A}, \mathcal{L}^n(\mathcal{A}, \mathfrak{X}')) \cong \mathcal{H}^1(\mathcal{A}, \mathcal{Y}') = \{0\}.$$

The result is obvious for the case $n = 1$. \square

Thus, by introducing a seemingly extraneous construction, we obtain a surprising connection between higher order Hochschild cohomology groups and the first.

Multiplicative linear functionals

This appendix aims to show that all multiplicative linear functionals are bounded, with norm less than or equal to 1.

A Banach algebra may or may not have an identity. However, we can always embed a Banach algebra in an algebra with an identity as follows:

Definition A.1. The *unitization* of a Banach algebra \mathcal{A} , denoted by $\mathcal{A} + \mathbb{C}$, is the Banach algebra consisting of the set $\mathcal{A} \times \mathbb{C}$ with the obvious coordinatewise addition, scalar multiplication and then the product defined by $(x, \alpha)(y, \beta) = (xy + \alpha y + \beta x, \alpha\beta)$ for all $x, y \in \mathcal{A}$, $\alpha, \beta \in \mathbb{C}$, together with the norm defined by $\|(x, \alpha)\| = \|x\| + |\alpha|$.

It is seen by routine checking that $\mathcal{A} + \mathbb{C}$ actually is a Banach algebra; furthermore, it has a unit element $(0, 1)$ with norm 1, and the mapping $a \mapsto (a, 0)$ is an isometric algebra isomorphism of \mathcal{A} onto a subalgebra of $\mathcal{A} + \mathbb{C}$.

The unitization of an algebra is a good aid, as it allows us to define certain properties for algebras without unit elements by using their unitizations instead. We define the spectrum $\sigma(\mathcal{A}, a)$ of an element a of an algebra \mathcal{A} with a unit element 1 as usual:

$$\sigma(\mathcal{A}, a) = \{\lambda \in \mathbb{C} \mid \lambda 1 - a \in \text{Sing}(\mathcal{A})\}.$$

We now define the spectrum $\sigma(\mathfrak{B}, b)$ of an element b of a non-unital algebra \mathfrak{B} to be the set

$$\sigma(\mathfrak{B}, b) = \sigma(\mathfrak{B} + \mathbb{C}, (b, 0)).$$

For any element a of an algebra \mathcal{A} , we then define the *resolvent set* $\rho(\mathcal{A}, a) = \mathbb{C} \setminus \sigma(\mathcal{A}, a)$. Finally, for any Banach algebra \mathcal{A} we define the *spectral radius* $r_{\mathcal{A}}(a)$ for $a \in \mathcal{A}$ by

$$r_{\mathcal{A}}(a) = \inf_{n \in \mathbb{N}} \left\{ \|a^n\|^{1/n} \right\}.$$

Lemma A.2. *Let \mathcal{A} be a Banach algebra and $a \in \mathcal{A}$. Then $r_{\mathcal{A}}(a) = \lim_{n \rightarrow \infty} \|a^n\|^{1/n}$.*

Proof. Let $\varepsilon > 0$. Then there exists $k \in \mathbb{N}$ such that $\|a^k\|^{1/k} < r_{\mathcal{A}}(a) + \varepsilon$ (if there didn't, $r_{\mathcal{A}}(a)$ wouldn't be a greatest lower bound). Every $n \in \mathbb{N}$ can be written in the form $n = p_n k + q_n$ with $p_n, q_n \in \mathbb{N}_0$, $q_n \leq k - 1$. Now $\frac{1}{n} q_n \rightarrow 0$ for $n \rightarrow \infty$, so $\frac{1}{n} p_n k \rightarrow 1$, and thus there exists $m \in \mathbb{N}$ such that $n \geq m$ implies $\left| \|a^k\|^{1/n p_n} \|a\|^{1/n q_n} - \|a^k\|^{1/k} \right| < \varepsilon$. This gives us

$$r_{\mathcal{A}}(a) \leq \|a^n\|^{1/n} = \|(a^k)^{p_n} a^{q_n}\|^{1/n} \leq \|a^k\|^{1/n p_n} \|a\|^{1/n q_n} < \|a^k\|^{1/k} + \varepsilon < r_{\mathcal{A}}(a) + 2\varepsilon.$$

Hence $\left| \|a^n\|^{1/n} - r_{\mathcal{A}}(a) \right| < 2\varepsilon$ for $n \geq m$. □

Lemma A.3. *Let \mathcal{A} be a unital Banach algebra and let $a \in \mathcal{A}$. Assume that $r_{\mathcal{A}}(a) < 1$. Then $1 - a$ is invertible, and $(1 - a)^{-1} = \sum_{n=0}^{\infty} a^n$, where $a^0 := 1$.*

Proof. Choose $r_{\mathcal{A}}(a) < \eta < 1$. Then there exists $m \in \mathbb{N}$ such that $\|a^n\|^{1/n} \leq \eta$ and thus $\|a^n\| \leq \eta^n$ for $n \geq m$, because of the preceding lemma, so the series $\sum_{n=0}^{\infty} \|a^n\|$ converges in \mathbb{R} . Define $b_m = \sum_{n=0}^m a^n$ for $m \in \mathbb{N}$. (b_m) is a Cauchy sequence; for $m > k$

$$\|b_m - b_k\| = \left\| \sum_{n=k+1}^m a^n \right\| \leq \sum_{n=k+1}^m \|a^n\| \leq \sum_{n=k+1}^{\infty} \|a^n\|,$$

and so (b_m) converges in \mathcal{A} to $b = \sum_{n=0}^{\infty} a^n$. Now $(1-a)b_m = 1 - a^{m+1} = b_m(1-a)$ for all $m \in \mathbb{N}$. By continuity of multiplication, then because $\|a^n\| \rightarrow 0$ and thus $a^n \rightarrow 0$ for $n \rightarrow \infty$, we obtain $(1-a)b = b(1-a) = 1$. \square

Theorem A.4. *Let φ be a multiplicative linear functional on a Banach algebra \mathcal{A} . Then φ is bounded, i.e. $\varphi \in \mathcal{A}'$, and $\|\varphi\| \leq 1$.*

Proof. We prove the result first in the case where \mathcal{A} is unital and then generalize it. If \mathcal{A} has a unit element 1, then $\varphi(1) = 1$, so $\varphi(x)\varphi(x^{-1}) = 1$ for any invertible element $x \in \mathcal{A}$ and $\varphi(x) \neq 0$ for all invertible elements $x \in \mathcal{A}$.

Let $a \in \mathcal{A}$. Then for any $\lambda \in \rho(\mathcal{A}, a)$ we have $\lambda - \varphi(a) = \varphi(\lambda 1 - a) \neq 0$, so $\varphi(a)$ can't be contained in $\rho(\mathcal{A}, a)$. Let $a \in \mathcal{A}$ and $\lambda \in \mathbb{C}$. If $|\lambda| > r_{\mathcal{A}}(a)$, then it is obvious that $1 > r_{\mathcal{A}}(\lambda^{-1}a)$. This in turn gives that $\lambda 1 - a = \lambda(1 - \lambda^{-1}a)$ is invertible (because of Lemma A.3 and the assumption that \mathcal{A} is Banach) and $\lambda \in \rho(\mathcal{A}, a)$. Therefore for any $a \in \mathcal{A}$, $|\varphi(a)| \leq r_{\mathcal{A}}(a) \leq \|a\|$ since $\varphi(a) \notin \rho(\mathcal{A}, a)$, so φ is bounded and $\|\varphi\| \leq 1$ for unital \mathcal{A} .

If \mathcal{A} is non-unital, let $\mathcal{A} + \mathbb{C}$ denote the unitization of \mathcal{A} (which is a Banach algebra). Define $\tilde{\varphi} : \mathcal{A} + \mathbb{C} \rightarrow \mathbb{C}$ by $\tilde{\varphi}(a, \lambda) = \varphi(a) + \lambda$ for $a \in \mathcal{A}$, $\lambda \in \mathbb{C}$; then $\tilde{\varphi}$ is linear, and moreover

$$\tilde{\varphi}((a, \lambda)(b, \eta)) = \varphi(ab + \eta a + \lambda b) + \lambda\eta = (\varphi(a) + \lambda)(\varphi(b) + \eta) = \tilde{\varphi}(a, \lambda)\tilde{\varphi}(b, \eta),$$

so $\tilde{\varphi}$ is multiplicative. $\tilde{\varphi}$ is then bounded and $|\tilde{\varphi}(a, \lambda)| \leq \|a\| + |\lambda|$ for any $a \in \mathcal{A}$, $\lambda \in \mathbb{C}$, so $|\varphi(a)| = |\tilde{\varphi}(a, 0)| \leq \|a\|$ for any $a \in \mathcal{A}$ and φ is bounded. Thus in all cases $\|\varphi\| \leq 1$. \square

Hilbert space results

A subset S of a normed space \mathfrak{X} is *totally bounded* if for every $\varepsilon > 0$ there exists a finite number n of ε -balls with centres $x_1, \dots, x_n \in S$ such that every $x \in S$ satisfies $\|x - x_i\| < \varepsilon$ for some $i = 1, \dots, n$.

Lemma B.1. *Let \mathfrak{X} be a normed space and let $S \subseteq \mathfrak{X}$ be non-empty. If \overline{S} is compact, then S is totally bounded. If \mathfrak{X} is a Banach space, then S is totally bounded iff \overline{S} is compact.*

Proof. Let $x_1 \in S$. If all $x \in S$ satisfy $\|x - x_1\| < \varepsilon$, we are done. Otherwise, let $x_2 \in S$ such that $\|x_2 - x_1\| \geq \varepsilon$. If every $x \in S$ satisfies $\|x - x_1\| < \varepsilon$ or $\|x - x_2\| < \varepsilon$, we are done. If not, choose x_3 that doesn't satisfy this and continue inductively for $i \geq 4$. The lemma asserts that we stop after a finite number of steps: indeed if not, we would have obtained a sequence (x_i) in \overline{S} with all elements having distance at least ε from each other, thus having no convergent subsequence, so that \overline{S} is not compact.

Assume that \mathfrak{X} is Banach and S is totally bounded, and let (s_n) be a sequence of S . Let Λ_n be a finite collection of 2^{-n} -balls covering S . At least one of the balls of Λ_1 contains infinitely many elements of (s_n) ; choose one such, S_1 , and obtain a subsequence (s_n^1) by throwing out all s_n not in S_1 . Now, at least one of the balls of Λ_2 contains infinitely many elements of (s_n^1) ; choose one such, S_2 , and obtain a subsequence (s_n^2) by throwing out all s_n^1 not in S_2 . Continue this way inductively for all $n \in \mathbb{N}$. Let z_m be the first element of s_n^m . It is clear that (z_m) is Cauchy, since for any $\varepsilon > 0$, there is $n \in \mathbb{N}$ such that $2^{-n} < \varepsilon$, so that for $k, m \geq n + 1$, $\|z_k - z_m\| \leq 2 \cdot 2^{-(n+1)} < \varepsilon$. By completeness of \mathfrak{X} , (z_m) is convergent, and \overline{S} is compact. \square

Theorem B.2. *Let \mathcal{H} be a Hilbert space. If $A \in K(\mathcal{H})$, then $A^* \in K(\mathcal{H})$.*

Proof. Let $\varepsilon > 0$; we only need to show that $A^*((\mathcal{H})_1)$ is totally bounded by Lemma B.1. Since A is compact, then by Lemma B.1, choose $x_1, \dots, x_n \in \mathcal{H}_1$ such that each $x \in (\mathcal{H})_1$ satisfies $\|A(x) - A(x_i)\| < \frac{\varepsilon}{4}$ for some $i = 1, \dots, n$. Define the mapping $B : \mathcal{H} \rightarrow \mathbb{C}^n$ by

$$B(y) = [(A(x_1), y), \dots, (A(x_n), y)].$$

Since $\text{rk } B \leq n$, B is finite rank and thus compact. Choose $y_1, \dots, y_m \in (\mathcal{H})_1$ by Lemma B.1 so that any $y \in (\mathcal{H})_1$ satisfies $\|B(y) - B(y_j)\| < \frac{\varepsilon}{4}$ for a $j \in \{1, \dots, m\}$. Then any $z \in (\mathcal{H})_1$ satisfies

$$|(A(x_i), z) - (A(x_i), z_j)| \leq \|B(z) - B(z_j)\| < \frac{\varepsilon}{4}$$

for some $j \in \{1, \dots, m\}$ and all $i = 1, \dots, n$. Let $z \in (\mathcal{H})_1$ and choose $j \in \{1, \dots, m\}$ so that this applies. Furthermore, let $x \in \mathcal{H}$ and choose $i \in \{1, \dots, n\}$ so that $\|A(x) - A(x_i)\| < \frac{\varepsilon}{4}$. Then

$$\begin{aligned} |(A(x), z) - (A(x), y_j)| &\leq |(A(x), z) - (A(x_i), z)| + |(A(x_i), z) - (A(x_i), y_j)| \\ &\quad + |(A(x_i), y_j) - (A(x), y_j)| \\ &\leq 2\|A(x) - A(x_i)\| + \frac{\varepsilon}{4}. \end{aligned}$$

Thus $|(x, A^*(z) - A^*(y_j))| < \frac{3\varepsilon}{4}$ for all $x \in (\mathcal{H})_1$, and $\|A^*(z) - A^*(y_j)\| < \varepsilon$ by Riesz-Frechet [13, Theorem 6.8]. Hence $A^*((\mathcal{H})_1)$ is totally bounded. \square

The next lemma proves that $K(\mathcal{H}) + \mathbb{C}I$ is closed in a Hilbert space \mathcal{H} , I denoting the identity operator on \mathcal{H} .

Lemma B.3. *Let S be a closed subspace and F be a finite-dimensional subspace of a Banach space \mathfrak{X} for which $S \cap F = \{0\}$. Then $S \oplus F$ is closed.*

Proof. Let $a_n = s_n + f_n \in S \oplus F$ converge to $a \in \mathfrak{X}$. Assume $\|f_n\| \rightarrow \infty$. Then

$$\|f_n\|^{-1}(s_n + f_n) \rightarrow 0;$$

because F_1 is compact since it is finite-dimensional, there exists a subsequence $(\|f_{n_p}\|^{-1}f_{n_p})$ of the sequence $(\|f_n\|^{-1}f_n)$ that converges to an f with norm 1. But then $\|f_{n_p}\|^{-1}s_{n_p} \rightarrow -f$. Since S is closed, $-f \in S \cap F = \{0\}$, so $f = 0$ – a contradiction. Thus (f_n) is bounded, since anything else would imply that a subsequence tending to infinity in norm would exist. The linear projection $P : S \oplus F \rightarrow F$ defined by $s + f \mapsto f$, $s \in S$, $f \in F$, is therefore bounded, and so $\|f_n - f_m\| \leq \|P\|\|a_n - a_m\|$, making (f_n) Cauchy and thus convergent to an element ϕ , whereupon the sequence $(s_n) = (a_n - f_n)$ converges to $a - \phi$ contained in S , because S is closed. Thus $a \in S \oplus F$. □

A condition implying group amenability

This appendix deals with a quite peculiar, small detail in the proof of Theorem 2.11.

Lemma C.1. *Let ϕ be a bounded linear functional on $\ell^\infty(G)$ such that $\phi(1) = 1$ and ϕ is invariant under left translations. Then G is amenable, i.e. $\ell^\infty(G)$ has an invariant mean.*

We see here that the requirement of positivity is exchanged with an initial requirement that the linear functional is bounded (which is indeed essential as we shall see).

In the following proof, we will therefore provide a construction of an invariant mean as well as a very thorough and somewhat complicated explanation of why it is. Let Ra and Ia denote the real and imaginary parts of a complex number a , respectively.

Proof. Let $M = \ell_+^\infty(G)$, and for any $x \in \ell_+^\infty(G)$, let

$$\Theta(x) = \{\text{R}\phi(y) \mid y \in \ell^\infty(G), |y| \leq x\} \subseteq \mathbb{R}$$

and define $\theta : M \rightarrow \mathbb{R}$ by $\theta(x) = \sup \Theta(x)$. It is well-defined since for $x \in M$, then

$$|\text{R}\phi(y)| \leq |\phi(y)| \leq \|\phi\| \|y\|_\infty \leq \|\phi\| \|x\|_\infty$$

for all $y \in \ell^\infty(G)$ with $|y| \leq x$. We will show that θ is (1) positive, (2) homogeneous, (3) additive, (4) left translation invariant and then (5) extend θ to $\ell^\infty(G)$ by using properties of the complex numbers.

(1) *θ is positive.* For any $y \in \ell^\infty(G)$ such that $|y| \leq x$, $\Theta(x)$ contains both $\text{R}(\phi(y))$ and $\text{R}(\phi(-y)) = -\text{R}\phi(y)$, and therefore $2\theta(x) \geq \text{R}\phi(y) + (-\text{R}\phi(y)) = 0$. Also, $\theta(1) \geq \text{R}(\phi(1)) = 1$.

(2) *θ is homogeneous.* It is clear that $\theta(0) = 0$. Let $x \in M$. For $t > 0$ and $y \in \ell^\infty(G)$ with $|y| \leq tx$, then $\text{R}\phi(y) = t\text{R}\phi(t^{-1}y) = t\text{R}\phi(a)$ for $a = t^{-1}y \in \ell^\infty(G)$ which then satisfies $|a| \leq x$. Conversely, for $t > 0$ and $a \in \ell^\infty(G)$ with $|a| \leq x$, then $t\text{R}(\phi(a)) = \text{R}\phi(ta) = \text{R}(\phi(y))$ for $y = ta \in \ell^\infty(G)$ which then satisfies $|y| \leq tx$. We thus obtain $t\Theta(x) = \Theta(tx)$ and $\theta(tx) = t\theta(x)$ for all $t \geq 0$.

(3) *θ is additive.* Indeed, let $a, b \in M$. For $y_1 \in \ell^\infty(G)$ with $|y_1| \leq a$ and $y_2 \in \ell^\infty(G)$ with $|y_2| \leq b$, then $\text{R}\phi(y) = \text{R}\phi(y_1) + \text{R}\phi(y_2)$ for $y = y_1 + y_2 \in \ell^\infty(G)$ which then satisfies $|y| \leq a + b$. Hence $\text{R}\phi(y_1) + \text{R}\phi(y_2) \leq \theta(a + b)$ for all such y_1 and y_2 , so taking supremums gives $\theta(a) + \theta(b) \leq \theta(a + b)$.

The other way around is trickier. Let $y \in \ell^\infty(G)$ with $|y| \leq a + b$, and define $y_1(x) = y(x)$ for all $x \in G$ such that $|y(x)| \leq a(x)$, $y_1(x) = a(x)e^{i\arg(y(x))}$ for the case $|y(x)| > a(x)$ and finally $y_2 = y - y_1$. Then $y = y_1 + y_2$ and $|y_1| \leq a$. For $x \in G$ such that $|y(x)| > a(x)$, it follows that

$$|y_2(x)| = |y(x) - y_1(x)| = \left| (|y(x)| - a(x))e^{i\arg(y(x))} \right| = ||y(x)| - a(x)| = |y(x)| - a(x) \leq b(x),$$

so $|y_2| \leq b$. Now $\text{R}\phi(y) = \text{R}\phi(y_1) + \text{R}\phi(y_2)$ for these $y_1, y_2 \in \ell^\infty(G)$, and $\text{R}\phi(y) \leq \theta(a) + \theta(b)$, so $\theta(a + b) \leq \theta(a) + \theta(b)$.

(4) *Finally, θ is left translation invariant.* Indeed, let $h \in G$, $x \in \ell_+^\infty(G)$. If $z \in \Theta(x)$, then $z = \mathbf{R}\phi(\tau_h(y))$ for a $y \in \ell^\infty(G)$ with $|y| \leq x$, and letting $a = \tau_h(y) \in \ell^\infty(G)$, we have $|(\tau_{h^{-1}}(a))(g)| \leq x(g)$ and $|a(g)| = |a(hh^{-1}g)| = |(\tau_{h^{-1}}(a))(h^{-1}g)| \leq x(h^{-1}g) = (\tau_h(x))(g)$ for all $g \in G$, so $|a| \leq \tau_h(x)$; hence $z = \mathbf{R}\phi(a)$ for a $a \in \ell^\infty(G)$ with $|a| \leq \tau_h(x)$ and $z \in \Theta(\tau_h(x))$.

If similarly $z = \mathbf{R}\phi(y)$ for a $y \in \ell^\infty(G)$ with $|y| \leq \tau_h(x)$, and letting $b = \tau_{h^{-1}}(y) \in \ell^\infty(G)$, we have $|b(g)| = |y(hg)| \leq (\tau_h(x))(hg) = x(g)$ for all $g \in G$, so $|b| \leq \tau_h(x)$; therefore $z = \mathbf{R}\phi(b)$ for a $b \in \ell^\infty(G)$ with $|b| \leq x$ and $z \in \Theta(x)$. We thus have $\Theta(x) = \Theta(\tau_h(x))$, meaning that θ is translation invariant.

(5) *The extension.* It is clear that for any $f \in \ell^\infty(G)$, $f = (\mathbf{R}f)_+ - (\mathbf{R}f)_- + i(\mathbf{I}f)_+ - i(\mathbf{I}f)_-$, with all four functions themselves bounded and positive. Therefore $\ell^\infty(G) = M - M + iM - iM$. This inspires us to extend θ to $\ell^\infty(G)$ by defining

$$\tilde{\theta}(f) := \frac{1}{\theta(1)} (\theta((\mathbf{R}f)_+) - \theta((\mathbf{R}f)_-) + i\theta((\mathbf{I}f)_+) - i\theta((\mathbf{I}f)_-)).$$

It is first and foremost clear that $\tilde{\theta}(1) = 1$ and that $\tilde{\theta}$ is positive. Additionally, $\tilde{\theta}$ is translation invariant because θ is: indeed, for $h \in G$ and $f \in \ell^\infty(G)$,

$$\begin{aligned} (\mathbf{R}\tau_h(f))_+(g) &= \begin{cases} \mathbf{R}f(h^{-1}g) & \text{if } \mathbf{R}f(h^{-1}g) \geq 0 \\ 0 & \text{otherwise} \end{cases} \\ &= (\mathbf{R}f)_+(h^{-1}g) \\ &= \tau_h(\mathbf{R}f)_+(g), \end{aligned}$$

and similar results follow for $(\mathbf{R}f)_-$, $(\mathbf{I}f)_+$ and $(\mathbf{I}f)_-$, implying $\tilde{\theta}(\tau_h(f)) = \tilde{\theta}(f)$.

It only remains to show that $\tilde{\theta}$ is a linear functional. Let $\lambda \in \mathbb{C}$ and $x \in \ell^\infty(G)$. We split λ into four separate non-negative real numbers a, b, c, d such that $\lambda = a - b + ic - id$. Then for $g \in G$,

$$\begin{aligned} \mathbf{R}(\lambda x(g)) &= (a - b)((\mathbf{R}x)_+(g) - (\mathbf{R}x)_-(g)) - (c - d)((\mathbf{I}x)_+(g) - (\mathbf{I}x)_-(g)) \quad \text{and} \\ \mathbf{I}(\lambda x(g)) &= (c - d)((\mathbf{R}x)_+(g) - (\mathbf{R}x)_-(g)) + (a - b)((\mathbf{I}x)_+(g) - (\mathbf{I}x)_-(g)), \end{aligned}$$

so

$$\begin{aligned} (\mathbf{R}\lambda x)_+(g) &= a\mathbf{R}x_+(g) + b\mathbf{R}x_-(g) + d(\mathbf{I}x)_+(g) + c(\mathbf{I}x)_-(g), \\ (\mathbf{R}\lambda x)_-(g) &= b\mathbf{R}x_+(g) + a\mathbf{R}x_-(g) + c(\mathbf{I}x)_+(g) + d(\mathbf{I}x)_-(g), \\ (\mathbf{I}\lambda x)_+(g) &= c\mathbf{R}x_+(g) + d\mathbf{R}x_-(g) + b(\mathbf{I}x)_+(g) + a(\mathbf{I}x)_-(g) \quad \text{and} \\ (\mathbf{I}\lambda x)_-(g) &= d\mathbf{R}x_+(g) + c\mathbf{R}x_-(g) + a(\mathbf{I}x)_+(g) + b(\mathbf{I}x)_-(g). \end{aligned}$$

We thus obtain, with grace and with homogeneity of θ , that

$$\begin{aligned} \tilde{\theta}(\lambda x) &= [a\theta((\mathbf{R}x)_+) + b\theta((\mathbf{R}x)_-) + d\theta((\mathbf{I}x)_+) + c\theta((\mathbf{I}x)_-)] \\ &\quad + [-b\theta((\mathbf{R}x)_+) - a\theta((\mathbf{R}x)_-) - c\theta((\mathbf{I}x)_+) - d\theta((\mathbf{I}x)_-)] \\ &\quad + [ic\theta((\mathbf{R}x)_+) + id\theta((\mathbf{R}x)_-) + ia\theta((\mathbf{I}x)_+) + ib\theta((\mathbf{I}x)_-)] \\ &\quad + [-id\theta((\mathbf{R}x)_+) - ic\theta((\mathbf{R}x)_-) - ib\theta((\mathbf{I}x)_+) - ia\theta((\mathbf{I}x)_-)] \\ &= \lambda [\theta((\mathbf{R}x)_+) - \theta((\mathbf{R}x)_-) + i\theta((\mathbf{I}x)_+) - i\theta((\mathbf{I}x)_-)] \\ &= \lambda \tilde{\theta}(x). \end{aligned}$$

Let $x, y \in \ell^\infty(G)$. By decomposing $x + y$ in positive functions in two different ways, we obtain

$$\begin{aligned} &(\mathbf{R}(x + y))_+ - (\mathbf{R}(x + y))_- + i(\mathbf{I}(x + y))_+ - i(\mathbf{I}(x + y))_- \\ &= ((\mathbf{R}x)_+ - (\mathbf{R}x)_-) + i((\mathbf{I}x)_+ - (\mathbf{I}x)_-) + ((\mathbf{R}y)_+ - (\mathbf{R}y)_-) + i((\mathbf{I}y)_+ - (\mathbf{I}y)_-). \end{aligned}$$

Hence by “removing minuses”

$$\begin{aligned} & (\mathbf{R}(x+y))_+ + (\mathbf{R}x)_- + (\mathbf{R}y)_- + i[(\mathbf{I}(x+y))_+ + (\mathbf{I}x)_- + (\mathbf{I}y)_-] \\ = & (\mathbf{R}x)_+ + (\mathbf{R}y)_+ + (\mathbf{R}(x+y))_- + i[(\mathbf{I}(x+y))_- + (\mathbf{I}x)_+ + (\mathbf{I}y)_+]. \end{aligned}$$

By additivity of θ , we thus obtain

$$\begin{aligned} \theta((\mathbf{R}(x+y))_+) + \theta((\mathbf{R}x)_-) + \theta((\mathbf{R}y)_-) &= \theta((\mathbf{R}(x+y))_-) + \theta((\mathbf{R}x)_+) + \theta((\mathbf{R}y)_+) \quad \text{and} \\ \theta((\mathbf{I}(x+y))_+) + \theta((\mathbf{I}x)_-) + \theta((\mathbf{I}y)_-) &= \theta((\mathbf{I}(x+y))_-) + \theta((\mathbf{I}x)_+) + \theta((\mathbf{I}y)_+). \end{aligned}$$

By these two equations, we obtain $\tilde{\theta}(x+y) = \tilde{\theta}(x) + \tilde{\theta}(y)$. Therefore $\tilde{\theta}$ is an invariant mean on $\ell^\infty(G)$. □

Outside theorems

Here we list some of the theorems used in the thesis for which proofs are not provided; for most of the theorems, a reference to a proof from literature used in the thesis is given, and Google Books should be the best of friends for the missing proofs.

Theorem D.1. *Let \mathcal{A} be a Banach algebra with a bounded net (e_v) such that $\langle e_v a, \varphi \rangle \rightarrow \langle a, \varphi \rangle$ for all $a \in \mathcal{A}$ and $\varphi \in \mathcal{A}'$. Then \mathcal{A} has a bounded left approximate identity.*

Proof. Consult [1, Proposition 11.4]: replacing $e_v a$ with $a e_v$ in Theorem D.1 likewise yields a bounded right approximate identity. \square

Theorem D.2 (Banach-Alaoglu's theorem). *Let \mathfrak{X} be a Banach space. The unit ball $(\mathfrak{X}')_1$ of the Banach dual \mathfrak{X}' is compact in the w^* topology.*

Theorem D.3 (Hahn-Banach's theorem). *Let \mathfrak{B} be a subspace of the Banach space \mathfrak{X} . If $\varphi \in \mathfrak{B}'$, there exists $\Phi \in \mathfrak{X}'$ such that $\Phi(b) = \varphi(b)$ for all $x \in \mathfrak{B}$ and $\|\Phi\| = \|\varphi\|$.*

Theorem D.4 (Stone-Weierstrass). *Let \mathfrak{X} be a compact Hausdorff space. If \mathcal{U} is a closed subalgebra of $C(\mathfrak{X})$ that is self-adjoint (i.e. $f \in \mathcal{U}$ implies that \mathcal{U} contains the complex conjugate \bar{f}), contains the constant function 1 and separates the points of \mathfrak{X} (i.e. if $x, y \in \mathfrak{X}$ and $x \neq y$, then there exists a function $f \in \mathcal{U}$ such that $f(x) \neq f(y)$), then $\mathcal{U} = C(\mathfrak{X})$.*

Proofs of theorems D.2, D.3 and D.4. Consult [4, Propositions 1.23, 1.26 and 2.40]. \square

Theorem D.5 (Goldstine's theorem). *Let \mathfrak{X} be a Banach space. The unit ball of \mathfrak{X} is w^* -dense in the unit ball of \mathfrak{X}'' , in the sense that $\{\hat{x} \mid x \in (\mathfrak{X})_1\}$ is w^* -dense in $(\mathfrak{X}'')_1$.*

Proof. Omitted. \square

Theorem D.6 (Markov-Kakutani fixed point theorem). *Let K be a non-empty convex compact subset of a Hausdorff topological space X . Let \mathcal{F} be a set of commuting continuous affine maps $f : K \rightarrow K$. Then there exists a point in K which is fixed by all elements of \mathcal{F} .*

Proof. Consult [2, Theorem C.1.1]. \square

Theorem D.7 (the Urysohn lemma). *Let \mathfrak{X} be a normal space; let \mathcal{A} and \mathfrak{B} be disjoint closed subsets of \mathfrak{X} . Let $[a, b]$ be a closed real interval. There exists a continuous map $f : \mathfrak{X} \rightarrow [a, b]$ such that $f(x) = a$ for all $x \in \mathcal{A}$ and $f(x) = b$ for all $x \in \mathfrak{B}$.*

Proof. Consult [9, Theorem 33.1]. \square

We use in the thesis that compact Hausdorff spaces are normal; this is proved in [9, Theorem 32.3].

BIBLIOGRAPHY

- [1] F. F. Bonsall and J. Duncan. *Complete Normed Algebras*, volume 80 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer-Verlag Berlin Heidelberg New York, 1973.
- [2] T. Ceccherini-Silberstein and M. Coornaert. *Cellular Automata and Groups*. Springer-Verlag, 2010.
- [3] J. Diestel, J. Fourie, and J. Swart. The Projective Tensor Product I. In A. Kamińska, editor, *Trends in Banach spaces and operator theory*. American Mathematical Society, 2003.
- [4] R. G. Douglas. *Banach Algebra Techniques in Operator Theory*, volume 179 of *Graduate Texts in Mathematics*. Springer-Verlag Berlin Heidelberg New York, second edition, 1972.
- [5] F. P. Greenleaf. *Invariant Means on Topological Groups and Their Applications*, volume 16 of *Van Nostrand Mathematical Studies*. Van Nostrand–Reinhold Company, American Book Company, 1969.
- [6] E. Hansen. *Measure theory*. Department of Mathematical Sciences, University of Copenhagen, fourth edition, 2009.
- [7] B. E. Johnson. *Cohomology in Banach algebras*, volume 127 of *Memoirs of the American Mathematical Society*. American Mathematical Society, 1972.
- [8] B. E. Johnson. *Introduction to Cohomology in Banach Algebras*. Academic Press, London, 1975.
- [9] J. R. Munkres. *Topology*. Pearson Education International, Prentice Hall Incorporated, second edition, 1975.
- [10] V. Runde. *Lectures on Amenability*, volume 1774 of *Lecture notes in mathematics*. Springer-Verlag, 2002.
- [11] V. Runde. Lectures on amenability (errata and updates). <http://www.math.ualberta.ca/~runde/files/errata.pdf>, March 2010.
- [12] M. Schechter. *Principles of functional analysis*, volume 36 of *Graduate Studies in Mathematics*. American Mathematical Society, 2002.
- [13] N. Young. *An introduction to Hilbert space*. Cambridge University Press, 1988.