HomAlg, Blok 2, 2011. Assignment 1

Due Tuesday December 20

(1) Let $A^\vee = \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$ be the Pontryagin dual of an abelian group A.
  (a) Prove that $A = 0$ if and only if $A^\vee = 0$.
  (b) Prove that a homomorphism $\alpha\colon A' \to A''$ is zero if and only if the homomorphism $\alpha^*\colon (A'')^\vee \to (A')^\vee$ is zero.
  (c) Prove that a sequence of abelian groups

$$0 \to A' \to A \to A'' \to 0$$

is exact if and only if the sequence

$$0 \to (A'')^\vee \to A^\vee \to (A')^\vee \to 0$$

is exact.

(2) Recall that projective modules are flat. The converse is not true in general, but in this exercise we will prove that *finitely presented* flat modules are projective. A Λ-module A is finitely presented if there is an exact sequence

$$\Lambda^n \to \Lambda^m \to A \to 0.$$

  (a) Write down a natural transformation

$$\eta_{B,A}\colon B^\vee \otimes_\Lambda A \to \mathrm{Hom}_\Lambda(A, B)^\vee$$

between functors $(\mathfrak{M}^\ell_\Lambda)^{op} \times \mathfrak{M}^\ell_\Lambda \to \mathfrak{M}^\ell_{\mathbb{Z}}$ with the property that $\eta_{B,\Lambda}$ is an isomorphism for every B.
  (b) Prove that $\eta_{B,A}$ is an isomorphism for all B and all finitely presented modules A.
  (c) Suppose that A is finitely presented and flat. Prove that A is projective.

(3) (a) Consider a commutative diagram of Λ-modules with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A' & \xrightarrow{\mu} & A & \xrightarrow{\epsilon} & A'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha'} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha''} & & \\ 0 & \longrightarrow & B' & \xrightarrow{\mu'} & B & \xrightarrow{\epsilon'} & B'' & \longrightarrow & 0. \end{array}$$

Show that α' is an isomorphism if and only if the sequence

$$0 \longrightarrow A \xrightarrow{\binom{\alpha}{\epsilon}} B \oplus A'' \xrightarrow{(\epsilon',\, -\alpha'')} B'' \longrightarrow 0$$

is exact. Here, the first homomorphism sends a to $(\alpha(a), \epsilon(a))$ and the second homomorphism sends (b, a'') to $\epsilon'(b) - \alpha''(a'')$.
  (b) Let A be a Λ-module. Given two short exact sequences

$$0 \longrightarrow A \longrightarrow I \longrightarrow C \longrightarrow 0$$

$$0 \longrightarrow A \longrightarrow I' \longrightarrow C' \longrightarrow 0$$

with I and I' injective, show that $I \oplus C' \cong I' \oplus C$. Show that C is injective if and only if C' is injective.

(4) (a) Given a commutative diagram in any category

$$\begin{array}{ccccc} D & \longrightarrow & E & \longrightarrow & F \\ \downarrow & & \downarrow & & \downarrow \\ A & \longrightarrow & B & \longrightarrow & C \end{array}$$

show that if both squares are pullbacks, then so is the outer rectangle. Show also that if the outer rectangle and the right square are pullbacks, then so is the left square.

(b) Consider a pullback diagram in the category of sets

$$\begin{array}{ccc} E & \longrightarrow & F \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ B & \xrightarrow{f} & C. \end{array}$$

Show that there is a bijection $p^{-1}(b) \cong q^{-1}(f(b))$ for every $b \in B$.