Geometry 2

Test

Rasmus Sylvester Bryder

January 6, 2012

1

(Exercise 16 from Chapter 1.) Consider the set M of points (x, y, z) in $\mathbb{R}^3$ which satisfy both equations

$$x^2 + y^2 + z^2 = 1 \text{ and } x^2 = yz^2.$$

Show that $P = (0, -1, 0)$ is isolated in M (that is, there is a neighbourhood in $\mathbb{R}^3$ in which P is the only point from M). Find another point Q in M, such that $M \setminus \{P, Q\}$ is a manifold in $\mathbb{R}^3$. Prove that $M \setminus \{P\}$ is not a manifold in $\mathbb{R}^3$.

Consider the open ball $B(P, 1) = \{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + (y + 1)^2 + z^2 < 1\}$ with radius 1; it is a neighbourhood of P. If $(x, y, z) \in B(P, 1) \cap M$, then $x^2 + (y + 1)^2 + z^2 < 1$, $x^2 + y^2 + z^2 = 1$ and $x^2 = yz^2$. Since $x^2 + y^2 + 2y + 1 + z^2 < 1$ by the inequality and $x^2 + y^2 + z^2$, we obtain $2y + 2 < 1$, or $y < \frac{1}{2}$. Since x^2 and z^2 are both non-negative, the only way the equality $x^2 = yz^2$ is satisfied is if $x = z = 0$ (if one were different from zero, we would obtain a contradiction immediately). Thus $y^2 = 1$, and since $y < \frac{1}{2}$, we must have $y = -1$. Since $P \in M$ (which is easily seen) and $P \in B(P, 1)$, we obtain $B(P, 1) \cap M = \{P\}$, so P is isolated in M.

Let $Q = (0, 1, 0)$. It is clear that $Q \in M$, and we claim that $M \setminus \{P, Q\}$ is a 1-dimensional manifold in $\mathbb{R}^3$. Define $f : \mathbb{R}^3 \to \mathbb{R}^2$ by $f(x, y, z) = (x^2 + y^2 + z^2, x^2 - yz^2)$; it is clear that f is smooth since all its coordinates are smooth functions over $\mathbb{R}^3$, and

$$Df(x, y, z) = \begin{pmatrix} 2x & 2y & 2z \\ 2x & -z^2 & -2yz \end{pmatrix}.$$

Let $c = (1, 0)$. Then $M = \{(x, y, z) \in \mathbb{R}^3 \,|\, f(x, y, z) = c\}$. We will show that P and Q are the only points of M with $\text{rank}Df(x, y, z)$ not equal to 2.

Assume that $(x, y, z) \in M$. Then finding the rank of $Df(x, y, z)$ is to find the rank of

$$A(x, y, z) = \begin{pmatrix} 0 & 2y + z^2 & 2z(y + 1) \\ 2x & -z^2 & -2yz \end{pmatrix}$$

by using row operations. If $2y + z^2 = 0$, then we have $x^2 = yz^2 = -2y^2$, implying $x = y = 0$ which in turn yields $z = 0$ (since $z^2 = -2y$); thus $x^2 + y^2 + z^2 = 0$, a contradiction. Therefore $2y + z^2 \neq 0$.

Assume now that $\text{rank}Df(x, y, z) < 2$. Then $x = 0$, since if it weren't, the two rows would be linearly independent since $2y + z^2$ is never 0 and the first coordinate of the first row always is. Assuming that $z \neq 0$, then $y = 0$ since $0 = x^2 = yz^2$, so

$$A(x, y, z) = \begin{pmatrix} 0 & z^2 & 2z \\ 0 & -z^2 & 0 \end{pmatrix},$$

which has rank 2, since $z^2 \neq 0$; thus $z = 0$. Since $x^2 + y^2 + z^2 = 1$, we obtain $x = (0, \pm 1, 0)$, i.e $x = P$ or $x = Q$. On the other hand, since $A(P)$ and $A(Q)$ contain a zero row, their rank is strictly less than 2. Thus the only points $(x, y, z) \in M$ with $\text{rank}Df(x, y, z) < 2$ are P and Q. Since the rank of $Df(x, y, z)$ is always less than or equal to 2 (it has two rows), all points $(x, y, z) \in M \setminus \{P, Q\}$ satisfy $\text{rank}Df(x, y, z) = 2$, and thus it follows by ((1), Theorem 1.6) that $M \setminus \{P, Q\}$ (a non-empty subset of $\mathbb{R}^3$ since it contains $(0, 0, 1)$) is a 1-dimensional manifold in $\mathbb{R}^3$.

By using ((1), Lemma 1.6), it is seen that $M \setminus \{P\}$ cannot be a manifold, since for any neighbourhood W in $\mathbb{R}^3$ around Q, it is not possible to make $W \cap M \setminus \{P\}$ into the graph of a function over an open subset of $\mathbb{R}^n$, since $W \cap M \setminus \{P\}$ will always contain two "branches" going through Q ($M \setminus \{P\}$ is a projection of the figure eight onto the 2-sphere).

2

(Exercise 12 from Chapter 2.) Let S^2 denote the unit sphere in $\mathbb{R}^3$, and define a map $f : S^2 \to S^2$ by

$$f(x, y, z) = (x\cos(z) + y\sin(z), x\sin(z) - y\cos(z), z).$$

Prove that f is a diffeomorphism.

Note first that for any $(x, y, z) \in S^2$, we have $f(x, y, z) \in S^2$; indeed,

$$\begin{aligned}
|f(x,y,z)|^2 &= (x\cos(z) + y\sin(z))^2 + (x\sin(z) - y\cos(z))^2 + z^2 \\
&= x^2\cos^2(z) + y^2\sin^2(x) + 2xy\cos(z)\sin(z) \\
&\qquad +x^2\sin^2(z) + y^2\cos^2(x) - 2xy\cos(z)\sin(z) + z^2 \\
&= x^2 + y^2 + z^2 \\
&= 1.
\end{aligned}$$

Additionally, for $(x, y, z) \in S^2$, note that

$$\begin{aligned}
f(f(x,y,z)) &= f(x\cos(z) + y\sin(z), x\sin(z) - y\cos(z), z) \\
&= \begin{pmatrix} (x\cos(z) + y\sin(z))\cos(z) + (x\sin(z) - y\cos(z))\sin(z) \\ (x\cos(z) + y\sin(z))\sin(z) - (x\sin(z) - y\cos(z))\cos(z) \\ z \end{pmatrix} \\
&= (x\cos^2(z) + x\sin^2(z), y\sin^2(z) + y\cos^2(z), z) \\
&= (x, y, z).
\end{aligned}$$

Thus f is a bijection $S^2 \to S^2$ with $f^{-1} = f$.

In order to check that f is smooth, we will use ((1), Definition 2.6.2) (which doesn't conflict with ((1), Definition 2.7.1) for general abstract manifolds). By ((1), Examples 1.6.5 and 2.2), $S^2 \subseteq \mathbb{R}^3$ is a manifold. Letting $p \in S^2 \subseteq \mathbb{R}^3$ and defining $F : \mathbb{R}^3 \to \mathbb{R}^3$ to be the extension of f to $\mathbb{R}^3$ (with the same coordinate functions), it's clear that F is smooth since its coordinate functions are smooth in all variables, and it clearly coincides with f on $S^2 \cap \mathbb{R}^3 = S^2$. Thus f is smooth at all points $p \in S^2$ by ((1), Definition 2.6.1), and by ((1), Definition 2.6.2), $f : S^2 \to S^2$ and $f^{-1} = f : S^2 \to S^2$ are smooth. Thus f is a diffeomorphism.

3

Consider the cylinder and sphere of radius 1

$$M = \{x \in \mathbb{R}^3 \mid x_1^2 + x_2^2 = 1\},$$
$$N = \{x \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\}.$$

a.

Prove that M and N have the same tangent space at each point $p \in M \cap N$.

$N = S^2$ is a 2-dimensional manifold by ((1), Example 1.6.5), and M is a 2-dimensional manifold in $\mathbb{R}^3$ by ((1), Theorem 1.6), since the only critical points of the smooth function $C : \mathbb{R}^3 \to \mathbb{R}$ given by $C(x, y, z) = x^2 + y^2$ are the points $(0, 0, z)$ for $z \in \mathbb{R}$, none of which is contained in the non-empty set M.

Let $p = (x, y, z) \in M \cap N$; then $(x, y, z) \neq (0, 0, 0)$ and since $z^2 = 1 - y^2 - x^2 = 0$, z must be 0. Since $DC(x, y, z) = (2x, 2y, 0)$, we have

$$T_pM = \{v \in \mathbb{R}^3 \mid DC(p)v = 0\} = \{v \in \mathbb{R}^3 \mid (2x, 2y, 0)v = 0\}$$

by ((1), Example 3.2.2). N is in the same manner as above given by the equation $S(x,y,z)=1$, where $S:\mathbb{R}^3\to\mathbb{R}$ is the smooth function given by $S(x,y,z)=x^2+y^2+z^2$ with $DS(x,y,z)=(2x,2y,2z)$, and so in the same way, we get

$$T_pN=\{v\in\mathbb{R}^3\mid DS(p)v=0\}=\{v\in\mathbb{R}^3\mid(2x,2y,2z)v=0\},$$

but since $z=0$, we obtain $T_pN=\{v\in\mathbb{R}^3\mid(2x,2y,0)v=0\}=T_pM$.

b.

Show that the map $M\to N$ given by $x\mapsto x/\|x\|$ is smooth and determine its differential at $p\in M\cap N$.

Define $f:M\to N$ by $f(x)=x/\|x\|$. It is clear that f is well-defined and maps into N. For showing that f is smooth, we can use ((1), Definition 2.6.2) again, since M and N are manifolds in $\mathbb{R}^3$ by our previous observations. Letting $a\in M\subseteq\mathbb{R}^3$ and defining $F:\mathbb{R}^3\setminus\{0\}\to\mathbb{R}^3$ by

$$F(x,y,z)=(x(x^2+y^2+z^2)^{-1/2},y(x^2+y^2+z^2)^{-1/2},z(x^2+y^2+z^2)^{-1/2}),$$

it's clear that F is smooth at the interior point a since its coordinate functions are smooth for $(x,y,z)\neq(0,0,0)$, and it clearly coincides with f on $M\cap\mathbb{R}^3\setminus\{0\}=M$. Thus f is smooth at all points $a\in M$ by ((1), Definition 2.6.1), and by ((1), Definition 2.6.2), f is smooth.

The differential is determined by using the smooth map F as defined above on the open subset $W=\mathbb{R}^3\setminus\{0\}$ of $\mathbb{R}^3$. By ((1), p. 42), the differential dF_p of F at $p=(x_0,y_0,z_0)\in W$ is the linear map $\mathbb{R}^3\to\mathbb{R}^3$ having $DF(p)$ as its matrix, i.e.

$$DF(p)=\begin{pmatrix}\frac{y_0^2+z_0^2}{(x_0^2+y_0^2+z_0^2)^{3/2}} & -\frac{x_0y_0}{(x_0^2+y_0^2+z_0^2)^{3/2}} & -\frac{x_0z_0}{(x_0^2+y_0^2+z_0^2)^{3/2}}\\ -\frac{x_0y_0}{(x_0^2+y_0^2+z_0^2)^{3/2}} & \frac{x_0^2+z_0^2}{(x_0^2+y_0^2+z_0^2)^{3/2}} & -\frac{y_0z_0}{(x_0^2+y_0^2+z_0^2)^{3/2}}\\ -\frac{x_0z_0}{(x_0^2+y_0^2+z_0^2)^{3/2}} & -\frac{y_0z_0}{(x_0^2+y_0^2+z_0^2)^{3/2}} & \frac{x_0^2+y_0^2}{(x_0^2+y_0^2+z_0^2)^{3/2}}\end{pmatrix}.$$

Let $p=(x_0,y_0,z_0)\in M\cap N$. By ((1), Lemma 3.8.2), since M and N are manifolds in $\mathbb{R}^3$ and f is smooth (since F is a (local) smooth extension of f at p), we obtain that df_p equals the restriction of dF_p to T_pM (by **a**). That is, $df_p:T_pM\to T_{f(p)}N=T_pN=T_pM$ is the linear map given by the matrix

$$dF_p=\begin{pmatrix}y_0^2 & -x_0y_0 & 0\\ -x_0y_0 & x_0^2 & 0\\ 0 & 0 & 1\end{pmatrix},$$

since $z_0=0$ and $x_0^2+y_0^2=x_0^2+y_0^2+z_0^2=1$.

4

(Exercise 9 from Chapter 4.) Show that S^3 can be given a structure as a Lie group.

Consider the group SU(2) and the embedding $g:\mathbb{C}^2\to\mathrm{M}(2,\mathbb{C})$ given by

$$\mathrm{SU}(2)=\left\{\begin{pmatrix}\alpha & \beta\\ -\overline{\beta} & \overline{\alpha}\end{pmatrix}:\ \alpha,\beta\in\mathbb{C},\ |\alpha|^2+|\beta|^2=1\right\},\quad g(\alpha,\beta)=\begin{pmatrix}\alpha & \beta\\ -\overline{\beta} & \overline{\alpha}\end{pmatrix}$$

by considering $\mathbb{C}^2$ as $\mathbb{R}^4$ and $\mathrm{M}(2,\mathbb{C})$ as $\mathbb{R}^8$. g is clearly smooth (its coordinate functions are just the coordinates of (α,β) with altering signs) and for $(\alpha,\beta)\in{}^3$, $|\alpha|^2+|\beta|^2=1$, so $g(S^3)\subseteq\mathrm{SU}(2)$. Thus the restriction $\tilde{g}:S^3\to\mathrm{SU}(2)$ to the manifold $S^3\subseteq\mathbb{C}^2$ is also smooth by ((1), Definition 2.6.2). $\tilde{g}$ has an inverse given by

$$\varphi\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}=(\alpha,\beta)$$

which is clearly also smooth (it is the projection onto the first four coordinates considering SU(2) as a subset of $\mathbb{R}^8$), so $\tilde{g}$ is a diffeomorphism.

We now endow S^3 with a group structure by considering the group structure of SU(2). Thus define

$$(\alpha,\beta)\cdot(\alpha',\beta'):=\varphi(\tilde{g}(\alpha,\beta)\tilde{g}(\alpha',\beta'))=\varphi\begin{pmatrix}\alpha\alpha'-\beta\overline{\beta'} & \alpha\beta'+\overline{\alpha'}\beta\\ -\alpha'\overline{\beta}-\overline{\alpha}\overline{\beta'} & -\overline{\beta}\beta'+\overline{\alpha}\overline{\alpha'}\end{pmatrix}=(\alpha\alpha'-\beta\overline{\beta'},\alpha\beta'+\overline{\alpha'}\beta),$$

which is clearly smooth, since each coordinate function is just a sum of products of the coordinates. It is associative, since

$$((\alpha,\beta)\cdot(\alpha',\beta'))\cdot(\alpha'',\beta'') = (\alpha\alpha'\alpha'' - \beta\overline{\beta'}\alpha'' - \alpha\beta'\overline{\beta''} - \overline{\alpha'}\beta\overline{\beta''}, \alpha\alpha'\beta'' - \beta\overline{\beta'}\beta'' + \alpha\beta'\overline{\alpha''} + \overline{\alpha'}\beta\overline{\alpha''})$$

and

$$(\alpha,\beta)\cdot((\alpha',\beta')\cdot(\alpha'',\beta'')) = (\alpha\alpha'\alpha'' - \alpha\beta'\overline{\beta''} - \beta\overline{\alpha'}\overline{\beta''} - \beta\alpha''\overline{\beta'}, \alpha\alpha'\beta'' + \alpha\overline{\alpha''}\beta' + \overline{\alpha'}\overline{\alpha''}\beta - \overline{\beta'}\beta''\beta).$$

The neutral element of S^3 is then $(1,0)$, and the inverse operation is given by $(\alpha,\beta)^{-1} = (\overline{\alpha}, -\beta)$, which is clearly smooth as well (once again, just the coordinates of (α,β) with different signs in some places). Thus S^3 becomes a Lie group.

5

(Exercise 14 from Chapter 4.) Let $F : M \to N$ be a diffeomorphism. If $D \subseteq M$ is a domain with smooth boundary, verify that so is $F(D)$, and $\partial(F(D)) = F(\partial D)$. Furthermore, if M and N are oriented, and F is orientation-preserving/reversing, then so is its restriction $\partial D \to \partial(F(D))$. Prove finally that the rotation of S^{n-1} by an element $A \in \mathrm{O}(n)$ is orientation-preserving if $\det A = 1$ and orientation-reversing if $\det A = -1$.

We will show first that $\partial(F(D)) = F(\partial D)$. Assume that $p \in F(\partial D)$ and let W be an open neighbourhood of p; we claim that W intersects both $F(D)$ and $F(D)^c$. Let $q = F^{-1}(p) \in \partial D$ and consider $V = F^{-1}(W)$ which is an open neighbourhood of q. Since $q \in \partial D$, there must exist $z \in V \cap D$ and $z' \in V \cap D^c$, so $F(z) \in F(V) \cap F(D) = W \cap F(D)$ and $F(z') \in F(V) \cap F(D^c) = W \cap F(D)^c$ since F is a bijection. Therefore W intersects $F(D)$ and $F(D)^c$, so $p \in \partial F(D)$, so $\partial(F(D)) \supseteq F(\partial D)$. From this, we obtain $\partial F(D) = F(F^{-1}\partial F(D)) \subseteq F(\partial F^{-1}(F(D))) = F(\partial D)$, and we are done.

For any injection $f : A \to B$, then $f(U \cap V) = f(U) \cap f(V)$ for all $U, V \subseteq A$: $\subseteq$ is obvious, and if $x \in f(U) \cap f(V)$, then $x = f(u) = f(v)$ for some $u \in U$ and $v \in V$, but then $u = v$, so $x = f(u) \in f(U \cap V)$. We will use this in the following.

We now assume that $D \subseteq M$ is a domain with smooth boundary and will prove that $F(D)$ is as well. First of all, $F(D)$ is open, since F is a homeomorphism, and non-empty since D is. Let $p \in \partial F(D) = F(\partial D)$ and let $q = F^{-1}(p) \in \partial D$. Since D is a domain with smooth boundary, there exists an open neighbourhood $V \backslash M$ of q and a smooth function $g : V \to \mathbb{R}$ for which 0 is a regular value (that is, all $r \in g^{-1}(0)$ have surjective differentials dg_r), $g(q) = 0$ and $V \cap D = \{x \in V \mid g(x) > 0\}$.

Now, let $W = F(V)$ which is an open neighbourhood of p, and let $f = g \circ F^{-1} : W \to \mathbb{R}$ where we restrict F^{-1} to W. For all $s \in f^{-1}(0)$, we have $df_s = d(g \circ F^{-1})_s = dg_{F^{-1}(s)} \circ dF_s^{-1}$. dF_s^{-1} is bijective since F^{-1} is a diffeomorphism, and $dg_{F^{-1}(s)}$ is surjective, since $F^{-1}(s) \in g^{-1}(0)$ by assumption. Thus df_s is a composite of two surjective maps and thus surjective itself. Moreover, $f(p) = g(q) = 0$ and

$$\begin{aligned} W \cap F(D) &= F(V) \cap F(D) \\ &= F(V \cap D) \\ &= \{y \in N \mid F^{-1}(y) \in V \cap D\} \\ &= \{y \in W \mid F^{-1}(y) \in V \cap D\} \\ &= \{y \in W \mid g \circ F^{-1}(y) > 0\} \\ &= \{y \in W \mid f(y) > 0\}, \end{aligned}$$

so we conclude that $F(D)$ is a domain with smooth boundary.

Let $p \in \partial D$ and assume that M and N are oriented. Let $v_1, \ldots, v_{m-1}$ be a positive basis for $T_p\partial D$; then there exists $v_0 \in T_pM$ such that $v_0, v_1, \ldots, v_{m-1}$ is a positive basis for T_pM. v_0 is chosen by considering a neighbourhood W of p and a smooth function $g : W \to \mathbb{R}$ with 0 being a regular value, so that there exists $v_0 \in T_pM$ satisfying $dg_p(v_0) < 0$, as per ((1), p. 60). Assume that the diffeomorphism F is orientation-preserving; then $dF_pv_0, dF_pv_1, \ldots, dF_pv_{m-1}$ is a positive basis for $T_{F(p)}N$ by ((1), Definition 3.10). Now, letting $f = g \circ F^{-1}$ where F^{-1} is restricted to $F(W)$, we observe that $df_{F(p)}(dF_p(v_0)) = dg_{f(F(p))}dF^{-1}_{f(p)}dF_p(v_0) = dg_p(v_0) < 0$, so the remaining elements $dF_pv_1, \ldots, dF_pv_{m-1}$ are a basis of the tangent space $T_{F(p)}F(\partial D) = T_{F(p)}\partial F(D)$, since the dimension of $T_{F(p)}\partial F(D)$ is one lower than $T_{F(p)}N$ and $dF_p(v_0)$ points out from $F(D)$. For orientation-reversingness, just replace the word "positive" with "negative".

For $A \in \mathrm{O}(n)$, let $F : \mathbb{R}^n \to \mathbb{R}^n$ be the linear diffeomorphism given by the matrix A; the differential of F is just F itself. Assuming that we have chosen an orientation of $\mathbb{R}^n$ such that $e_1, \dots, e_n$ is a positive basis of $\mathbb{R}^n$, we want to determine whether the basis $Ae_1, \dots, Ae_n$ is positive or negative depending on A. The transition matrix from $e_1, \dots, e_n$ to $Ae_1, \dots, Ae_n$ is A^{-1} which has positive determinant if $\det A = 1$ and negative if $\det A = -1$. Alas F is orientation-preserving if $\det A = 1$ and orientation-reversing if $\det A = -1$.

6

Let X be a compact topological space, and let $A \subseteq X$ be a subset with the property that each element $x \in X$ has a neighbourhood which contains at most one point from A. Prove that A is finite.

For each $x \in X$, let N'_x be a neighbourhood that contains at most one point from A. For all $a \in A$, let N_a be the union of all N'_x, $x \in X$, such that $a \in N'_x$, and let N_0 be the union of all N'_x, $x \in X$, satisfying $N'_x \cap A = \emptyset$; this implies $N_0 \cap A = \emptyset$. Letting $I = A \cup \{0\}$, $\{N_i\}_{i \in I}$ is an open covering of X: N_i is open for all $i \in I$ since it is a union of open sets, and for each $x \in X$, then either N'_x contains no point from A, in which case $x \in N'_x \subseteq N_0 \subseteq \bigcup_{i \in I} N_i$, or N'_x contains a point a from A in which case $x \in N'_x \subseteq N_a \subseteq \bigcup_{i \in I} N_i$.

We claim now that $\{N_i\}_{i \in I}$ has no finite subcovering if A is infinite. Let $\{N_j\}_{j \in J}$, $J \subseteq I$, be a finite subcollection of $\{N_i\}_{i \in I}$; we will find a point of X that is not contained in $\bigcup_{j \in J} N_j$ so that $\{N_j\}_{j \in J}$ does not cover X: thus no finite subcollection of $\{N_i\}_{i \in I}$ covers X if A is infinite, contradicting the compactness of X, meaning that A is finite.

Assuming that A is infinite, J cannot contain all of A, so there must exist some $a \in A$ such that $a \notin J$. Since $J' := J \setminus \{0\} \subseteq A$ and J is finite, we must have $J' = \{a_1, \dots, a_n\}$ for some $a_i \in A$, $i \in \{1, \dots, n\}$. Assuming that $a \in N_{a_i}$ for some $i \in \{1, \dots, n\}$, then there must exist some $x \in X$ satisfying $a_i \in N'_x$ *and* $a \in N'_x$ (by definition of N_{a_i}), but this is impossible since N'_x contains at most one point from A, and a and a_i are different points because $a \notin J$. Alas $a \notin N_{a_i}$ for all $i \in \{1, \dots, n\}$ meaning that $a \notin \bigcup_{j \in J'} N_j$. If $0 \notin J$, then $J' = J$, so $a \notin \bigcup_{j \in J} N_j$; if $0 \in J$, then since $N_0 \cap A = \emptyset$, $a \notin N_0$, so a is not contained in $\bigcup_{j \in J'} N_j \cup N_0 = \bigcup_{j \in J} N_j$. Thus we have reached the desired contradiction, so A must be finite.

7

Let M be an abstract manifold and assume that there exists a locally finite atlas for M. Let $M = \bigcup_{\beta \in B} E_\beta$ be an arbitrary open covering. Show that M has a partition of unity $(f_i)_{i \in I}$, such that for each i the support $\mathrm{supp} f_i$ *is compact and contained in E_β for some $\beta \in B$.*

This is basically the same statement as ((1), Theorem 5.5), aside from the theorem not stating that the supports are compact. We need only note that in the proof of ((1), Theorem 5.5), the $(g_i)_{i \in I}$ used to construct the partition of unity already have compact support; then the members of the partition of unity $f_i = g_i/g$ also have compact support since $f_i(x) = 0$ if and only if $g_i(x) = 0$. Indeed, for $i \in I$, g_i is a smooth function on M taking values in $[0,1]$ such that $g_i(q) = 1$ for q inside the image of a ball $A \subseteq \mathbb{R}^n$ under a chart σ, and $g_i(q) = 0$ outside the image of a ball $B \subseteq \mathbb{R}^n$ under σ, existence of which is proved in ((1), Lemma 5.4). Then the set of $q \in M$ such that $g_i(q) \neq 0$ is contained in $\sigma(B)$. Taking closures of these sets, $\mathrm{supp} g_i \subseteq \overline{\sigma(B)} = \sigma(\overline{B})$ because charts are homeomorphisms onto their images. But now, $\overline{B}$ is compact and compactness is preserved by continuous functions ((1), Lemma 5.1.1), so $\sigma(\overline{B})$ is compact. Since $\mathrm{supp} g_i$ is closed in $\sigma(\overline{B})$, it is compact as well ((1), Lemma 5.1.2), and we are done.

References

[1] H. Schlichtkrull, *Differentiable Manifolds: Lecture Notes for Geometry 2*, Department of Mathematical Sciences, University of Copenhagen, 2011.