HomAlg, Blok 2, 2011-2012. Assignment 2

Due Tuesday January 10

(1) Let k be a field, let $\Lambda = k[x, y, z]$ and let $\Gamma = \Lambda/(xz^2, yz^2, xyz)$.
   (a) Construct a free resolution of Γ over Λ and use this to calculate $\mathrm{Tor}_i^\Lambda(\Gamma, k)$ for all i.
   (b) Calculate the homology of the Koszul complex $H_i(K_\bullet^\Gamma(x, y, z))$ for all i.
   (c) Compare your answers in (a) and (b) and explain the observations you make.

(2) Let $\varphi\colon C_\bullet \to D_\bullet$ be a morphism of chain complexes and let $E(\varphi)$ denote the mapping cone of φ (see Exercise IV.1.2 in [HS]).
   (a) Establish a short exact sequence of chain complexes
$$0 \to D_\bullet \to E(\varphi) \to \Sigma C_\bullet \to 0$$
   and identify the connecting homomorphism associated to this sequence.
   (b) Prove that φ is a quasi-isomorphism if and only if $E(\varphi)$ is exact.
   (c) Prove that φ is a homotopy equivalence if and only if $E(\varphi)$ is split exact.
   (d) Prove that a morphism between bounded below chain complexes of projective modules is a quasi-isomorphism if and only if it is a homotopy equivalence.

(3) Consider a commutative diagram of chain complexes with exact rows
$$\begin{array}{ccccccccc}
0 & \longrightarrow & C'_\bullet & \xrightarrow{\mu} & C_\bullet & \xrightarrow{\epsilon} & C''_\bullet & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\
0 & \longrightarrow & D'_\bullet & \xrightarrow{\mu'} & D_\bullet & \xrightarrow{\epsilon'} & D''_\bullet & \longrightarrow & 0
\end{array}$$
   (a) Show that if two out of α, β, γ are quasi-isomorphisms, then so is the third.
   (b) Is the same true if you replace 'quasi-isomorphism' by 'homotopy equivalence'? If yes, provide a proof, if no provide a counterexample.

(4) (a) Let $F\colon \mathcal{C} \leftrightarrows \mathcal{D} : G$ be an adjunction (F left adjoint to G). Prove that if F preserves monomorphisms then G preserves injective objects.
   (b) Prove that if A is a flat right Λ-module and D is a divisible abelian group then $\mathrm{Hom}_{\mathbb{Z}}(A, D)$ is an injective left Λ-module.