Homological Algebra

Assignment 1

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(1)

Let $A^\vee = \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$ be the Pontryagin dual of an abelian group A.

Before dealing with the problems, we shall for a moment deal with constructing homomorphisms from any abelian group (or $\mathbb{Z}$-module) into the injective $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$ – it is dealt with in ((1), Proposition I.7.4) – since it will prove to be very helpful in the following.

Lemma 1. *Any abelian group $A \neq 0$ allows a non-trivial ($\mathbb{Z}$-module) homomorphism $A \to \mathbb{Q}/\mathbb{Z}$.*

Proof. Let $0 \neq a \in A$ and let (a) denote the subgroup of A generated by a. Define $\alpha : (a) \to \mathbb{Q}/\mathbb{Z}$ as follows: if the order of a is infinite, let $\alpha(a)$ be arbitrary and different from 0; if the order of a is finite, say equal to $n \geq 2$, choose a non-zero element $\beta \in \mathbb{Q}/\mathbb{Z}$ with order dividing n and let $\alpha(a) = \beta$. Since $\mathbb{Z}$ is a PID and $\mathbb{Q}/\mathbb{Z}$ is a divisible and thus injective $\mathbb{Z}$-module ((1), example (b) p. 31 and Theorem I.7.1), there exists a map $\omega : A \to \mathbb{Q}/\mathbb{Z}$ such that $\omega\iota = \alpha$, where ι denotes the inclusion homomorphism $(a) \to A$. ω is non-trivial, since $\omega(a) = \alpha(a) \neq 0$. □

(a)

Prove that $A = 0$ if and only if $A^\vee = 0$.

This follows almost directly from Lemma 1. If $A = 0$, then there is only the trivial homomorphism $A \to \mathbb{Q}/\mathbb{Z}$, so $A^\vee = 0$. If $A \neq 0$, then it follows from Lemma 1 that $A^\vee \neq 0$, and we are done.

(b)

Prove that a homomorphism $\alpha : A' \to A''$ is zero if and only if the homomorphism $\alpha^* : (A'')^\vee \to (A')^\vee$ is zero.

Assuming that α is zero, then $\alpha^*(\varphi)(a') = \varphi(\alpha(a')) = \varphi(0) = 0$ for all $\varphi \in (A'')^\vee$ and $a' \in A'$, so $\alpha^*(\varphi)$ is the trivial homomorphism for all $\varphi \in (A'')^\vee$, meaning that α^* is zero. If α is non-zero, then there exists $a' \in A'$ such that $\alpha(a) \neq 0$, implying that A'' is non-trivial. As in the proof of Lemma 1, define a homomorphism $\beta : (\alpha(a')) \to \mathbb{Q}/\mathbb{Z}$, and by injectivity of the $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$, extend β to a homomorphism $\omega : A'' \to \mathbb{Q}$. Then $\alpha^*(\omega)(a') = \omega(\alpha(a')) \neq 0$, so $\alpha^*(\omega) \neq 0$, meaning that α^* is non-zero.

(c)

Prove that a sequence of abelian groups

$$0 \longrightarrow A' \xrightarrow{\mu} A \xrightarrow{\varepsilon} A'' \longrightarrow 0$$

is exact if and only if the sequence

$$0 \longrightarrow (A'')^\vee \xrightarrow{\varepsilon^*} A^\vee \xrightarrow{\mu^*} (A')^\vee \longrightarrow 0$$

is exact.

The forward implication follows from ((1), Theorem I.8.4) (we will take a look at this later in 2(b)), or the dualization of ((1), Theorem I.4.7), since $\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module. Assume now that the second sequence is exact; we will prove in steps that the first is as well.

(1) μ is injective. If $A' = 0$, then μ is clearly injective. If $A' \neq 0$, then let $0 \neq a' \in A'$. As in the proof of Lemma 1 we can find a homomorphism $\omega \in (A')^\vee$ such that $\omega(a') \neq 0$. Since μ^* is surjective, there exists $\varphi \in A^\vee$ such that $\varphi\mu = \mu^*(\varphi) = \omega$, so that $\varphi(\mu(a')) \neq 0$, implying $\mu(a') \neq 0$. Alas μ is injective.

(2) $\mathrm{im}\mu \subseteq \ker\varepsilon$. Since $(\varepsilon\mu)^* = \mu^*\varepsilon^* : (A'')^\vee \to (A')^\vee$ is zero by exactness of the second sequence, it follows from (b) that $\varepsilon\mu : A' \to A''$ is zero as well, and we are done.

(3) $\ker\varepsilon \subseteq \mathrm{im}\mu$. If $\mathrm{im}\mu = A$, the inclusion is trivial. Assume that there is $a \in A$ such that $a \notin \mathrm{im}\mu$. Defining $r : A \to A/\mathrm{im}\mu$ to be the canonical surjective mapping, we obtain that $r(a) \neq 0$. Thus $A/\mathrm{im}\mu \neq 0$ and there is a homomorphism $\varphi \in (A/\mathrm{im}\mu)^\vee$ such that $\varphi(r(a)) \neq 0$ (obtained in the way of Lemma 1). Since $r\mu = 0$, we obtain that $\mu^*(\varphi r) = \varphi r\mu = 0$, so $\varphi r \in \ker\mu^* = \mathrm{im}\varepsilon^*$. Therefore, there exists $\psi \in (A'')^\vee$ such that $\psi\varepsilon = \varepsilon^*(\psi) = \varphi r$. Assuming that $a \in \ker\varepsilon$ leads to a contradiction, since then $\psi(\varepsilon(a)) = 0$ since $a \in \ker\varepsilon$ and $\varphi(r(a)) \neq 0$ by construction. Thus we obtain $\ker\varepsilon \subseteq \mathrm{im}\mu$.

(4) ε is surjective. We will show that $A''/\mathrm{im}\varepsilon = 0$ which can be done by showing that $(A''/\mathrm{im}\varepsilon)^\vee = 0$, using 1(a). Let $\varphi \in (A''/\mathrm{im}\varepsilon)^\vee$. Defining $r : A'' \to A''/\mathrm{im}\varepsilon$ to be the canonical surjective mapping, then since $r\varepsilon = 0$ we obtain $\varepsilon^*(\varphi r) = \varphi r\varepsilon = 0$. Since ε^* is injective, we obtain $\varphi r = 0$. Letting $b \in A''/\mathrm{im}\varepsilon$, then since r is surjective, there exists $a \in A''$ such that $r(a) = b$, but then we obtain $\varphi(b) = \varphi(r(a)) = 0$. Thus $\varphi = 0$, and we are done.

(2)

Recall that projective modules are flat. The converse is not true in general, but in this exercise we will prove that finitely presented flat modules are projective. A Λ-module A is finitely presented if there is an exact sequence

$$\Lambda^n \to \Lambda^m \to A \to 0.$$

(a)

Write down a natural transformation

$$\eta_{B,A} : B^\vee \otimes_\Lambda A \to \mathrm{Hom}_\Lambda(A, B)^\vee$$

between functors $(\mathfrak{M}^\ell_\Lambda)^{\mathrm{op}} \times \mathfrak{M}^\ell_\Lambda \to \mathfrak{M}^\ell_\mathbb{Z}$ with the property that $\eta_{B,\Lambda}$ is an isomorphism for every B.

Denote $(\mathfrak{M}^\ell_\Lambda)^{\mathrm{op}} \times \mathfrak{M}^\ell_\Lambda$ by $\mathfrak{C}$. Note that for all Λ-modules A and B,

$$\mathfrak{C}((B, A), (B', A')) = (\mathfrak{M}^\ell_\Lambda)^{\mathrm{op}}(B, B') \times \mathfrak{M}^\ell_\Lambda(A, A') = \mathfrak{M}^\ell_\Lambda(B', B) \times \mathfrak{M}^\ell_\Lambda(A, A'),$$

so any morphism $f : (B, A) \to (B', A')$ of the category $\mathfrak{C}$ is of the form $f = (f_1, f_2)$ where $f_1 : B' \to B$ and $f_2 : A \to A'$. $B^\vee$ is made into a right Λ-module by defining $(\varphi\lambda)(b) = \varphi(\lambda b)$ for all $\varphi \in B^\vee$, $\lambda \in \Lambda$ and $b \in B$. We will now define two functors $F, G : \mathfrak{C} \to \mathfrak{M}^\ell_\mathbb{Z}$ so that we will be able to define $\eta_{B,A}$ as wanted. This is done as follows:

In the case of F, let A, A', B and B' be Λ-modules and $f_1 : B' \to B$, $f_2 : A \to A'$ be Λ-module homomorphisms. Define the map $\psi_{f_1,f_2} : B^\vee \times A \to (B')^\vee \otimes_\Lambda A'$ given by $\psi(\varphi, a) = (\varphi \circ f_1) \otimes f_2(a)$. By the identifications used in constructing the tensor product, ψ is clearly a homomorphism in both variables and additionally, since

$$(\varphi\lambda)(f_1(b')) = \varphi(\lambda f_1(b')) = \varphi(f_1(\lambda b')) = ((\varphi \circ f_1)\lambda)(b')$$

for $\varphi \in B'$, $\lambda \in \Lambda$ and $b' \in B'$, we have

$$\psi(\varphi\lambda, a) = ((\varphi\lambda) \circ f_1) \otimes f_2(a) = (\varphi \circ f_1)\lambda \otimes f_2(a) = (\varphi \circ f_1) \otimes \lambda f_2(a) = (\varphi \circ f_1) \otimes f_2(\lambda a) = \psi(\varphi, \lambda a)$$

for all $\varphi \in B^\vee$, $\lambda \in \Lambda$ and $a \in A$. By definition of the tensor product, there is a unique homomorphism $\tilde{\psi} : B^\vee \otimes_\Lambda A \to (B')^\vee \otimes_\Lambda A'$ with $\tilde{\psi}_{f_1,f_2}(\varphi \otimes a) = (\varphi \circ f_1) \otimes f_2(a)$ (it is only necessary to define homomorphisms for elementary tensors).

We can now define F easily. Let $F(B, A) = B^\vee \otimes_\Lambda A$ and for $f = (f_1, f_2) \in \mathfrak{C}((B, A), (B', A'))$, define $F(f) = \tilde{\psi}_{f_1,f_2}$; since $\tilde{\psi}_{f_1,f_2}$ is a homomorphism, $F(f)$ becomes a $\mathbb{Z}$-module homomorphism. It's clear that $F(1_{(B,A)}) = 1_{B^\vee \otimes_\Lambda A}$, and furtherly, for f as above and $g = (g_1, g_2) \in \mathfrak{C}((B', A'), (B'', A''))$, we have

$$F(g \circ f)(\varphi \otimes a) = (\varphi \circ f_1 \circ g_1) \otimes g_2(f_2(a)) = F(g)((\varphi \circ f_1) \otimes f_2(a)) = (F(g) \circ F(f))(\varphi \otimes a).$$

F is thus a co-variant functor $\mathfrak{C} \to \mathfrak{M}^\ell_\mathbb{Z}$.

In the case of G, define $G(B, A) = \mathrm{Hom}_\Lambda(A, B)^\vee$ and for $f = (f_1, f_2) \in \mathfrak{C}((B, A), (B', A'))$, define $G(f)(\varphi)(\psi) = (\varphi \circ (f_1)_* \circ (f_2)^*)(\psi) = \varphi(f_1 \circ \psi \circ f_2)$ for $\psi \in \mathrm{Hom}_\Lambda(A', B')$, inspired by the diagram

$$\mathrm{Hom}_\Lambda(A', B') \xrightarrow{(f_2)^*} \mathrm{Hom}_\Lambda(A, B') \xrightarrow{(f_1)_*} \mathrm{Hom}_\Lambda(A, B) \xrightarrow{\varphi} \mathbb{Q}/\mathbb{Z}.$$

$G(f)$ is clearly a $\mathbb{Z}$-module homomorphism for all f. Obviously, $G(1_{(B,A)})(\varphi) = \varphi$ and for f as above and $g = (g_1, g_2) \in \mathfrak{C}((B', A'), (B'', A''))$, we have

$$\begin{aligned} G(g \circ f)(\varphi) &= \varphi \circ (f_1 \circ g_1)_* \circ (g_2 \circ f_2)^* \\ &= \varphi \circ (f_1)_* \circ (g_1)_* \circ (f_2)^* \circ (g_2)^* \\ &= \varphi \circ (f_1)_* \circ (f_2)^* \circ (g_1)_* \circ (g_2)^* \\ &= (G(g) \circ G(f))(\varphi), \end{aligned}$$

since for any Λ-module homomorphism $\omega : A' \to B''$, we have

$$(g_1)_*((f_2)^*(\omega)) = g_1 \circ \omega \circ f_2 = (f_2)^*((g_1)^*(\omega)).$$

Thus G is also a co-variant functor $\mathfrak{C} \to \mathfrak{M}^\ell_\mathbb{Z}$. We will be using F and G throughout Problem 2.

For Λ-modules A and B, define $\zeta_{B,A} : B^\vee \times A \to \mathrm{Hom}_\Lambda(A, B)^\vee$ by $\zeta_{B,A}(\varphi, a)(\psi) = \varphi(\psi(a))$ for all $\varphi \in B^\vee$, $a \in A$ and $\psi \in \mathrm{Hom}_\Lambda(A, B)$.

Note that indeed $\zeta_{B,A}(\varphi, a) \in \mathrm{Hom}_\Lambda(A, B)^\vee$ for all $\varphi \in B^\vee$ and $a \in A$, since for any $\psi_1, \psi_2 \in \mathrm{Hom}_\Lambda(A, B)$, we clearly have $\zeta_{B,A}(\varphi, a)(\psi_1 + \psi_2) = \zeta_{B,A}(\varphi, a)(\psi_1) + \zeta_{B,A}(\varphi, a)(\psi_2)$ because φ is a homomorphism. Additionally, $\zeta_{B,A}$ is bilinear, as it is clearly a homomorphism in both variables and

$$\zeta_{B,A}(\varphi\lambda, a)(\psi) = (\varphi\lambda)(\psi(a)) = \varphi(\lambda\psi(a)) = \varphi(\psi(\lambda a)) = \zeta_{B,A}(\varphi, \lambda a)(\psi)$$

for all $\lambda \in \Lambda$, $\varphi \in B^\vee$, $a \in A$ and $\psi \in \mathrm{Hom}_\Lambda(A, B)$. By definition of the tensor product, there exists a unique homomorphism $\eta_{B,A} : B^\vee \otimes_\Lambda A \to \mathrm{Hom}_\Lambda(A, B)^\vee$ with

$$\eta_{B,A}(\varphi \otimes a)(\psi) = \zeta_{B,A}(\varphi, a)(\psi) = \varphi(\psi(a)) = (e_a \circ \varphi_*)(\psi),$$

where e_a denotes taking the image of a. Note that for Λ-modules A, A', B and B', $f = (f_1, f_2) \in \mathfrak{C}((B, A), (B', A'))$, $\varphi \in B^\vee$, $a \in A$ and $\psi \in \mathrm{Hom}_\Lambda(A', B')$, we have

$$\begin{aligned} G(f)(\eta_{B,A}(\varphi \otimes a))(\psi) &= G(f)(e_a \circ \varphi_*)(\psi) \\ &= (e_a \circ \varphi_* \circ (f_1)_* \circ (f_2)^*)(\psi) \\ &= \varphi(f_1(\psi(f_2(a)))) \\ &= \eta_{B',A'}((\varphi \circ f_1) \otimes f_2(a))(\psi) \\ &= \eta_{B',A'}(F(f)(\varphi \otimes a))(\psi). \end{aligned}$$

Thus $G(f) \circ \eta_{B,A} = \eta_{B',A'} \circ F(f)$. Thus we have a natural transformation η by associating $\eta_{B,A}$ to any $(B, A) \in \mathrm{Obj}(\mathfrak{C})$.

Let $A = \Lambda$ (Λ being considered as a Λ-module). We claim that $\eta_{B,\Lambda} : B^\vee \otimes_\Lambda \Lambda \to \mathrm{Hom}_\Lambda(\Lambda, B)^\vee$ still defined by $\eta_{B,\Lambda}(\varphi \otimes \lambda)(\psi) = \varphi(\psi(\lambda))$ is an isomorphism. The homomorphism $\alpha : B^\vee \otimes_\Lambda \Lambda \to B^\vee$ given by $\alpha(\psi \otimes \lambda) = \psi\lambda$ is an isomorphism ((1), p. 109). Additionally, define $\beta : \mathrm{Hom}_\Lambda(\Lambda, B) \to B$

by $\beta(\psi) = \psi(1)$. β is clearly a homomorphism; for any $b \in B$, then by defining $\psi(\lambda) = \lambda b$, we see that β is surjective, and if $\psi(1) = 0$ for some $\psi : \Lambda \to B$, then $\psi(\lambda) = \lambda\psi(1) = 0$, so $\psi = 0$. Thus β is an isomorphism, making the sequence

$$0 \longrightarrow 0 \longrightarrow \mathrm{Hom}_\Lambda(\Lambda, B) \xrightarrow{\beta} B \longrightarrow 0.$$

exact. By 1(c) and 1(a), the sequence

$$0 \longrightarrow B^\vee \xrightarrow{\beta^*} \mathrm{Hom}_\Lambda(\Lambda, B)^\vee \longrightarrow 0^\vee (= 0) \longrightarrow 0.$$

is exact as well, so $\beta^* : B^\vee \to \mathrm{Hom}_\Lambda(\Lambda, B)^\vee$ given by $\beta^*(\varphi)(\psi) = \varphi(\beta(\psi)) = \varphi(\psi(1))$ is an isomorphism. Therefore the composition $\beta^* \circ \alpha : B^\vee \otimes_\Lambda \Lambda \to \mathrm{Hom}_\Lambda(\Lambda, B)^\vee$ is an isomorphism as well, but lo and behold! For $\varphi \in B^\vee$, $\lambda \in \Lambda$ and $\psi \in \mathrm{Hom}_\Lambda(\Lambda, B)$ we have

$$\beta^*(\alpha(\varphi \otimes \lambda))(\psi) = \beta^*(\varphi\lambda)(\psi) = (\varphi\lambda)(\psi(1)) = \varphi(\lambda\psi(1)) = \varphi(\psi(\lambda)) = \eta_{B,\Lambda}(\varphi \otimes \lambda)(\psi),$$

so $\eta_{B,\Lambda} = \beta^* \circ \alpha$ is an isomorphism.

(b)

Prove that $\eta_{B,A}$ is an isomorphism for all B and all finitely presented modules A.

Assume that $\eta_{B,A}$ is an isomorphism for Λ-modules A and B. We will prove that η_{B,A^n} ($A^n = \bigoplus_1^n A$) is an isomorphism, so that η_{B,Λ^n} is an isomorphism for all Λ-modules B (because of the property of $\eta_{B,\Lambda}$ found in 2(a)). Note first that we have four isomorphisms defined as follows:

1. There is an isomorphism $\tau : B^\vee \otimes_\Lambda A^n \to (B^\vee \otimes_\Lambda A)^n$ given by $\tau(\varphi \otimes (a_1, \ldots, a_n)) = (\varphi \otimes a_1, \ldots, \varphi \otimes a_n)$ (it is properly defined by considering the mapping $B^\vee \times A^n \to (B^\vee \otimes_\Lambda A)^n$ defined by $(\varphi, (a_1, \ldots, a_n)) \mapsto (\varphi \otimes a_1, \ldots, \varphi \otimes a_n)$; it is clearly bilinear and can therefore be made into the above homomorphism by the definition of the tensor product). The maps $\tau_i' : B^\vee \times A \to B^\vee \otimes_\Lambda A^n$ given by $\tau_i'(\varphi, a) = \varphi \otimes (0, \ldots, 0, a, 0, \ldots, 0)$ (a in the i'th coordinate), $i = 1, \ldots, n$, are clearly bilinear and extend to homomorphisms over $B^\vee \otimes_\Lambda A$. Thus we obtain a homomorphism $\tau' : (B^\vee \otimes_\Lambda A)^n \to B^\vee \otimes_\Lambda A^n$ given by $\tau'(\varphi_1 \otimes a_1, \ldots, \varphi_n \otimes a_n) = \sum_{i=1}^n \varphi_i \otimes (0, \ldots, 0, a_i, 0, \ldots, 0)$ by the property of the direct sum. We claim that τ' is the inverse of τ. Indeed, $\tau'(\tau(\varphi \otimes (a_1, \ldots, a_n))) = \sum_{i=1}^n \varphi \otimes (0, \ldots, 0, a_i, 0, \ldots, 0) = \varphi \otimes (a_1, \ldots, a_n)$ and
$$\begin{aligned} \tau(\tau'(\varphi_1 \otimes a_1, \ldots, \varphi_n \otimes a_n)) &= \sum_{i=1}^n \tau(\varphi_i \otimes (0, \ldots, 0, a_i, 0, \ldots, 0)) \\ &= (\varphi_1 \otimes a_1 + \ldots + \varphi_n \otimes 0, \ldots, \varphi_1 \otimes 0 + \ldots + \varphi_n \otimes a_n) \\ &= (\varphi_1 \otimes a_1 + 0 \otimes a_1, \ldots, 0 \otimes a_n + \varphi_n \otimes a_n) \\ &= (\varphi_1 \otimes a_1, \ldots, \varphi_n \otimes a_n), \end{aligned}$$
since for instance $\varphi_i \otimes 0 = \varphi_i \otimes (0a_1) = (\varphi_i 0) \otimes a_1 = 0 \otimes a_1$ for all $i = 2, \ldots, n$ (we know that $0a = (1-1)a = a - a = 0$ for all $a \in A$).
2. The mapping $\bigoplus \eta_{B,A} : (B^\vee \otimes_\Lambda A)^n \to (\mathrm{Hom}_\Lambda(A, B)^\vee)^n$ obtained by taking coordinatewise images $(\varphi_1 \otimes a_1, \ldots, \varphi_n \otimes a_n) \mapsto (\eta_{B,A}(\varphi_1 \otimes a_1), \ldots, \eta_{B,A}(\varphi_n \otimes a_n))$ is an isomorphism because $\eta_{B,A}$ is.
3. By ((1), Proposition I.3.4), we obtain an isomorphism $\theta : (\mathrm{Hom}_\Lambda(A, B)^\vee)^n \to (\mathrm{Hom}_\Lambda(A, B)^n)^\vee$ defined by
$$\theta(\psi_1, \ldots, \psi_n)(\alpha_1, \ldots, \alpha_n) = \sum_{i=1}^n \psi_i(\alpha_i),$$
for $\psi_1, \ldots, \psi_n \in \mathrm{Hom}_\Lambda(A, B)^\vee$ and $\alpha_1, \ldots, \alpha_n \in \mathrm{Hom}_\Lambda(A, B)$.
4. Lastly, by ((1), Proposition I.3.4) again, we obtain an isomorphism $\rho : \mathrm{Hom}_\Lambda(A^n, B) \to \mathrm{Hom}_\Lambda(A, B)^n$ by defining $\rho(\psi) = (\psi\iota_1, \ldots, \psi\iota_n)$, ι_j denoting the usual inclusion from the j'th copy of $\mathrm{Hom}_\Lambda(A, B)$. By 1(c), we obtain an isomorphism $\rho^* : (\mathrm{Hom}_\Lambda(A, B)^n)^\vee \to \mathrm{Hom}_\Lambda(A^n, B)^\vee$ defined by $\rho^*(\beta)(\psi) = (\beta \circ \rho)(\psi) = \beta(\psi\iota_1, \ldots, \psi\iota_n)$ for all homomorphisms $\beta : \mathrm{Hom}_\Lambda(A, B)^n \to \mathbb{Q}/\mathbb{Z}$ and $\psi : A^n \to B$.

By composing all these isomorphisms, we obtain the following diagram

$$\begin{array}{ccccc}
B^\vee \otimes_\Lambda A^n & & \xrightarrow{\eta_{B,A^n}} & & \mathrm{Hom}_\Lambda(A^n, B)^\vee \\
\cong \downarrow \tau & & & & \rho^* \uparrow \cong \\
(B^\vee \otimes_\Lambda A)^n & \underset{\cong}{\xrightarrow{\bigoplus \eta_{B,A}}} & (\mathrm{Hom}_\Lambda(A,B)^\vee)^n & \underset{\cong}{\xrightarrow{\theta}} & (\mathrm{Hom}_\Lambda(A,B)^n)^\vee.
\end{array}$$

The question remains: does it commute? As matter of fact, it does. Let $\varphi \in B^\vee$, $a_1, \dots, a_n \in A$. Then $\bigoplus \eta_{B,A}(\tau(\varphi, (a_1, \dots, a_n))) = \bigoplus \eta_{B,A}(\varphi \otimes a_1, \dots, \varphi \otimes a_n) = (e_{a_1} \circ \varphi_*, \dots, e_{a_n} \circ \varphi_*)$. Taking the image under θ of this yields

$$\theta(e_{a_1} \circ \varphi_*, \dots, e_{a_n} \circ \varphi_*)(\alpha_1, \dots, \alpha_n) = \sum_{i=1}^n e_{a_i} \circ \varphi_*(\alpha_i) = \varphi\left(\sum_{i=1}^n \alpha_i(a_i)\right)$$

for all $\alpha_1, \dots, \alpha_n \in \mathrm{Hom}_\Lambda(A, B)$. Finally, for all homomorphisms $\psi : A^n \to B$, we obtain

$$\begin{aligned}
\rho^*(\theta(e_{a_1} \circ \varphi_*, \dots, e_{a_n} \circ \varphi_*))(\psi) &= \theta(e_{a_1} \circ \varphi_*, \dots, e_{a_n} \circ \varphi_*)(\psi\iota_1, \dots, \psi\iota_n) \\
&= \varphi\left(\sum_{i=1}^n \psi\iota_i(a_i)\right) \\
&= \varphi\left(\sum_{i=1}^n \psi(0, \dots, a_i, \dots, 0)\right) \\
&= \varphi(\psi(a_1, \dots, a_n)) \\
&= \eta_{B,A^n}(\varphi \otimes (a_1, \dots, a_n))(\psi),
\end{aligned}$$

so the diagram commutes, and η_{B,A^n} is an isomorphism.

Suppose now that B is a Λ-module and A is a finitely presented Λ-module, so that there exists $n, m \in \mathbb{N}$ such that there is an exact sequence

$$\Lambda^n \xrightarrow{\mu} \Lambda^m \xrightarrow{\varepsilon} A \to 0.$$

Since η is a natural transformation from F to G as defined in 2(a), we obtain a commutative diagram

$$\begin{array}{ccccccccc}
B^\vee \otimes_\Lambda \Lambda^n & \xrightarrow{F(f)} & B^\vee \otimes_\Lambda \Lambda^m & \xrightarrow{F(g)} & B^\vee \otimes_\Lambda A & \longrightarrow & 0 & \longrightarrow & 0 \\
\cong \downarrow \eta_{B,\Lambda^n} & & \cong \downarrow \eta_{B,\Lambda^n} & & \downarrow \eta_{B,A} & & \cong \downarrow & & \cong \downarrow \\
\mathrm{Hom}_\Lambda(\Lambda^n, B)^\vee & \xrightarrow{G(f)} & \mathrm{Hom}_\Lambda(\Lambda^m, B)^\vee & \xrightarrow{G(g)} & \mathrm{Hom}_\Lambda(A, B)^\vee & \longrightarrow & 0 & \longrightarrow & 0.
\end{array}$$

with the morphisms $f = (1_B, \mu) : (B, \Lambda^n) \to (B, \Lambda^m)$ and $g = (1_B, \varepsilon) : (B, \Lambda^m) \to (B, A)$. We aim at using the Five Lemma to prove that the middle homomorphism is an isomorphism, requiring the rows to be exact.

We have $F(f)(\varphi \otimes a) = \varphi \otimes \mu(a) = \mu_*(\varphi \otimes a)$, $F(g)(\varphi \otimes a) = \varphi \otimes \varepsilon(a) = \varepsilon_*(\varphi \otimes a)$ (with μ_* and ε_* as defined in ((1), pp. 109-110) and $G(f)(\varphi) = \varphi \circ \mu^* = (\mu^*)^*(\varphi)$ and $G(g)(\varphi) = \varphi \circ \varepsilon^* = (\varepsilon^*)^*(\varphi)$. The upper row is then exact by ((1), Proposition III.7.3(ii)). It follows from ((1), Theorem I.2.2) that the sequence

$$0 \longrightarrow \mathrm{Hom}_\Lambda(A, B) \xrightarrow{\varepsilon^*} \mathrm{Hom}_\Lambda(\Lambda^m, B) \xrightarrow{\mu^*} \mathrm{Hom}_\Lambda(\Lambda^n, B) \qquad (1)$$

is exact. In order to check that the lower row is exact, we need the following theorem:

Theorem 2. *Let I be an injective Λ-module and $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ be an exact sequence of Λ-modules. Then the sequence*

$$\mathrm{Hom}_\Lambda(C, I) \xrightarrow{\beta^*} \mathrm{Hom}_\Lambda(B, I) \xrightarrow{\alpha^*} \mathrm{Hom}_\Lambda(A, I) \longrightarrow 0$$

is exact.

Proof. For any $\omega \in \mathrm{Hom}_\Lambda(A, I)$, then since $\alpha : A \to B$ is injective, there exists $\rho \in \mathrm{Hom}_\Lambda(B, I)$ such that $\alpha^*(\rho) = \rho\alpha = \omega$ because I is injective, so α^* is surjective. For any $\gamma \in \mathrm{Hom}_\Lambda(C, I)$ we have $\alpha^*\beta^*(\gamma) = (\beta\alpha)^*(\gamma) = \gamma\beta\alpha = 0$, so $\mathrm{im}\beta^* \subseteq \ker \alpha^*$. Assuming that $\zeta \in \mathrm{Hom}_\Lambda(B, I)$ is not contained in $\ker \alpha^*$, there exists $a \in A$ such that $\zeta(\alpha(a)) = \alpha^*(\zeta)(a) \neq 0$. If some $\gamma \in \mathrm{Hom}_\Lambda(C, I)$ were to satisfy $\gamma\beta = \beta^*(\gamma) = \zeta$, then $\zeta(\alpha(a)) = \gamma(\beta(\alpha(a))) = 0$, so ζ is not contained in the image of β^*. □

The above theorem resembles one of the parts of ((1), Theorem I.8.4) quite a bit – if β were surjective, then β^* is easily seen to be injective. Since $\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module, we deduce the sequence (1) and by the above theorem (by putting $\alpha = \varepsilon^*$ and $\beta = \mu^*$) that the lower row in the diagram is exact, and the Five Lemma yields that $\eta_{B,A}$ is an isomorphism.

(c)

Suppose that A is finitely presented and flat. Prove that A is projective.

Let $0 \to B' \overset{\mu}{\to} B \overset{\varepsilon}{\to} B'' \to 0$ be a short exact sequence of Λ-modules. Define morphisms f and g in $\mathfrak{C}$ by $f = (\mu, 1_A) : (B, A) \to (B', A)$ and $g = (\varepsilon, 1_A) : (B'', A) \to (B, A)$. From 1(c), we obtain a short exact sequence

$$0 \longrightarrow (B'')^\vee \xrightarrow{\varepsilon^*} B^\vee \xrightarrow{\mu^*} (B')^\vee \longrightarrow 0.$$

Because A is flat, the induced sequence

$$0 \longrightarrow (B'')^\vee \otimes_\Lambda A \xrightarrow{\varepsilon'} B^\vee \otimes_\Lambda A \xrightarrow{\mu'} (B')^\vee \otimes_\Lambda A \longrightarrow 0$$

is exact, where ε' and μ' for any $a \in A$ are given by $\varepsilon'(\varphi_1 \otimes a) = \varepsilon^*(\varphi_1) \otimes a = (\varphi_1 \circ \varepsilon) \otimes a = F(g)(\varphi_1 \otimes a)$ for $\varphi_1 \in (B'')^\vee$ and $\mu'(\varphi_2 \otimes a) = \mu^*(\varphi_2) \otimes a = F(f)(\varphi_2 \otimes a)$ for $\varphi_2 \in B^\vee$ ((1), Proposition III.7.3(ii)). Since A is finitely presented, then from (b) we obtain a commutative diagram with exact rows and isomorphisms between them:

$$\begin{array}{ccccccccc}
0 \longrightarrow & (B'')^\vee \otimes_\Lambda A & \xrightarrow{F(g)} & B^\vee \otimes_\Lambda A & \xrightarrow{F(f)} & (B')^\vee \otimes_\Lambda A & \longrightarrow 0 \\
& \cong \downarrow \eta_{B'',A} & & \cong \downarrow \eta_{B,A} & & \cong \downarrow \eta_{B',A} & \\
0 \longrightarrow & \mathrm{Hom}_\Lambda(A, B'')^\vee & \xrightarrow{G(g)} & \mathrm{Hom}_\Lambda(A, B)^\vee & \xrightarrow{G(f)} & \mathrm{Hom}_\Lambda(A, B')^\vee & \longrightarrow 0.
\end{array}$$

The lower row is exact because it is connected by isomorphisms to the upper row which is exact. For any $\varphi_1 \in \mathrm{Hom}_\Lambda(A, B'')^\vee$ and $\varphi_2 \in \mathrm{Hom}_\Lambda(A, B)^\vee$, we have $G(g)(\varphi_1) = \varphi_1 \circ \varepsilon_* = (\varepsilon_*)^*(\varphi_1)$ and $G(f)(\varphi_2) = \varphi_2 \circ \mu_* = (\mu_*)^*(\varphi_2)$. Thus we obtain an exact sequence

$$0 \longrightarrow \mathrm{Hom}_\Lambda(A, B'')^\vee \xrightarrow{(\varepsilon_*)^*} \mathrm{Hom}_\Lambda(A, B)^\vee \xrightarrow{(\mu_*)^*} \mathrm{Hom}_\Lambda(A, B')^\vee \longrightarrow 0.$$

By 1(c), the sequence

$$0 \longrightarrow \mathrm{Hom}_\Lambda(A, B') \xrightarrow{\mu_*} \mathrm{Hom}_\Lambda(A, B) \xrightarrow{\varepsilon_*} \mathrm{Hom}_\Lambda(A, B'') \longrightarrow 0$$

is exact as well, but this is equivalent to A being projective ((1), Theorem I.4.7).

(3)

(a)

Consider a commutative diagram of Λ-modules with exact rows

$$\begin{array}{ccccccccc}
0 \longrightarrow & A' & \xrightarrow{\mu} & A & \xrightarrow{\varepsilon} & A'' & \longrightarrow 0 \\
& \downarrow \alpha' & & \downarrow \alpha & & \downarrow \alpha'' & \\
0 \longrightarrow & B' & \xrightarrow{\mu'} & B & \xrightarrow{\varepsilon'} & B'' & \longrightarrow 0.
\end{array}$$

Show that α' is an isomorphism if and only if the sequence

$$0 \longrightarrow A \xrightarrow{\binom{\alpha}{\varepsilon}} B \oplus A'' \xrightarrow{(\varepsilon', -\alpha'')} B'' \longrightarrow 0 \tag{2}$$

is exact. Here, the first homomorphism sends a to $(\alpha(a), \varepsilon(a))$ and the second homomorphism sends (b, a'') to $\varepsilon'(b) - \alpha''(a'')$.

Assume first that α' is an isomorphism. We prove in steps that the sequence (2) is exact (only the first two steps use the assumption about α'):

(1) $\binom{\alpha}{\varepsilon}$ is injective. Let $a \in A$ such that $(\alpha(a), \varepsilon(a)) = (0, 0)$. Since $a \in \ker \varepsilon = \mathrm{im}\mu$, there exists $a' \in A'$ such that $\mu(a') = a$. Since $\mu'(\alpha'(a')) = \alpha(\mu(a')) = \alpha(a) = 0$ by commutativity of the left square, then since μ' and α' are injective, we obtain $a' = 0$ so $a = \mu(a') = 0$.

(2) $\ker(\varepsilon', -\alpha'') \subseteq \mathrm{im}\binom{\alpha}{\varepsilon}$. Let $(b, a'') \in B \oplus A''$ such that $\varepsilon'(b) - \alpha''(a'') = 0$. Then because ε is surjective, there exists $a \in A$ such that $\varepsilon(a) = a''$, and therefore $\varepsilon'(b) = \alpha''(a'') = \alpha''(\varepsilon(a)) = \varepsilon'(\alpha(a))$ by commutativity of the right square. Thus $b - \alpha(a) \in \ker \varepsilon' = \mathrm{im}\mu'$, so there exists $b' \in B'$ such that $\mu'(b') = b - \alpha(a)$ and additionally because α' is surjective, there exists $a' \in A'$ such that $\alpha'(a') = b'$. Then $\alpha(\mu(a')) = \mu'(\alpha'(a')) = b - \alpha(a)$. We finally obtain $\alpha(a + \mu(a')) = \alpha(a) + \alpha(\mu(a')) = b$ and $\varepsilon(\mu(a') + a) = \varepsilon(a) = a''$ since $\mathrm{im}\mu \subseteq \ker \varepsilon$, so $(\alpha(a + \mu(a'), \varepsilon(a + \mu(a')) = (b, a'')$.

(3) $\ker(\varepsilon', -\alpha'') \supseteq \mathrm{im}\binom{\alpha}{\varepsilon}$. This is clear, since for all $a \in A$, we have $\varepsilon(\alpha(a)) - \alpha''(\varepsilon(a)) = 0$ by commutativity of the right square, implying $(\alpha(a), \varepsilon(a)) \in \ker(\varepsilon', -\alpha'')$ for all $a \in A$.

(4) $(\varepsilon', -\alpha'')$ is surjective. For any $b \in B''$, there exists $b \in B$ such that $\varepsilon'(b) = b''$, but then $(\varepsilon', -\alpha'')(b, 0) = \varepsilon'(b) - \alpha''(0) = b''$, implying the wanted property.

Assume now that the sequence (2) is exact. We will prove that α' is injective and surjective. For $a' \in A'$ satisfying $\alpha'(a') = 0$, then $\alpha(\mu(a')) = \mu'(\alpha'(a')) = 0$ by commutativity of the left square and $\varepsilon(\mu(a')) = 0$ by exactness of the upper row. Thus $(\alpha(\mu(a')), \varepsilon(\mu(a'))) = (0, 0)$, implying $\mu(a') = 0$ by injectivity of $\binom{\alpha}{\varepsilon}$ and $a' = 0$ by injectivity of μ'. Thus α' is injective. Letting $b' \in B'$, we have that $\varepsilon'(\mu'(b')) - \alpha''(0) = 0$ by exactness of the lower row, implying $(\mu'(b'), 0) \in \ker(\varepsilon', -\alpha'') = \mathrm{im}\binom{\alpha}{\varepsilon}$. Thus there exists $a \in A$ such that $\alpha(a) = \mu'(b')$ and $\varepsilon(a) = 0$. Since $a \in \ker \varepsilon = \mathrm{im}\mu$, there exists $a' \in A'$ such that $\mu(a') = a$ and therefore $\mu'(\alpha'(a')) = \alpha(\mu(a')) = \alpha(a) = \mu'(b')$ by commutativity of the left square, and finally $\alpha'(a') = b'$ by injectivity of μ'. Therefore α' is an isomorphism.

(b)

Let A be a Λ-module. Given two short exact sequences

$$0 \longrightarrow A \xrightarrow{\mu} I \xrightarrow{\varepsilon} C \longrightarrow 0$$

$$0 \longrightarrow A \xrightarrow{\mu'} I' \xrightarrow{\varepsilon'} C' \longrightarrow 0$$

with I and I' injective Λ-modules, show that $I \oplus C' \simeq I' \oplus C$. Show that C is injective if and only if C' is injective.

We want to make the two short exact sequences into a commutative diagram like the one in 3(a). For this, we need homomorphisms $A \to A$, $I \to I'$ and $C \to C'$. For the homomorphism $A \to A$, take the identity. Since $\mu : A \to I$ is a monomorphism and $\mu' : A \to I'$ is a homomorphism, then because I' is an injective module, there exists a homomorphism $\alpha : I \to I'$ such that $\alpha\mu = \mu'$. Finally, we need a homomorphism $C \to C'$ and it is defined as follows: for any $c \in C$, then because ε is surjective, there exists $b \in I$ such that $\varepsilon(b) = c$. Now define $\gamma(c) = \varepsilon'(\alpha(b))$. In this way, we obtain a map $\gamma : C \to C'$; in order to check that it is well-defined, we need to show that the image of c does not depend on the choice of $b \in I$. Alas, let $b' \in I$ such that $\varepsilon(b') = c$. Since $\varepsilon(b') = \varepsilon(b)$, we have $b - b' \in \ker \varepsilon = \mathrm{im}\mu$ by exactness of the upper sequence. Pick $a \in A$ such that $\mu(a) = b - b'$. Then $0 = \varepsilon'(\mu'(a)) = \varepsilon'(\alpha(\mu(a))) = \varepsilon'(\alpha(b - b')) = \varepsilon'(\alpha(b)) - \varepsilon'(\alpha(b'))$ by exactness of the lower sequence, so $\varepsilon'(\alpha(b)) = \varepsilon'(\alpha(b'))$.

Finally, γ is a homomorphism: let $c_1, c_2 \in C$ and choose $b_1, b_2 \in I$ such that $\varepsilon(b_1) = c_1$ and $\varepsilon(b_2) = c_2$; then $\varepsilon(b_1+b_2) = c_1+c_2$, so $\gamma(c_1+c_2) = \varepsilon'(\alpha(b_1+b_2)) = \varepsilon'(\alpha(b_1)+\varepsilon'(\alpha(b_2)) = \gamma(c_1)+\gamma(c_2)$ and for $\lambda \in \Lambda$, then since $\varepsilon(\lambda b_1) = \lambda c_1$, we obtain $\gamma(\lambda c_1) = \varepsilon'(\alpha(\lambda c_1)) = \lambda\gamma(c_1)$.

Thus we obtain a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\mu} & I & \xrightarrow{\varepsilon} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 1_A} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & A & \xrightarrow{\mu'} & I' & \xrightarrow{\varepsilon'} & C' & \longrightarrow & 0 \end{array}$$

that commutes because $\alpha(\mu(a)) = \mu'(a) = \mu'(1_A(a))$ for all $a \in A$, and $\gamma(\varepsilon(b)) = \varepsilon'(\alpha(b))$ by construction. Since 1_A is an isomorphism, we obtain a short exact sequence by using 3(a):

$$0 \longrightarrow I \xrightarrow{\binom{\alpha}{\varepsilon}} I' \oplus C \xrightarrow{(\varepsilon', -\gamma)} C' \longrightarrow 0$$

Because I is injective and $\binom{\alpha}{\varepsilon} : I \to I' \oplus C$ is a monomorphism, we obtain by ((1), Theorem I.8.4) that there exists $\beta : I' \oplus C \to I$ such that $\beta\binom{\alpha}{\varepsilon} = 1_I$. The short exact sequence therefore splits, and by the Splitting Lemma ((1), Exercise I.3.7), we obtain that $I' \oplus C \simeq I \oplus C'$.

Finally, under assumption that both I and I' are injective, we obtain from ((1), Proposition I.6.3) the following equivalences, since finite direct products are direct sums and isomorphisms of modules preserve injectivity:

$$C \text{ is injective} \Leftrightarrow I' \oplus C \text{ is injective} \Leftrightarrow I \oplus C' \text{ is injective} \Leftrightarrow C \text{ is injective.}$$

(4)

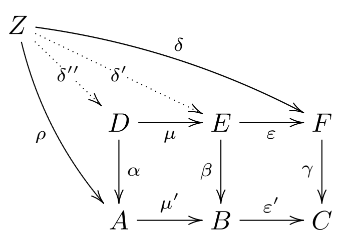
(a)

Given a commutative diagram in any category

$$\begin{array}{ccccc} D & \xrightarrow{\mu} & E & \xrightarrow{\varepsilon} & F \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} \\ A & \xrightarrow{\mu'} & B & \xrightarrow{\varepsilon'} & C, \end{array}$$

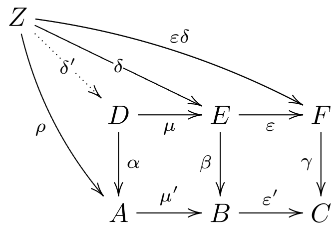
show that if both squares are pullbacks, then so is the outer rectangle. Show also that if the outer rectangle and the right square are pullbacks, then so is the left square.

Suppose that both squares are pullbacks and that Z is an object of the given category with morphisms $\delta : Z \to F$ and $\rho : Z \to A$ such that $\gamma\delta = \varepsilon'\mu'\rho$. Because the right square is a pullback, then because $\gamma\delta = \varepsilon'\mu'\rho$ by assumption there is a unique morphism $\delta' : Z \to E$ such that $\varepsilon\delta' = \delta$ and $\beta\delta' = \mu'\rho$. Additionally because the left square is a pullback, then because $\beta\delta' = \mu'\rho$, there exists a unique $\delta'' : Z \to D$ such that $\mu\delta'' = \delta'$ and $\alpha\delta'' = \rho$. Thus we obtain the following commutative diagram:



Now, $\delta'' : Z \to D$ satisfies $\alpha\delta'' = \rho$ and $\varepsilon\mu\delta'' = \varepsilon\delta'' = \delta$. Furthermore, δ'' is unique: assume that $\omega : Z \to D$ satisfies $\alpha\omega = \rho$ and $\varepsilon\mu\omega = \delta$. Because ω satisfies $\beta\mu\omega = \mu'\alpha\omega = \mu'\rho$ and $\varepsilon\mu\omega = \delta$, then by uniqueness of δ', we obtain $\mu\omega = \delta'$, and now, since ω satisfies $\mu\omega = \delta'$ and $\alpha\omega = \rho$, then we obtain $\omega = \delta''$ by uniqueness of δ''. Thus the outer rectangle is a pullback.

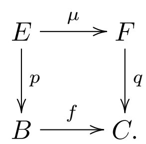
Suppose now that the outer rectangle is a pullback along with the right square, and that Z is an object of the given category with morphisms $\delta : Z \to E$ and $\rho : Z \to A$ such that $\beta\delta = \mu'\rho$. Because the outer rectangle is a pullback and $\gamma\varepsilon\delta = \varepsilon'\beta\delta = \varepsilon'\mu'\rho$, there is a unique morphism $\delta' : Z \to D$ such that $\varepsilon\mu\delta' = \varepsilon\delta$ and $\alpha\delta' = \rho$. Now, because the right square is a pullback and $\varepsilon'\mu'\rho = \gamma\varepsilon\delta$, there is a unique morphism $\delta'' : Z \to E$ such that $\varepsilon\delta'' = \varepsilon\delta$ and $\beta\delta'' = \mu'\rho$. However, since δ clearly satisfies these equalities as well, and $\varepsilon(\mu\delta') = \varepsilon\delta$ and $\beta(\mu\delta') = \mu'\alpha\delta' = \mu\rho$, we obtain $\delta = \mu\delta' = \delta''$ by uniqueness of δ''. We thus have the following commutative diagram:



Now, since $\alpha\delta' = \rho$ and $\mu\delta' = \delta$, $\delta' : Z \to D$ satisfies the wanted properties. Furthermore, δ' is unique with this property: if $\omega : Z \to D$ satisfies $\alpha\omega = \rho$ and $\mu\omega = \delta$, then $\varepsilon\mu\omega = \varepsilon\delta$, so that $\omega = \delta'$ by the uniqueness of δ' from the pullback of the outer rectangle.

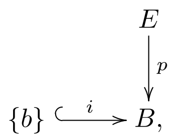
(b)

Consider a pullback diagram in the category of sets

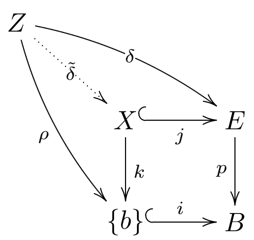


Show that there is a bijection $p^{-1}(\{b\}) \simeq q^{-1}(\{f(b)\})$ for every $b \in B$.

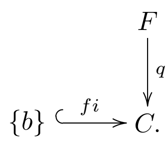
In the category of sets, the objects are sets and the morphisms are maps between sets. Let $b \in B$ and define the sets $X = p^{-1}(\{b\}) \subseteq E$ and $Y = q^{-1}(\{f(b)\}) \subseteq F$. Consider the diagram



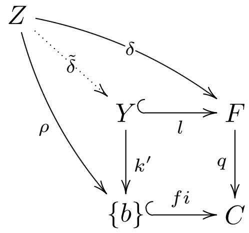
where i is the inclusion. We claim that X with the inclusion $j : X \hookrightarrow E$ and the constant map $k : x \mapsto b$ is a pullback of the above diagram. Indeed, note first that $ik(x) = b = pj(x)$ for all $x \in X$. Let Z be a set with functions $\delta : Z \to E$ and $\rho : Z \to \{b\}$ such that $i\rho = p\delta$. For all $z \in Z$, $p\delta(z) = i\rho(z) = b$ as an element in B, so $\delta(z) \in p^{-1}(\{b\})$ for all $z \in Z$. Define $\tilde{\delta} : Z \to p^{-1}(\{b\})$ by $\tilde{\delta}(z) = \delta(z)$. Then obviously $k\tilde{\delta} = \rho$ and $j\tilde{\delta} = \delta$, making the following diagram commute:



If $\omega : Z \to p^{-1}(b)$ satisfies $k\omega = \rho$ and $j\omega = \delta$, then by the second equality, we obtain $\omega = \tilde{\delta}$ (anything else would be absurd since j is the inclusion), so $\tilde{\delta}$ is unique with this property. Thus (X, j, k) is a pullback. By 4(a), we obtain that $(X, \varepsilon j, k)$ is a pullback of the diagram

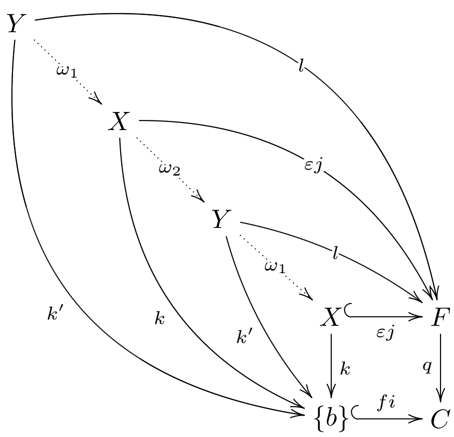


However, we claim now that (Y, l, k') is also a pullback of the above diagram, where $l : Y \hookrightarrow F$ is the inclusion and $k' : y \mapsto b$ is the constant map. Indeed, note first that for all $y \in Y$, then $q(l(y)) = q(y) = f(b) = f(k'(y))$. For any set Z with maps $\delta : Z \to F$ and $\rho : Z \to \{b\}$ such that $fi\rho = q\delta$, note that $q\delta(z) = fi\rho(z) = f(b)$ for all $z \in Z$ so that $\delta(z) \in q^{-1}(\{f(b)\})$. Define $\tilde{\delta} : Z \to Y$ by $\tilde{\delta}(z) = \delta(z)$. Clearly, $l\tilde{\delta} = \delta$ and $k'\tilde{\delta} = \rho$, so the diagram



commutes. $\tilde{\delta}$ is in the same way as before seen to be unique with this property, so (Y, l, k') is also a pullback.

Now consider the following (large!) commutative diagram



where $\omega_1 : Y \to X$ is the unique map such that $\varepsilon j\omega_1 = l$ and $k\omega_1 = k'$ and $\omega_2 : X \to Y$ is the unique map such that $l\omega_2 = \varepsilon j$ and $k'\omega_2 = k$, both existing because $(X, \varepsilon j, k)$ and (Y, l, k') are pullbacks. From these equations we obtain that *(1)* $l\omega_2\omega_1 = l$ and $k'\omega_2\omega_1 = k'$ and *(2)* $\varepsilon j\omega_1\omega_2 = \varepsilon j$ and $k\omega_1\omega_2 = k$. From *(1)* we deduce that since (Y, l, k') is a pullback, there is a unique homomorphism $g : Y \to Y$ such that $k'g = k'$ and $lg = l$, and since both 1_Y and $\omega_2\omega_1$ satisfy these equations, then $\omega_2\omega_1 = 1_Y$. In the same way, we obtain from *(2)* that $\omega_1\omega_2 = 1_X$. Therefore $\omega_2 : X \to Y$ is a bijection – which is what we wanted.

References

[1] P. J. Hilton and U. Stammbach, *A Course in Homological Algebra*, volume 4 of *Graduate Texts in Mathematics*, Springer Verlag, second edition, 1971.