Homological Algebra

Assignment 3

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(1)

Let Γ be a ring, let $x \in \Gamma$ be a central non-zero divisor and let $\Lambda = \Gamma/x\Gamma$.

(a)

Suppose that A and B are Γ-modules on which multiplication by x is zero respectively injective. Prove that there is an isomorphism

$$\mathrm{Ext}^n_\Lambda(A, B/xB) \simeq \mathrm{Ext}^{n+1}_\Gamma(A, B).$$

First of all, A is easily made into a Λ-module by defining $[\omega]a := \omega a$ for all $\omega \in \Gamma$ and $a \in A$, $[\omega]$ denoting the equivalence class of ω; the multiplication is well-defined, since if $[\omega] = [\omega']$, then $\omega - \omega' = x\gamma$ for some $\gamma \in \Gamma$. Since x is contained in the center of Γ, we obtain $\omega a - \omega' a = (\omega - \omega')a = (x\gamma)a = \gamma(xa) = 0$ for any $a \in A$. B/xB is made into a Λ-module as well by defining $[\omega][b] := [\omega b]$ for all $\omega \in \Gamma$ and $b \in B$; this multiplication is also well-defined, since if $\omega - \omega' = x\gamma$ for some $\gamma \in \Gamma$ and $b - b' = x\beta$ for some $\beta \in B$, then $\omega b - \omega' b' = (\omega - \omega')b + \omega'(b - b') = x(\gamma b) + \omega'(x\beta) = x(\gamma b + \omega'\beta) \in xB$, so $[\omega b] = [\omega' b']$.

We can make any Λ-module $\mathfrak{X}$ into a Γ-module by defining $\gamma x := [\gamma]x$ for all $\gamma \in \Gamma$ and $x \in \mathfrak{X}$. This technique becomes helpful in this next lemma.

Lemma 1. *If there is a surjective ring homomorphism $\xi : \Lambda \to \Lambda'$ and Λ'-modules P and Q, it holds that* $\mathrm{Hom}_\Lambda(P, Q) = \mathrm{Hom}_{\Lambda'}(P, Q)$.

Proof. P is made into a Λ-module by defining $\lambda p := \xi(\lambda)p$ for $\lambda \in \Lambda$, $p \in P$; likewise for Q. If $\varphi' \in \mathrm{Hom}_{\Lambda'}(P, Q)$, then $\varphi'(\lambda p) = \varphi'(\xi(\lambda)p) = \xi(\lambda)\varphi'(p) = \lambda\varphi'(p)$ for all $p \in P$ and $\lambda \in \Lambda$, so $\varphi' \in \mathrm{Hom}_{\Lambda'}(P, Q)$. If $\varphi \in \mathrm{Hom}_\Lambda(P, Q)$, then for any $\lambda' \in \Lambda'$, there exists $\lambda \in \Lambda$ such that $\xi(\lambda) = \lambda'$ by surjectivity, so $\varphi(\lambda' p) = \varphi(\lambda p) = \lambda\varphi(p) = \lambda'\varphi(p)$ for all $p \in P$. □

We run through the ordeal by induction, and will prove that for all Γ-modules A and B as described in the problem, there is an isomorphism $\mathrm{Ext}^n_\Lambda(A, B/xB) \simeq \mathrm{Ext}^{n+1}_\Gamma(A, B)$ for all $n \geq -1$ (the statement holds trivially for all $n < -1$).

Base cases ($n = -1, 0, 1$). If $\varphi : A \to B$ is a Γ-module homomorphism, then $x\varphi(a) = \varphi(xa) = \varphi(0) = 0$ for any $a \in A$, so $\varphi = 0$ by injectivity of multiplication by x in B and $\mathrm{Hom}_\Gamma(A, B) = 0$.

Step 1: $n = -1$**.** Since $\mathrm{Ext}^{-1}_\Lambda(A, B/xB) = 0$ by definition and $\mathrm{Ext}^0_\Gamma(A, B) \simeq \mathrm{Hom}_\Gamma(A, B) = 0$, the result follows.

Step 2: $n = 0$**.** Consider the short exact sequence of Γ-modules

$$0 \to B \xrightarrow{\delta} B \to B/xB \to 0,$$

where $\delta : B \to B$ is given by $\delta(b) = xb$. It induces a long exact Ext-sequence in the second variable by ((1), Proposition IV.7.5):

$$\mathrm{Hom}_\Gamma(A, B) \longrightarrow \mathrm{Hom}_\Gamma(A, B/xB) \xrightarrow{\omega_0} \mathrm{Ext}^1_\Gamma(A, B) \xrightarrow{\delta'_*} \mathrm{Ext}^1_\Gamma(A, B) \longrightarrow \cdots$$

We claim that the homomorphism δ'_* induced by δ is actually the zero map. Once that is proved, then it follows from the equality $\mathrm{Hom}_\Gamma(A,B)=0$ that the connecting homomorphism ω_0 is an isomorphism. By the balancing of Ext we can consider an injective resolution $I^\bullet$ of B with homomorphisms ∂^i, $i\in\mathbb{N}_0$, yielding a commutative diagram

$$\begin{array}{ccccccc}
\cdots \xleftarrow{\partial^2_*} & \mathrm{Hom}_\Gamma(A,I^2) & \xleftarrow{\partial^1_*} & \mathrm{Hom}_\Gamma(A,I^1) & \xleftarrow{\partial^0_*} & \mathrm{Hom}_\Gamma(A,I^0) \\
 & \downarrow{\scriptstyle \delta^*} & & \downarrow{\scriptstyle \delta^*} & & \downarrow{\scriptstyle \delta^*} \\
\cdots \xleftarrow{\partial^2_*} & \mathrm{Hom}_\Gamma(A,I^2) & \xleftarrow{\partial^1_*} & \mathrm{Hom}_\Gamma(A,I^1) & \xleftarrow{\partial^0_*} & \mathrm{Hom}_\Gamma(A,I^0).
\end{array}$$

The homomorphism induced from δ under homology is the same as the homomorphism δ'_*, but note that $\delta^*(\varphi)(a)=\varphi(xa)=0$ for all $\varphi\in\mathrm{Hom}_\Gamma(A,I^i)$ and $a\in A$. Thus $\delta'_*([\varphi])=[\delta^*(\varphi)]=[0]$ for all $[\varphi]\in\mathrm{Ext}^1_\Gamma(A,B)$, implying $\delta'_*=0$. Thus $\mathrm{Ext}^1_\Gamma(A,B)\simeq\mathrm{Hom}_\Gamma(A,B/xB)$, but by Lemma 1, we then get

$$\mathrm{Ext}^1_\Gamma(A,B)\simeq\mathrm{Hom}_\Gamma(A,B/xB)=\mathrm{Hom}_\Lambda(A,B/xB)\simeq\mathrm{Ext}^0_\Lambda(A,B/xB),$$

since the Γ-module structure induced by the quotient mapping $\Gamma\to\Lambda$ is in fact the original Γ-module structure of A.

Step 3: $n=1$**.** Choose a Λ-free presentation of A, i.e. a short exact sequence of Λ-modules

$$0\hookrightarrow K\to P\to A\to 0$$

with $P\simeq\Lambda^{(S)}$ for some set S. The Λ-module structure induces a Γ-module structure on the above modules; in particular, multiplication on K and P by x is zero, yielding two long exact Ext-sequences in the first variable:

$$\begin{array}{ccccccc}
\mathrm{Ext}^0_\Lambda({}_\Lambda P,B/xB) & \longrightarrow & \mathrm{Ext}^0_\Lambda({}_\Lambda K,B/xB) & \longrightarrow & \mathrm{Ext}^1_\Lambda({}_\Lambda A,B/xB) & \longrightarrow & \mathrm{Ext}^1_\Lambda({}_\Lambda P,B/xB) \\
\downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} & & & & \\
\mathrm{Ext}^1_\Gamma({}_\Gamma P,B) & \longrightarrow & \mathrm{Ext}^1_\Gamma({}_\Gamma K,B) & \longrightarrow & \mathrm{Ext}^2_\Gamma({}_\Gamma A,B) & \longrightarrow & \mathrm{Ext}^2_\Gamma({}_\Gamma P,B)
\end{array}$$

The last term in the upper row is zero by ((1), Proposition IV.7.2), and the two downward isomorphisms in the diagram is given by the induction hypothesis. Since $P\simeq\Gamma^{(S)}/x\Gamma^{(S)}$ as Γ-modules (see 1(c) for a proof), we have an exact sequence

$$0\longrightarrow\Gamma^{(S)}\xrightarrow{\cdot x}\Gamma^{(S)}\xrightarrow{\pi}{}_\Gamma P\longrightarrow 0,$$

of Γ-modules, π denoting the quotient mapping. This yields a long exact Ext-sequence in the first variable

$$\cdots\longrightarrow\mathrm{Ext}^{s-1}_\Gamma(\Gamma^{(S)},B)\longrightarrow\mathrm{Ext}^s_\Gamma({}_\Gamma P,B)\longrightarrow\mathrm{Ext}^s_\Gamma(\Gamma^{(S)},B)\longrightarrow\cdots,$$

and since the outer two terms are zero by ((1), Proposition IV.7.2) for $s\geq 2$, $\mathrm{Ext}^s_\Gamma({}_\Gamma P,B)=0$ for all $s\geq 2$. We therefore obtain a commutative diagram

$$\begin{array}{ccccccc}
\mathrm{Ext}^0_\Lambda({}_\Lambda P,B/xB) & \longrightarrow & \mathrm{Ext}^0_\Lambda({}_\Lambda K,B/xB) & \longrightarrow & \mathrm{Ext}^1_\Lambda({}_\Lambda A,B/xB) & \longrightarrow & 0 \\
\downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} & & & & \downarrow{\scriptstyle\cong} \\
\mathrm{Ext}^1_\Gamma({}_\Gamma P,B) & \longrightarrow & \mathrm{Ext}^1_\Gamma({}_\Gamma K,B) & \longrightarrow & \mathrm{Ext}^2_\Gamma({}_\Gamma A,B) & \longrightarrow & 0
\end{array}$$

Indeed, the diagram is commutative if and only if the first square is, but since $\mathrm{Ext}^0_\Lambda({}_\Lambda P,B/xB)\simeq\mathrm{Ext}^0_\Gamma({}_\Gamma P,B/xB)$ and $\mathrm{Ext}^0_\Lambda({}_\Lambda K,B/xB)\simeq\mathrm{Ext}^0_\Gamma({}_\Gamma K,B/xB)$ by Lemma 1, and the diagram

$$\begin{array}{ccc}
\mathrm{Ext}^0_\Gamma({}_\Gamma P,B/xB) & \longrightarrow & \mathrm{Ext}^0_\Gamma({}_\Gamma K,B/xB) \\
\downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} \\
\mathrm{Ext}^1_\Gamma({}_\Gamma P,B) & \longrightarrow & \mathrm{Ext}^1_\Gamma({}_\Gamma K,B)
\end{array}$$

is commutative by ((1), Proposition IV.7.6, diagram (7.5) with α being the inclusion $K\to P$), then the desired commutativity follows. In other to prove that there is an arrow between the third terms that is an isomorphism, we prove the following lemma:

Lemma 2. *For any commutative diagram of Λ-modules with exact rows*

$$\begin{array}{ccccccc} A & \xrightarrow{\mu_1} & B & \xrightarrow{\mu_2} & C & \longrightarrow & 0 \\ \cong\downarrow\alpha & & \cong\downarrow\beta & & & & \\ A' & \xrightarrow{\mu_1'} & B' & \xrightarrow{\mu_2'} & C' & \longrightarrow & 0, \end{array}$$

there is an induced arrow $\gamma : C \to C'$ that is an isomorphism.

Proof. For any $c \in C$, take $b \in B$ such that $\mu_2(b) = c$ and consider $\mu_2'\beta(b)$. If another $b_0 \in B$ satisfies $\mu_2(b_0) = c$, then $b - b_0 \in \ker \mu_2 = \mathrm{im}\mu_1$, i.e. $\mu_1(a) = b - b_0$ for some $a \in A$. We now have

$$\mu_2'\beta(b) - \mu_2'\beta(b_0) = \mu_2'\beta(b - b_0) = \mu_2'\beta\mu_1(a) = \mu_2'\mu_1'\alpha(a) = 0,$$

so by defining $\gamma(c) = \mu_2'\beta(b)$, we obtain a well-defined mapping. It is a homomorphism, since if $c, c_0 \in C$ and $\lambda \in \Lambda$, then by picking $b, b_0 \in C$ such that $\mu_2(b) = c$ and $\mu_2(b_0) = c_0$, then $\mu_2(b+b_0) = c+c_0$ and $\gamma(c+c_0) = \mu_2'\beta(b+b_0) = \gamma(c)+\gamma(c_0)$, and additionally $\gamma(\lambda c) = \mu_2'\beta(\lambda b) = \lambda\gamma(c)$. By adding an extra column of zeros at the end, we obtain a commutative diagram with isomorphisms in the last two columns, the Five Lemma yielding that γ is an isomorphism. $\square$

Alas $\mathrm{Ext}^1_\Lambda(A, B/xB) \simeq \mathrm{Ext}^2_\Gamma(A, B)$.

Inductive step. Assume that the theorem holds for $n = m$. Choose a Λ-free presentation of A, i.e. a short exact sequence of Λ-modules

$$0 \to K \hookrightarrow P \to A \to 0$$

with $P \simeq \Lambda^{(S)}$ for some set S. We induce the same Γ-module structure as in Step 3, so multiplication on K and P by x is zero, and we obtain a diagram

$$\begin{array}{ccccccc} \mathrm{Ext}^m_\Lambda({}_\Lambda P, B/xB) & \longrightarrow & \mathrm{Ext}^m_\Lambda({}_\Lambda K, B/xB) & \longrightarrow & \mathrm{Ext}^{m+1}_\Lambda({}_\Lambda A, B/xB) & \longrightarrow & \mathrm{Ext}^{m+1}_\Lambda({}_\Lambda P, B/xB) \\ \downarrow\cong & & \downarrow\cong & & & & \\ \mathrm{Ext}^{m+1}_\Gamma({}_\Gamma P, B) & \longrightarrow & \mathrm{Ext}^{m+1}_\Gamma({}_\Gamma K, B) & \longrightarrow & \mathrm{Ext}^{m+2}_\Gamma({}_\Gamma A, B) & \longrightarrow & \mathrm{Ext}^{m+2}_\Gamma({}_\Gamma P, B) \end{array}$$

The outer terms in the upper row is zero by ((1), Proposition IV.7.2), and the two downward isomorphisms in the diagram are given by the induction hypothesis. Since $\mathrm{Ext}^s_\Gamma({}_\Gamma P, B) = 0$ for all $s \geq 2$ as proved in Step 3 of the induction start, then the two outer terms in the lower row are zero, and we obtain the desired isomorphism by composing

$$\begin{array}{ccc} \mathrm{Ext}^m_\Lambda({}_\Lambda K, B/xB) & \xrightarrow{\cong} & \mathrm{Ext}^{m+1}_\Lambda({}_\Lambda A, B/xB) \ . \\ \downarrow\cong & & \\ \mathrm{Ext}^{m+1}_\Gamma({}_\Gamma K, B) & \xrightarrow{\cong} & \mathrm{Ext}^{m+2}_\Gamma({}_\Gamma A, B) \end{array}$$

(b)

Let A be a Λ-module such that $\mathrm{pd}_\Lambda A = n < \infty$. Prove that there exists a free Λ-module F such that $\mathrm{Ext}^n_\Lambda(A, F) \neq 0$.

Let $P_\bullet \to A$ be a minimal Λ-projective resolution of A with homomorphisms $\partial^i : P_i \to P_{i-1}$, $i \in \mathbb{N}$. For any Λ-module B we obtain the induced sequence

$$\cdots \longrightarrow \mathrm{Hom}_\Lambda(P_{n-1}, B) \xrightarrow{\partial^n_*} \mathrm{Hom}_\Lambda(P_n, B) \xrightarrow{\partial^{n+1}_*} 0.$$

Since P_n is projective, there exists a Λ-module Q such that $F := P_n \oplus Q$ is free ((1), Theorem I.4.7). We now want to prove that ∂_n^* is not surjective in the case of this F, so that

$$\mathrm{Ext}^n_\Lambda(A, F) = H^n\mathrm{Hom}_\Lambda(P_\bullet, F) = \frac{\ker \partial^{n+1}_*}{\mathrm{im}\partial^n_*} = \frac{\mathrm{Hom}_\Lambda(P_n, F)}{\mathrm{im}\partial^n_*} \neq 0.$$

Assume that ∂_n^* is surjective and consider the inclusion $\iota : P_n \to F$ given by $\iota(p) = (p, 0)$. By the assumption of surjectivity, there exists $\varphi : P_{n-1} \to F$ such that $\varphi\partial_n = \partial_n^*\varphi = \iota$. Let $\pi : F \to P_n$ denote the standard projection; then $(\pi\varphi)\partial^n = \pi(\varphi\partial^n) = \pi\iota = 1_{P_n}$, so ∂_n has a left inverse. Consider now the short exact sequence

$$0 \longrightarrow P_n \xrightarrow{\partial^n} P_{n-1} \xrightarrow{\partial^{n-1}} \text{im}\partial^{n-1} \longrightarrow 0;$$

It splits, since ∂^n has a left inverse. Thus by the Splitting Lemma ((1), Exercise I.3.7), we obtain an isomorphism $P_n \oplus \text{im}\partial^{n-1} \simeq P_{n-1}$, so $\text{im}\partial^{n-1}$ is a projective Λ-module by ((1), Proposition I.4.5). Consider now an altered version $P'_\bullet$ of the chain complex $P_\bullet$:

$$P'_\bullet: \quad \cdots \longrightarrow 0 \longrightarrow \text{im}\partial^{n-1} \xrightarrow{i} P_{n-2} \xrightarrow{\partial^{n-2}} \cdots \xrightarrow{\partial^1} P_0 \longrightarrow 0,$$

i denoting the inclusion. Then the sequence

$$P'_\bullet \to A: \quad 0 \longrightarrow \text{im}\partial^{n-1} \xrightarrow{i} P_{n-2} \xrightarrow{\partial^{n-2}} \cdots \xrightarrow{\partial^1} P_0 \longrightarrow A \longrightarrow 0,$$

is clearly exact at $\text{im}\partial^{n-1}$ and at P_{n-2} since $\ker\partial^{n-2} = \text{im}\partial^{n-1} = i(\text{im}\partial^{n-1})$. It inherits exactness at all other P_i and at A from exactness of $P_\bullet \to A$. Thus $P'_\bullet$ is a Λ-projective resolution of A, but this contradicts the assumption that $P_\bullet$ was a minimal Λ-projective resolution of A, since this new projective resolution has length $n-1$. Alas ∂_*^n is not surjective, so we have found a free module F such that $\text{Ext}^n_\Lambda(A, F) \neq 0$.

(c)

Prove that $\text{gldim}\Gamma \geq \text{gldim}\Lambda + 1$ *if* Λ *has finite global dimension.*

Assume $\text{gldim}\Lambda = n < \infty$. Then there exists a Λ-module A such that $\text{pd}_\Lambda A = n$. By 1(b), there exists a free Λ-module F such that $\text{Ext}^n_\Lambda(A, F) \neq 0$. Define a Γ-module multiplication on A by $\gamma a := [\gamma]a$ for all $\gamma \in \Gamma$, $a \in A$. Thus A becomes a Γ-module such that multiplication by x is 0, since $x \in x\Gamma$.

Since F is free, $F \simeq \Lambda^{(S)}$ for some set S; $\Lambda^{(S)}$ obtains a Γ-module structure by defining $\gamma([\lambda_s])_{s\in S} = ([\gamma\lambda_s])_{s\in S}$ for all $\gamma, \lambda_s \in \Gamma$. Define a Γ-module $B = \Gamma^{(S)}$; then it is clear that the map $B/xB \to \Lambda^{(S)}$ given by $[(\gamma_s)_{s\in S}] \mapsto ([\gamma_s])_{s\in S}$ is a well-defined Γ-module homomorphism, since if $(\gamma_s)_s \in xB$, then each $\gamma_s \in x\Gamma$, so $([\gamma_s])_{s\in S} = 0$. It's also clearly bijective; surjectivitity is clear from the outset, and if $([\gamma_s])_{s\in S} = 0$, then $\gamma_s \in x\Gamma$ for all $s \in S$, so $(\gamma_s)_{s\in S} \in xB$, so it is injective as well. If $x(\gamma_s)_{s\in S} = (x\gamma_s)_{s\in S} = 0$, then because x is a non-zero divisor, we obtain $(\gamma_s)_{s\in S} = 0$, so multiplication by x on B is injective.

By 1(a), we now obtain $\text{Ext}^{n+1}_\Gamma(A, B) \simeq \text{Ext}^n_\Lambda(A, B/xB) \simeq \text{Ext}^n_\Lambda(A, F) \neq 0$. By the Projective Dimension Theorem, we obtain $\text{pd}_\Gamma A > n$, or $\text{pd}_\Gamma A \geq n+1$, but then

$$\text{gldim}\Gamma \geq \text{pd}_\Gamma A \geq n + 1 = \text{gldim}\Lambda + 1.$$

(2)

Let Λ be a ring and let $\Lambda[x]$ denote the polynomial ring over Λ. If A is a Λ-module then let $A[x] = \Lambda[x] \otimes_\Lambda A$. Clearly, $A[x]$ is a $\Lambda[x]$-module by defining the left multiplication as $Q \cdot (P \otimes a) := (QP \otimes a)$.

(a)

Prove that $\text{pd}_{\Lambda[x]}A[x] = \text{pd}_\Lambda A$ *for any* Λ*-module* A.

First a lemma:

Lemma 3. *If the right* Λ*-module* B *is flat, the functor* $B \otimes_\Lambda - : \mathfrak{M}^\ell_\Lambda \to \mathbf{Ab}$ *preserves long exact sequences.*

Proof. Let

$$A_\bullet : \quad \cdots \longrightarrow A_{n+1} \xrightarrow{\partial_{n+1}} A_n \xrightarrow{\partial_n} A_{n-1} \longrightarrow \cdots$$

be a long exact sequence. For $n \in \mathbb{Z}$, this gives rise to the short exact sequence

$$0 \longrightarrow \mathrm{im}\partial_{n+1} \xrightarrow{i} A_n \xrightarrow{\partial_n} \mathrm{im}\partial_n \longrightarrow 0$$

i denoting the inclusion. For any Λ-module homomorphism $\varphi : C_1 \to C_2$ and a right Λ-module R, it holds that the map $R \times \mathrm{im}\varphi \to \mathrm{im}(R \otimes_\Lambda \varphi)$ given by $(r, \varphi(c)) \mapsto r \otimes \varphi(c)$ is bilinear and thus allows for a homomorphism $R \otimes_\Lambda \mathrm{im}\varphi \to \mathrm{im}(R \otimes_\Lambda \varphi)$ given by $r \otimes \varphi(c) \mapsto r \otimes \varphi(c)$. It has an inverse given in the same way, so it is an isomorphism. With these considerations, we obtain a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & B \otimes_\Lambda \mathrm{im}\partial_{n+1} & \xrightarrow{B\otimes_\Lambda i} & B \otimes_\Lambda A_n & \xrightarrow{B\otimes_\Lambda \partial_n} & B \otimes_\Lambda \mathrm{im}\partial_n & \longrightarrow & 0 \\
 & & \downarrow \cong & & \downarrow = & & \downarrow \cong & & \\
0 & \longrightarrow & \mathrm{im}(B \otimes_\Lambda \partial_{n+1}) & \xrightarrow{\iota} & B \otimes_\Lambda A_n & \xrightarrow{B\otimes_\Lambda \partial_n} & \mathrm{im}(B \otimes_\Lambda \partial_n) & \longrightarrow & 0,
\end{array}$$

ι denoting an inclusion. Since B is flat, the upper and thus lower row is exact. Therefore

$$H_n(B \otimes_\Lambda A_\bullet) = \frac{\ker(B \otimes_\Lambda \partial_n)}{\mathrm{im}(B \otimes_\Lambda \partial_{n+1})} = \frac{\mathrm{im}(B \otimes_\Lambda \partial_{n+1})}{\mathrm{im}(B \otimes_\Lambda \partial_{n+1})} = 0,$$

so exactness is preserved. □

Let $P_\bullet \to A$ be a minimal Λ-projective resolution of A with homomorphisms $\partial_i : P_i \to P_{i-1}$, $i \in \mathbb{N}$. $\Lambda[x]$ is a free Λ-module, as it has generators $\{1, x, x^2, x^3, \ldots\}$, and therefore it is flat. Thus the sequence

$$\cdots \longrightarrow P_n[x] \xrightarrow{\Lambda[x]\otimes_\Lambda \partial_n} \cdots \xrightarrow{\Lambda[x]\otimes_\Lambda \partial_1} P_0[x] \longrightarrow A[x] \longrightarrow 0 \tag{1}$$

is exact by the above lemma. Each $P_i[x]$ is also a $\Lambda[x]$-module, and the homomorphisms $\Lambda[x] \otimes_\Lambda \partial_i$ are $\Lambda[x]$-module homomorphisms since for each $Q, P \in \Lambda[x]$ and $p \in P_i$, we have

$$(\Lambda[x] \otimes_\Lambda \partial_i)(Q(P \otimes p)) = QP \otimes \partial_i(p) = Q(P \otimes \partial_i(p)) = Q(\Lambda[x] \otimes_\Lambda \partial_i)(P \otimes p).$$

We now prove that each $P_i[x]$ is a projective $\Lambda[x]$-module. Let $i \in \mathbb{N}_0$ and $\varepsilon : B \to C$ and $\beta : P_i[x] \to C$ be $\Lambda[x]$-module homomorphisms with ε surjective. They are clearly also Λ-module homomorphisms, and defining $\hat{\beta} : P_i \to C$ by $\hat{\beta}(p) = \beta(1 \otimes p)$, it is clearly a Λ-module homomorphism, since

$$\hat{\beta}(\lambda p) = \beta(1 \otimes \lambda p) = \beta(\lambda \otimes p) = \lambda\beta(1 \otimes p) = \lambda\hat{\beta}(p), \quad \lambda \in \Lambda,\ p \in P_i.$$

By P_i being projective, there now exists $\hat{\rho} : P_i \to B$ such that $\varepsilon\hat{\rho} = \hat{\beta}$. Define $\tilde{\rho} : \Lambda[x] \times P_i \to B$ by $\tilde{\rho}(Q, p) = Q\hat{\rho}(p)$, it is clearly bilinear and so allows for a homomorphism $\rho : P_i[x] \to B$ defined by $\rho(Q \otimes p) = Q\hat{\rho}(p)$. It is a $\Lambda[x]$-module homomorphism, and since

$$\varepsilon\rho(Q \otimes p) = \varepsilon(Q\hat{\rho}(p)) = Q\varepsilon(\hat{\rho}(p)) = Q\hat{\beta}(p) = Q\beta(1 \otimes p) = \beta(Q \otimes p), \quad Q \in \Lambda[x],\ p \in P_i,$$

$P_i[x]$ is projective. The sequence (1) therefore yields $\mathrm{pd}_{\Lambda[x]}A[x] \leq \mathrm{pd}_\Lambda A$. In particular, if $\mathrm{pd}_{\Lambda[x]}A[x] = \infty$, then $\mathrm{pd}_{\Lambda[x]}A = \infty$, so it only remains to show the equality for $\mathrm{pd}_{\Lambda[x]}A[x]$ finite.

Assume now that $\mathrm{pd}_{\Lambda[x]}A[x] = n < \infty$, and let

$$P_\bullet \to A : \quad \cdots \longrightarrow P_{n+1} \xrightarrow{\partial_{n+1}} P_n \xrightarrow{\partial_n} \cdots \longrightarrow P_0 \longrightarrow A \longrightarrow 0$$

be a projective resolution of A. Letting $K_i = \ker \partial_i$ for all $i \in \mathbb{N}_0$, we obtain an exact sequence

$$0 \longrightarrow K_{n-1} \hookrightarrow P_{n-1} \xrightarrow{\partial_{n-1}} \cdots \longrightarrow P_0 \longrightarrow A \longrightarrow 0.$$

Now, by the Projective Dimension Theorem, $\mathrm{pd}_\Lambda A \le n$ if and only if K_{n-1} is projective. Construct a projective resolution of $A[x]$ as before, i.e.

$$\cdots \longrightarrow P_n[x] \xrightarrow{\Lambda[x]\otimes_\Lambda \partial_n} \cdots \xrightarrow{\Lambda[x]\otimes_\Lambda \partial_1} P_0[x] \longrightarrow A[x] \longrightarrow 0.$$

Since $\mathrm{pd}_{\Lambda[x]} A[x] = n$ by assumption, then by the Projective Dimension Theorem, $L = \ker(\Lambda[x] \otimes_\Lambda \partial_{n-1})$ is projective. We will now prove that K_{n-1} is projective using this fact. Let $\beta : K_{n-1} \to C$ and $\varepsilon : B \to C$ be Λ-module homomorphisms with ε surjective. K_{n-1}, B and C can be given a $\Lambda[x]$-module multiplication by defining

$$(\lambda_0 + \lambda_1 x + \cdots + \lambda_n x^n)b = \lambda_0 b, \quad \lambda_i \in \Lambda,\ b \in B;$$

likewise for the two others. Define the homomorphism $\hat\beta : L \to C$ by $\hat\beta(P \otimes a) = \beta(Pa)$ (possible by clear bilinearity). Letting P_0 and Q_0 denote the constant terms of the $P, Q \in \Lambda[x]$ respectively, we obtain

$$\hat\beta(Q(P\otimes a)) = \hat\beta(QP \otimes A) = \beta((QP)a) = Q_0\beta(P_0 a) = Q\beta(Pa) = \hat\beta(Q(P\otimes a)),$$

for all $a \in A$, so β is a $\Lambda[x]$-module homomorphism. ε is clearly a $\Lambda[x]$-module homomorphism as well, seen in the same way. Since L is projective, there exists $\hat\rho : L \to B$ such that $\varepsilon\hat\rho = \hat\beta$. If $a \in K_{n-1}$, then $(\Lambda[x] \otimes_\Lambda \partial_{n-1})(1 \otimes a) = 1 \otimes \partial_{n-1}(a) = 0$, and therefore it is possible to define a map $\rho : K_{n-1} \to B$ by $\rho(a) = \hat\rho(1 \otimes a)$. It is clearly additive and

$$\rho(\lambda a) = \hat\rho(1 \otimes (\lambda a)) = \hat\rho(\lambda \otimes a) = \lambda\hat\rho(1 \otimes a) = \lambda\rho(a) \quad a \in K_{n-1},\ \lambda \in \Lambda,$$

so it is a homomorphism and $\varepsilon\rho(a) = \varepsilon\hat\rho(1\otimes a) = \hat\beta(1\otimes a) = \beta(1a) = \beta(a)$ for all $a \in K_{n-1}$. Therefore K_{n-1} is projective, so $\mathrm{pd}_\Lambda A \le n = \mathrm{pd}_{\Lambda[x]} A[x]$.

(b)

Let A be a $\Lambda[x]$-module. Prove that there is a short exact sequence of $\Lambda[x]$-modules

$$0 \to A[x] \to A[x] \xrightarrow{\varepsilon} A \to 0,$$

where $\varepsilon(P \otimes a) = Pa$.

Note first that for any $P = \lambda_0 + \lambda_1 x + \cdots + \lambda_n x^n \in \Lambda[x]$ and $a \in A$, an elementary tensor of $A[x]$ can be written

$$P \otimes a = \sum_{i=0}^{n} \lambda_i x^i \otimes a = \sum_{i=0}^{n} x^i \otimes \lambda_i a.$$

Thus any element $\Psi \in A[x]$ is of the form $\Psi = \sum_{i=0}^n x^i \otimes a_i$ for some $n \in \mathbb{N}$ and $a_i \in A$, and we define $\deg(\Psi) = n$ in this case. It is clear that $\deg(\Psi) \ge 0$ for all $\Psi \in A[x]$.

Clearly, $\varepsilon(1 \otimes a) = a$ for all $a \in A$, so ε is surjective. Let $\tilde\mu : \Lambda[x] \times A \to A[x]$ be given by $\tilde\mu(P, a) = Px \otimes a - P \otimes xa$. $\tilde\mu$ is bilinear, since it is linear in both variables and

$$\tilde\mu(P, \lambda a) = Px \otimes \lambda a - P \otimes x(\lambda a) = (P\lambda)x \otimes a - (P\lambda) \otimes (xa) = \tilde\mu(P\lambda, a)$$

for all $P \in \Lambda[x]$, $a \in A$ and $\lambda \in \Lambda$. It thus allows for a homomorphism $\mu : A[x] \to A[x]$ given by $\mu(P\otimes a) = Px\otimes a - P\otimes xa$ that is clearly a $\Lambda[x]$-module homomorphism as well. Clearly $\mathrm{im}\mu \subseteq \ker\varepsilon$, since $\varepsilon(\mu(x^i \otimes a)) = x^{i+1}a - x^{i+1}a = 0$ for all $i \in \mathbb{N}$, $a \in A$. We will show that it is injective and onto $\ker\varepsilon$, so that we obtain a short exact sequence

$$0 \to A[x] \xrightarrow{\mu} A[x] \xrightarrow{\varepsilon} A \to 0.$$

Assume that μ doesn't map onto $\ker\varepsilon$; then $\ker\varepsilon \setminus \mathrm{im}\mu$ is non-empty, and we can define

$$n = \min\{\deg(\Psi) \mid \Psi \in \ker\varepsilon \setminus \mathrm{im}\mu\}.$$

Assume that $n \geq 1$. Then take $\Psi \in \ker \varepsilon \setminus \mathrm{im}\mu$ with $\deg(\Psi) = n$ and write $\Psi = \sum_{i=0}^{n} x^i \otimes a_i$. Now, define

$$\hat{\Psi} = \sum_{i=0}^{n-1} x^i \otimes a_i + x^{n-1} \otimes xa_n.$$

Since

$$\varepsilon(\hat{\Psi}) = \sum_{i=0}^{n-1} x^i a_i + x^{n-1} x a_n = \sum_{i=0}^{n} x^i a_i = \varepsilon(\Psi) = 0,$$

we have $\hat{\Psi} \in \ker \varepsilon$. Now, since $\Psi = \hat{\Psi} + x^n \otimes a_n - x^{n-1} \otimes xa_n = \hat{\Psi} + \mu(x^{n-1} \otimes a)$, then assuming $\hat{\Psi} \in \mathrm{im}\mu$ leads to an immediate contradiction, since $\Psi \notin \mathrm{im}\mu$ by assumption. Therefore $\hat{\Psi} \notin \mathrm{im}\mu$, but $\deg(\hat{\Psi}) = n - 1$, contradicting minimality of n. Therefore $n = 0$, so all elements $\Psi \in \ker \varepsilon \setminus \mathrm{im}\mu$ have degree 0, i.e. $\Psi = 1 \otimes a_0$ for some $a_0 \in A$, but then $0 = \varepsilon(\Psi) = \varepsilon(1 \otimes a_0) = 1a_0 = a_0$, so $\Psi = 1 \otimes 0 = 0 \in \mathrm{im}\mu$, another contradiction. Therefore $n < 0$ which is absurd, so we conclude that μ maps onto $\ker \varepsilon$.

Assuming that $\mu(\Psi) = 0$ for $\Psi \in A[x]$, then by writing $\Psi = \sum_{i=0}^{n} x^i \otimes a_i$, we obtain

$$\begin{aligned} 0 = \mu(\Psi) &= \sum_{i=0}^{n} \mu(x^i \otimes a_i) \\ &= \sum_{i=0}^{n} x^{i+1} \otimes a_i - \sum_{i=0}^{n} x^i \otimes xa_i \\ &= \sum_{i=1}^{n+1} x^i \otimes a_{i-1} - \sum_{i=0}^{n} x^i \otimes xa_i \\ &= x^{n+1} \otimes a_n + \sum_{i=1}^{n} x^i \otimes (a_{i-1} - xa_i) - 1 \otimes xa_0. \end{aligned}$$

Note that there is an isomorphism $A[x] \to \bigoplus_{i=0}^{\infty} A$ given by $\sum_i x^i \otimes a_i \mapsto (a_i)_{i=0}^{\infty}$ with the obvious inverse $(a_i)_{i=0}^{\infty} \mapsto \sum_i x^i \otimes a_i$; thus the above sum being equal to zero yield the equalities

$$a_n = 0, \quad a_{n-1} - xa_n = 0, \quad a_{n-2} - xa_{n-1} = 0, \quad \ldots, \quad a_0 - xa_1 = 0, \quad -xa_0 = 0,$$

where $a_n = 0$ implies $a_{n-1} = 0$ which in turn implies $a_{n-2} = 0$ and so on. Thus $\Psi = \sum_{i=0}^{n} x^i \otimes a_i = 0$, so μ is injective as well.

(c)

Prove that $\mathrm{gldim}\Lambda[x] \leq \mathrm{gldim}\Lambda + 1$.

Let A be a $\Lambda[x]$-module and $n := \mathrm{gldim}\Lambda$. A is clearly also a Λ-module, and thus from 2(a) we obtain that $\mathrm{pd}_{\Lambda[x]} A[x] = \mathrm{pd}_\Lambda A \leq n$. The short exact sequence from 2(b) yields a long exact Ext-sequence in the first variable for all $\Lambda[x]$-modules B ((1), p. 139):

$$\cdots \longrightarrow \mathrm{Ext}^{n+1}_{\Lambda[x]}(A[x], B) \longrightarrow \mathrm{Ext}^{n+2}_{\Lambda[x]}(A, B) \longrightarrow \mathrm{Ext}^{n+2}_{\Lambda[x]}(A[x], B) \longrightarrow \cdots$$

By the Projective Dimension Theorem, we obtain for all Λ-modules B that

$$\mathrm{Ext}^{n+1}_{\Lambda[x]}(A[x], B) = \mathrm{Ext}^{n+2}_{\Lambda[x]}(A[x], B) = 0$$

since $\mathrm{pd}_{\Lambda[x]} A[x] \leq n$, and therefore $\mathrm{Ext}^{n+2}_{\Lambda[x]}(A, B) = 0$. By the Projective Dimension Theorem, we obtain $\mathrm{pd}_{\Lambda[x]} A \leq n + 1$. Since A was arbitrarily chosen, we obtain

$$\mathrm{gldim}\Lambda[x] \leq n + 1 = \mathrm{gldim}\Lambda + 1.$$

Observe that (1) and (2) imply that $\mathrm{gldim}\Lambda[x] = \mathrm{gldim}\Lambda + 1$ for any ring Λ with finite global dimension. Indeed, since $\Lambda \simeq \Lambda[x]/x\Lambda[x]$ and x is a central non-zero divisor in $\Lambda[x]$, 1(c) yields $\mathrm{gldim}\Lambda[x] \leq \mathrm{gldim}\Lambda + 1$, the other inequality following from 2(c). As a corollary, we obtain Hilbert's syzygy theorem: $\mathrm{gldim} k[x_1, \ldots, x_n] = n$ if k is a field. This follows from the fact that all modules over fields are free and thus projective, so $\mathrm{gldim} k = 0$, and therefore

$$\mathrm{gldim} k[x_1, \ldots, x_n] = \mathrm{gldim}(k[x_1, \ldots, x_{n-1}])[x_n] = 1 + \mathrm{gldim} k[x_1, \ldots, x_{n-1}] = \cdots = n + \mathrm{gldim} k = n.$$

(3)

For a group G, let $B_\bullet G$ denote the chain complex $(E_\bullet G)_G$ where $E_\bullet G$ is some choice of a projective resolution of the trivial G-module $\mathbb{Z}$ (i.e. $\mathbb{Z}$ being viewed as a $\mathbb{Z}G$-module via augmentation: $(\sum_{g\in G} z_g g)z := (\sum_{g\in G} z_g)z$, $z \in \mathbb{Z}$).

(a)

Let G and H be groups. Prove that there is a homotopy equivalence of chain complexes of abelian groups $B_\bullet(G\times H) \simeq B_\bullet G \otimes_{\mathbb{Z}} B_\bullet H$.

Let $D = G\times H$ and choose free resolutions $E_\bullet G$, $E_\bullet H$ and $E_\bullet D$ of $\mathbb{Z}$ as a trivial G-, H- and D-module respectively; alas

$$\begin{array}{ll} E_\bullet G: & \cdots \to (\mathbb{Z}G)^{G_n} \to \cdots \to (\mathbb{Z}G)^{G_1} \to (\mathbb{Z}G)^{G_0} \to 0, \\ E_\bullet H: & \cdots \to (\mathbb{Z}H)^{H_n} \to \cdots \to (\mathbb{Z}H)^{H_1} \to (\mathbb{Z}H)^{H_0} \to 0, \end{array}$$

with generating sets G_i, H_i; recall that $H_0(E_\bullet G) \simeq \mathbb{Z} \simeq H_0(E_\bullet H)$ and $H_n(E_\bullet G) = H_n(E_\bullet H) = 0$ for all $n \geq 1$. Consider the chain complex of abelian groups $E_\bullet G \otimes_{\mathbb{Z}} E_\bullet H$. The above chain complexes $E_\bullet G$ and $E_\bullet H$ consist of free abelian groups, so by the Künneth theorem ((1), Theorem V.2.1) it follows that

$$\begin{aligned} H_n(E_\bullet G \otimes_{\mathbb{Z}} E_\bullet H) &\simeq \left(\bigoplus_{p+q=n} H_p(E_\bullet G)\otimes_{\mathbb{Z}} H_q(E_\bullet H)\right) \oplus \left(\bigoplus_{p+q=n-1} \operatorname{Tor}_1^{\mathbb{Z}}(H_p(E_\bullet G), H_q(E_\bullet H))\right) \\ &\simeq \begin{cases} 0 & n\geq 2 \\ \operatorname{Tor}_1^{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}) & n=1 \\ \mathbb{Z}\otimes_{\mathbb{Z}}\mathbb{Z} & n=0 \end{cases} \\ &\simeq \begin{cases} 0 & n\geq 1 \\ \mathbb{Z} & n=0 \end{cases} \end{aligned}$$

by $\mathbb{Z}$ being $\mathbb{Z}$ projective ((1), p. 161) and using properties of the tensor product. The abelian groups of the chain complex are given by

$$(E_\bullet G \otimes_{\mathbb{Z}} E_\bullet H)_n = \bigoplus_{p+q=n} (\mathbb{Z}G)^{G_p} \otimes_{\mathbb{Z}} (\mathbb{Z}H)^{H_q} \simeq \bigoplus_{p+q=n,\, g\in G_p\, h\in H_q} \mathbb{Z}G\otimes_{\mathbb{Z}}\mathbb{Z}H$$

for $n \geq 0$, using ((1), Proposition III.7.3). We can give $\mathbb{Z}G\otimes_{\mathbb{Z}}\mathbb{Z}H$ a D-module structure: for all $(g,h)\in D$, define maps $\bar{\Psi}_{g,h} : \mathbb{Z}G\times\mathbb{Z}H \to \mathbb{Z}G\otimes_{\mathbb{Z}}\mathbb{Z}H$ by $\bar{\Psi}_{g,h}(a_1,a_2) = (ga_1)\otimes(ha_2)$ for $a_1\in\mathbb{Z}G$, $a_2\in\mathbb{Z}H$. They are clearly $\mathbb{Z}$-bilinear and induce homomorphisms $\Psi_{g,h} : \mathbb{Z}G\otimes_{\mathbb{Z}}\mathbb{Z}H \to \mathbb{Z}G\otimes_{\mathbb{Z}}\mathbb{Z}H$. The desired D-module structure is then given by

$$(g,h)(a_1\otimes a_2) := \Psi_{g,h}(a_1\otimes a_2) = (ga_1)\otimes(ha_2), \quad g\in G,\ h\in H,\ a_1\in\mathbb{Z}G,\ a_2\in\mathbb{Z}H,$$

All abelian groups in $E_\bullet G\otimes_{\mathbb{Z}} E_\bullet H$ thus become D-modules and it is clear that the differentials on $E_\bullet G\otimes_{\mathbb{Z}} E_\bullet H$ respect this structure. Also, $\mathbb{Z}G\otimes_{\mathbb{Z}}\mathbb{Z}H$ is a free D-module: define the mapping over $\mathbb{Z}$-modules $\bar{\xi} : \mathbb{Z}G\times\mathbb{Z}H \to \mathbb{Z}D$ by

$$\bar{\xi}\left(\sum_{g\in G} z_g g, \sum_{h\in H} z_h h\right) = \sum_{(g,h)\in D} z_g z_h (g,h);$$

it is clearly bilinear, and so induces a homomorphism $\xi : \mathbb{Z}G\otimes_{\mathbb{Z}}\mathbb{Z}H \to \mathbb{Z}D$. It is a D-module homomorphism, and a composition of isomorphisms of direct sums and tensor products (again using (1), Proposition III.7.3) by considering the commutative diagram

$$\begin{array}{ccccc} \mathbb{Z}G\otimes_{\mathbb{Z}}\mathbb{Z}H & \xrightarrow{\ \xi\ } & \mathbb{Z}D & \xleftarrow{\ \simeq\ } & \bigoplus_{(g,h)\in D}\mathbb{Z} \\ \downarrow\simeq & & & & \uparrow\simeq \\ \bigoplus_{g\in G}\mathbb{Z}\otimes_{\mathbb{Z}}\bigoplus_{h\in H}\mathbb{Z} & \xrightarrow{\ \simeq\ } & \bigoplus_{h\in H}\left(\left(\bigoplus_{g\in G}\mathbb{Z}\right)\otimes_{\mathbb{Z}}\mathbb{Z}\right) & \xrightarrow{\ \simeq\ } & \bigoplus_{(g,h)\in D}\mathbb{Z}\otimes_{\mathbb{Z}}\mathbb{Z} \end{array}$$

Thus it follows that $(E_\bullet G \otimes_{\mathbb{Z}} E_\bullet H)_n$ is D-free as well, being a direct sum of free D-modules, so that $E_\bullet G \otimes_{\mathbb{Z}} E_\bullet H$ becomes a free D-resolution of $\mathbb{Z}$, using the above discussion. Since any two projective resolutions of a module are homotopy equivalent, it follows now that the chain complexes $E_\bullet G \otimes_{\mathbb{Z}} E_\bullet H$ and $E_\bullet D$ are homotopy equivalent. Since the functor $-_D$ is an additive covariant functor $\mathfrak{M}^\ell_D \to \mathbf{Ab}$, it then follows from ((1), Lemma IV.3.4) that the chain complexes of abelian groups $B_\bullet D$ and $(E_\bullet G \otimes_{\mathbb{Z}} E_\bullet H)_D$ are homotopy equivalent. As

$$\begin{aligned}
((E_\bullet G \otimes_{\mathbb{Z}} E_\bullet H)_n)_D &\simeq \mathbb{Z} \otimes_{\mathbb{Z}D} \left(\bigoplus_{p+q=n,\, g\in G_p,\, h\in H_q} \mathbb{Z}G \otimes_{\mathbb{Z}} \mathbb{Z}H \right) \\
&\simeq \mathbb{Z} \otimes_{\mathbb{Z}D} \left(\bigoplus_{p+q=n,\, g\in G_p,\, h\in H_q} \mathbb{Z}D \right) \\
&\simeq \bigoplus_{p+q=n,\, g\in G_p,\, h\in H_q} \mathbb{Z} \otimes_{\mathbb{Z}D} \mathbb{Z}D \\
&\simeq \bigoplus_{p+q=n,\, g\in G_p,\, h\in H_q} \mathbb{Z} \\
&\simeq \bigoplus_{p+q=n,\, g\in G_p,\, h\in H_q} \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} \\
&\simeq \bigoplus_{p+q=n,\, g\in G_p,\, h\in H_q} (\mathbb{Z} \otimes_{\mathbb{Z}G} \mathbb{Z}G) \otimes_{\mathbb{Z}} (\mathbb{Z} \otimes_{\mathbb{Z}H} \mathbb{Z}H) \\
&\simeq \bigoplus_{p+q=n,\, g\in G_p,\, h\in H_q} (\mathbb{Z}G)_G \otimes_{\mathbb{Z}} (\mathbb{Z}H)_H \\
&\simeq \bigoplus_{p+q=n} ((\mathbb{Z}G)_G)^{G_p} \otimes_{\mathbb{Z}} ((\mathbb{Z}H)_H)^{H_q} \\
&\simeq \bigoplus_{p+q=n} \left((\mathbb{Z}G)^{G_p}\right)_G \otimes_{\mathbb{Z}} \left((\mathbb{Z}H)^{H_q}\right)_H \\
&= (B_\bullet G \otimes_{\mathbb{Z}} B_\bullet H)_n,
\end{aligned}$$

by using *a lot* of tensor product isomorphisms of abelian groups and the fact that $-_A$ for all groups A is additive. One can verify that the isomorphism is also a chain map, and so we obtain the desired homotopy equivalence $B_\bullet D \simeq B_\bullet G \otimes_{\mathbb{Z}} B_\bullet H$ by composing this (complicated) isomorphism of chain complexes with the homotopy equivalences $B_\bullet D \simeq (E_\bullet G \otimes_{\mathbb{Z}} E_\bullet H)_D$.

(b)

Calculate the integral homology of the group $C_2 \times C_2$ in all dimensions.

We first calculate the integral homology of C_2 in all dimensions, as it will become useful in the following. Since $\mathbb{Z}C_2 = \mathbb{Z}[x]/(x^2-1)$, a free resolution of the trivial C_2-module $\mathbb{Z}$ is given by

$$E_\bullet C_2 \to \mathbb{Z}: \quad \cdots \longrightarrow \mathbb{Z}C_2 \xrightarrow{\cdot(x-1)} \mathbb{Z}C_2 \xrightarrow{\cdot(x+1)} \cdots \xrightarrow{\cdot(x+1)} \mathbb{Z}C_2 \xrightarrow{\cdot(x-1)} \mathbb{Z}C_2 \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0,$$

ε denoting the augmentation map. By the isomorphism of functors $-_{C_2} \cong \mathbb{Z} \otimes_{\mathbb{Z}C_2} -$, we then obtain

$$B_\bullet C_2: \quad \cdots \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \cdots \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$$

as we regard $\mathbb{Z}$ as a C_2-module via augmentation (so multiplication by $x \pm 1$ becomes multiplication by 1 ± 1). Thus the integral homology of C_2 is

$$H_n(C_2;\mathbb{Z}) = H_n((E_\bullet C_2)_{C_2}) = H_n(B_\bullet C_2) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}_2 & n > 0,\, n \text{ odd} \\ 0 & n > 0,\, n \text{ even and all } n < 0. \end{cases}$$

Let $D = C_2 \times C_2$; we now calculate $H_n(D;\mathbb{Z})$ for all $n \in \mathbb{Z}$. Take a D-projective resolution $E_\bullet D$ of $\mathbb{Z}$ and a D-projective resolution $E_\bullet D$ of $\mathbb{Z}$; then $H_n(D;\mathbb{Z}) = H_n((E_\bullet D)_D)$. Using the homotopy equivalence obtained in 3(a), we obtain

$$H_n(D;\mathbb{Z}) = H_n((E_\bullet D)_D) = H_n(B_\bullet D) \simeq H_n(B_\bullet C_2 \otimes_{\mathbb{Z}} B_\bullet C_2),$$

$B_\bullet C_2$ denoting the chain complex found earlier. Since $\mathbb{Z}$ is a PID and $B_\bullet C_2$ consists entirely of free and hence flat $\mathbb{Z}$-modules, by the Künneth theorem ((1), Theorem V.2.1), we obtain

$$H_n(D;\mathbb{Z}) \simeq \left(\bigoplus_{p+q=n} H_p(B_\bullet C_2) \otimes_{\mathbb{Z}} H_q(B_\bullet C_2)\right) \oplus \left(\bigoplus_{p+q=n-1} \mathrm{Tor}_1^{\mathbb{Z}}(H_p(B_\bullet C_2), H_q(B_\bullet C_2)\right).$$

If any of the above p or q are negative, then the corresponding terms in the sum are 0 by the integral homology of C_2; thus we need only consider p and q that are simultaneously non-negative. We consider the various cases:

- If $n < 0$, it's clear that $H_n(D;\mathbb{Z}) = 0$.
- If $n = 0$, the second term is 0; the first term boils down to $H_0(B_\bullet C_2) \otimes_{\mathbb{Z}} H_0(B_\bullet C_2)$ in the same way. As $H_0(B_\bullet C_2) \otimes_{\mathbb{Z}} H_0(B_\bullet C_2) = \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} \simeq \mathbb{Z}$, we obtain $H_0(D;\mathbb{Z}) \simeq \mathbb{Z}$.
- If $n = 1$, the second term boils down to $\mathrm{Tor}_1^{\mathbb{Z}}(H_0(B_\bullet C_2), H_0(B_\bullet C_2)) = \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}) = 0$, since $\mathbb{Z}$ is free; the first term boils down to

  $$(H_1(B_\bullet C_2) \otimes_{\mathbb{Z}} H_0(B_\bullet C_2)) \oplus (H_0(B_\bullet C_2) \otimes_{\mathbb{Z}} H_1(B_\bullet C_2)) \simeq (\mathbb{Z}_2 \otimes_{\mathbb{Z}} \mathbb{Z}) \otimes (\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}_2) \simeq (\mathbb{Z}_2)^2.$$

  Alas $H_1(D;\mathbb{Z}) \simeq (\mathbb{Z}_2)^2$.
- If $n > 1$ and n is odd, then $n-1$ is even. The second term has terms equal to 0 if and only if q is even and ≥ 0, as $\mathbb{Z}$ is free, and thus the second term boils down to ${}^{n-1}/_2$ copies of $\mathrm{Tor}_1^{\mathbb{Z}}(H_p(B_\bullet C_2), H_q(B_\bullet C_2))$ for p, q odd, i.e.

  $$\mathrm{Tor}_1^{\mathbb{Z}}(H_p(B_\bullet C_2), H_q(B_\bullet C_2)) = \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}_2, \mathbb{Z}_2).$$

  A $\mathbb{Z}$-free resolution of $\mathbb{Z}_2$ is $\mathbb{Z} \xrightarrow{2} \mathbb{Z}$ whose first homology group when tensored with $\mathbb{Z}_2$ is isomorphic to $\mathbb{Z}_2$; alas $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}_2,\mathbb{Z}_2) \simeq \mathbb{Z}_2$. The first term has terms equal to 0 if and only if $0 < p, q < n$, since if one is odd, then the other is even, the resulting group being 0. Thus it boils down to

  $$(H_0(B_\bullet C_2) \otimes_{\mathbb{Z}} H_n(B_\bullet C_2)) \oplus (H_n(B_\bullet C_2) \otimes_{\mathbb{Z}} H_0(B_\bullet C_2)) \simeq (\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}_2) \otimes (\mathbb{Z}_2 \otimes_{\mathbb{Z}} \mathbb{Z}) \simeq (\mathbb{Z}_2)^2.$$

  Therefore $H_n(D;\mathbb{Z}) \simeq (\mathbb{Z}_2)^2 \oplus (\mathbb{Z}_2)^{{}^{n-1}/_2} = (\mathbb{Z}_2)^{{}^{n+3}/_2}$.
- If $n > 1$ and n is even, then $n-1$ is odd. The second term then has terms equal to 0 if $q = 0$ or q is even (as $\mathbb{Z}$ is free), so we need only check the cases where q is odd. Thus p must be even, but for the term to be non-zero, then p must be 0. Thus the second term boils down to calculating $\mathrm{Tor}_1^{\mathbb{Z}}(H_0(B_\bullet C_2), H_{n-1}(B_\bullet C_2)) = \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}_2)$. Using the balancing of Tor and that $\mathbb{Z}$ is free, we then obtain $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}_2) = 0$, so there are no contributions from the second term. The first term has terms equal to 0 if and only if p and q are both even; summing over all p and q odd, we then obtain ${}^{n}/_2$ copies of $(H_p(B_\bullet C_2) \otimes_{\mathbb{Z}} H_q(B_\bullet C_2)) = \mathbb{Z}_2 \otimes_{\mathbb{Z}} \mathbb{Z}_2$. Since $\mathbb{Z}_2 \otimes_{\mathbb{Z}} \mathbb{Z}_2$ is an abelian group of order 2 (its only non-zero element being $1 \oplus 1$, out of the four "possible" elements), we obtain $H_n(D;\mathbb{Z}) \simeq (\mathbb{Z}_2)^{{}^{n}/_2}$.

To summarize:

$$H_n(C_2 \times C_2;\mathbb{Z}) \simeq \begin{cases} 0 & n < 0 \\ \mathbb{Z} & n = 0 \\ (\mathbb{Z}_2)^{{}^{n+3}/_2} & n > 0,\ n \text{ odd} \\ (\mathbb{Z}_2)^{{}^{n}/_2} & n > 0,\ n \text{ even.} \end{cases}$$

(4)

(a) (1), Exercise IV.9.4

Define addition in $\mathrm{Yext}^m_\Lambda(A,B)$*, independently of the equivalence* $\mathrm{Ext}^m_\Lambda \simeq \mathrm{Yext}^m_\Lambda$*. Describe a representative of* $0 \in \mathrm{Yext}^m_\Lambda(A,B)$*,* $m \geq 2$ *and show that* $\xi + 0 = \xi$*,* $\xi \in \mathrm{Yext}^m_\Lambda(A,B)$*.*

First follows an extensive explanation of $\mathrm{Yext}^m_\Lambda(A,B)$ and the tools needed to define addition. For Λ-modules A and B, $\mathrm{Yext}^m_\Lambda(A,B)$ consists of all equivalence classes (under a specific relation to be mentioned later) of m-extensions of A by B; these are exact sequences of Λ-modules

$$\mathfrak{E}: \quad 0 \longrightarrow B \xrightarrow{\mu} E_m \xrightarrow{\varepsilon_m} \cdots \longrightarrow E_1 \xrightarrow{\varepsilon} A \longrightarrow 0. \tag{2}$$

A morphism $f : \mathfrak{E} \to \mathfrak{E}'$ of m-extensions of A by B is a sequence of homomorphisms $f_i : E_i \to E'_i$, $i = 1, \ldots, m$ such that the diagram

$$\begin{array}{lccccccccccc}
\mathfrak{E}: & 0 & \longrightarrow & B & \longrightarrow & E_m & \longrightarrow \cdots \longrightarrow & E_1 & \longrightarrow & A & \longrightarrow & 0 \\
& & & \downarrow{=} & & \downarrow{f_m} & & \downarrow{f_1} & & \downarrow{=} & & \\
\mathfrak{E}': & 0 & \longrightarrow & B & \longrightarrow & E'_m & \longrightarrow \cdots \longrightarrow & E'_1 & \longrightarrow & A & \longrightarrow & 0
\end{array}$$

commutes. Two m-extensions $\mathfrak{E}$ and $\mathfrak{E}'$ of A of B are equivalent (we write $\mathfrak{E} \sim \mathfrak{E}'$) if there exists a sequence of morphisms between $\mathfrak{E}$ and $\mathfrak{E}'$, in the sense that there are a finite number of m-extensions $\mathfrak{X}_1, \ldots, \mathfrak{X}_n$ and morphisms such that we obtain a diagram

$$\mathfrak{E} \leftrightarrow \mathfrak{X}_1 \leftrightarrow \cdots \leftrightarrow \mathfrak{X}_n \leftrightarrow \mathfrak{E}',$$

$\leftrightarrow$ meaning that there is a morphism from one of the m-extensions to the other; this is clearly an equivalence relation, and $\mathrm{Yext}^n_\Lambda(A,B)$ is the set of its equivalence classes.

1. Induced homomorphisms in the first variable. If $\alpha : A' \to A$ is a homomorphism of Λ-modules, it induces a map $\alpha^* : \mathrm{Yext}^m_\Lambda(A,B) \to \mathrm{Yext}^m_\Lambda(A',B)$ as follows: given $[\mathfrak{E}] \in \mathrm{Yext}^m_\Lambda(A,B)$, we obtain a commutative diagram

$$\begin{array}{ccccccccc}
E_3 & \xrightarrow{\varepsilon_3} & E_2 & \xrightarrow{\varepsilon_2} & E_1 & \xrightarrow{\varepsilon} & A & \longrightarrow & 0 \\
& & & & \uparrow{\beta} & & \uparrow{\alpha} & & \\
& & & & E_1^\alpha & \xrightarrow{\varepsilon'} & A' & \longrightarrow & 0,
\end{array}$$

where E_1^α denotes the Λ-module $\{(e, a') \in E_1 \oplus A' \,|\, \varepsilon(e) = \alpha(a')\}$ and ε' and β are defined by $\varepsilon'(e,a') = a'$ and $\beta(e,a') = e$ for all $e \in E_1$, $a \in A'$. It is a pullback of the diagram since if $\gamma : Z \to A'$ and $\rho : Z \to E_1$ are morphisms such that $\alpha\gamma(z) = \varepsilon\rho(z)$ for all $z \in Z$, then $(\rho(z), \gamma(z)) \in E_1^\alpha$ for all $z \in Z$, so defining the map $(\rho, \gamma) : Z \to E_1^\alpha$ by $z \mapsto (\rho(z), \gamma(z))$, we obtain a morphism making the new triangles commute, and it is unique, since if a morphism $\zeta : Z \to E_1^\alpha$ satisfies $\varepsilon'\zeta = \gamma$ and $\beta\zeta = \rho$, then $\zeta = (\rho, \gamma)$; therefore $(E^1_\alpha; \beta, \varepsilon')$ is unique with this property (which is indeed necessary for the induced map to be well-defined). Note what follows by exactness of the upper row:

- ε' is surjective by surjectivity of ε, since if $a' \in A$, then $\varepsilon(e) = \alpha(a')$ for some $e \in E_1$, so $\varepsilon'(e,a') = a'$.
- Define a homomorphism $d_2 : E_2 \to E_1^\alpha$ by $d_2(e) = (\varepsilon_2(e), 0)$. Then $\beta d_2 = \varepsilon_2$ and it's clear that $\mathrm{im} d_2 \subseteq \ker \varepsilon'$; to prove the other inclusion, let $(e, a') \in \ker \varepsilon'$. Then $a' = 0$ and $\varepsilon(e) = \alpha(0) = 0$, so $e \in \ker \varepsilon = \mathrm{im}\varepsilon_2$ by exactness of the upper row; choosing $e' \in E_2$ such that $\varepsilon_2(e') = e$, then $d_2(e') = (e, a')$. Thus $\mathrm{im} d_2 = \ker \varepsilon'$. Note that for any $e'' \in E_3$, $\varepsilon_2(\varepsilon_3(e'')) = 0$, so $\varepsilon_3(e'') \in \ker d_2$. On the other hand, then if $d_2(e) = (\varepsilon_2(e), 0) = 0$, then $\varepsilon_2(e) = 0$, so $e \in \mathrm{im}\varepsilon_3$.

Thus, we obtain a commutative diagram with exact rows

$$\begin{array}{lccccccccccccc}
\mathfrak{E}: & 0 & \longrightarrow & B & \longrightarrow & E_m & \longrightarrow \cdots \xrightarrow{\varepsilon_3} & E_2 & \xrightarrow{\varepsilon_2} & E_1 & \xrightarrow{\varepsilon} & A & \longrightarrow & 0 \\
& & & \uparrow{=} & & \uparrow{=} & & \uparrow{=} & & \uparrow{\beta} & & \uparrow{\alpha} & & \\
\mathfrak{E}^\alpha: & 0 & \longrightarrow & B & \longrightarrow & E_m & \longrightarrow \cdots \xrightarrow{\varepsilon_3} & E_2 & \xrightarrow{d_2} & E_1^\alpha & \xrightarrow{\varepsilon'} & A' & \longrightarrow & 0,
\end{array}$$

that is unique by the property of the last square that is a pullback (note that this holds no matter what m is). If $\mathfrak{E}' \sim \mathfrak{E}'$, then there exist morphisms of extensions connecting them; i.e.

$$\mathfrak{E} \leftrightarrow \mathfrak{X}_1 \leftrightarrow \cdots \leftrightarrow \mathfrak{X}_n \leftrightarrow \mathfrak{E}',$$

we obtain $\mathfrak{E}^\alpha \leftrightarrow \mathfrak{X}_1^\alpha \leftrightarrow \cdots \leftrightarrow \mathfrak{X}_n^\alpha \leftrightarrow \mathfrak{E}'^\alpha$ by composing the morphisms with the homomorphisms obtained by the ones induced by α in the right way, depending on which ways the arrows go. Thus we obtain a well-defined homomorphism $\alpha^*([\mathfrak{E}]) = [\mathfrak{E}^\alpha]$.

2. Induced homomorphisms in the second variable. If $\beta : B \to B'$ is a homomorphism of Λ-modules, it induces a map $\beta_* : \mathrm{Yext}_\Lambda^m(A, B) \to \mathrm{Yext}_\Lambda^m(A, B')$ as follows: given $[\mathfrak{E}] \in \mathrm{Yext}_\Lambda^m(A, B)$, we obtain a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \xrightarrow{\mu} & E_m & \xrightarrow{\varepsilon_m} & E_{m-1} & \xrightarrow{\varepsilon_{m-1}} & E_{m-2} \\ & & \downarrow{\scriptstyle\beta} & & \vdots{\scriptstyle\omega} & & & & \\ 0 & \longrightarrow & B' & \overset{\mu'}{\dashrightarrow} & E_{m\beta}, & & & & \end{array}$$

where $E_{m\beta}$ denotes the Λ-module $E_m \oplus B'/\{(\mu(b), -\beta(b)) \,|\, b \in B\}$ and μ' and ω are defined by $\mu'(b') = [(0, b')]$ and $\omega(e) = [(e, 0)]$ for all $e \in E_m$, $b' \in B'$. It is a pushout of the diagram since if $\gamma : E_m \to Z$ and $\rho : B' \to Z$ are morphisms such that $\gamma\mu(b) = \rho\beta(b)$ for all $b \in B$, then consider the homomorphism $E_m \oplus B' \to Z$ given by $(e, b') \mapsto \gamma(e) + \rho(b')$. If $(e_1 - e_2, b_1' - b_2') = (\mu(b), -\beta(b))$ for $(e_1, b_1'), (e_2, b_2') \in E_m \oplus B'$ and some $b \in B$, then

$$\gamma(e_1) + \rho(b_1') - (\gamma(e_2) + \rho(b_2')) = \gamma(e_1 - e_2) - \rho(b_1' - b_2') = \gamma\mu(b) - \rho\beta(b) = 0$$

so defining the map $\psi : E_{m\beta} \to Z$ by $[(e, b')] \mapsto \gamma(e) + \rho(b')$, we obtain a well-defined homomorphism making the new triangles commute, and it is unique, since if a morphism $\zeta : E_{m\beta} \to Z$ satisfies $\zeta\mu' = \rho$ and $\zeta\omega = \gamma$, then

$$\zeta([(e, b')]) = \zeta([(e, 0)]) + \zeta([(0, b')]) = \rho(e) + \gamma(b) = \psi([(e, b')]);$$

therefore $(E_{m\beta}; \mu', \omega)$ is unique with this property (which is indeed necessary for the induced map to be well-defined). Note what follows by exactness of the upper row:

- μ' is injective, since $\mu'(b') = [(0, b')] = 0$, then $(0, b') = (\mu(b), -\beta(b))$ for some $b \in B$; since μ is injective, $b = 0$, so $b' = -\beta(b) = 0$.
- Define a homomorphism $d_m : E_{m\beta} \to E_{m-1}$ by $d_m([(e, b')]) = \varepsilon_m(e)$. It is well-defined, since if $(e_1 - e_2, b_1' - b_2') = (\mu(b), -\beta(b))$ for $(e_1, b_1'), (e_2, b_2') \in E_m \oplus B'$ and some $b \in B$, then $\varepsilon_m(e_1) - \varepsilon_m(e_2) = \varepsilon_m\mu(b) = 0$. It's clear that $d_m\omega = \varepsilon_m$ and that $\mathrm{im}\mu' \subseteq \ker d_m$. On the other hand, if $[(e, b')] \in \ker d_m$, then $\varepsilon_m(e) = 0$, so $e = \mu(b)$ for some $b \in B$; therefore $[(e, b')] = [(0, b' + \beta(b))]$ and $\mu'(b' + \beta(b)) = [(e, b')]$. It's clear that $\mathrm{im} d_m([(e, b')]) \subseteq \ker \varepsilon_{m-1}$; if $\varepsilon_{m-1}(e') = 0$, then $\varepsilon_{m_2}(e) = e'$ for some $e \in E_m$ and $d_m([e, 0]) = e'$.

Thus, we obtain a commutative diagram with exact rows

$$\begin{array}{rccccccccccccc} \mathfrak{E}: & 0 & \longrightarrow & B & \xrightarrow{\mu} & E_m & \xrightarrow{\varepsilon_m} & E_{m-1} & \xrightarrow{\varepsilon_{m-1}} \cdots \longrightarrow & E_1 & \longrightarrow & A & \longrightarrow & 0 \\ & & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\omega} & & \downarrow{\scriptstyle=} & & \downarrow{\scriptstyle=} & & \downarrow{\scriptstyle=} & & \\ \mathfrak{E}_\beta: & 0 & \longrightarrow & B' & \xrightarrow{\mu'} & E_{m\beta} & \xrightarrow{d_m} & E_{m-1} & \xrightarrow{\varepsilon_{m-1}} \cdots \longrightarrow & E_1 & \longrightarrow & A & \longrightarrow & 0, \end{array}$$

that is unique. In the same manner as before, we obtain a well-defined homomorphism $\beta_*([\mathfrak{E}]) = [\mathfrak{E}_\beta]$.

3. The addition. Note that for two m-extensions $\mathfrak{E} = (E_i)_{i=1}^m$, $\mathfrak{E}' = (E_i')_{i=1}^m$ of A by B, by taking the direct sum of the sequences, we obtain an m-extension of $A \oplus A$ by $B \oplus B$:

$$\mathfrak{E} \oplus \mathfrak{E}' : \quad 0 \longrightarrow B \oplus B \longrightarrow E_m \oplus E_m' \longrightarrow \cdots \longrightarrow E_1 \oplus E_1' \longrightarrow A \oplus A \longrightarrow 0.$$

By defining homomorphisms $\Delta : A \to A \oplus A$, $\Delta(a) = (a, a)$ and $\nabla : B \oplus B \to B$, $\nabla(b, b') = b + b'$, we can now define

$$[\mathfrak{E}] + [\mathfrak{E}'] := \Delta^*\nabla_*([\mathfrak{E} \oplus \mathfrak{E}']), \quad [\mathfrak{E}], [\mathfrak{E}'] \in \mathrm{Yext}_\Lambda^m(A, B),$$

which is well-defined by well-definedness of the induced homomorphisms. This composition in fact allows for an abelian group structure on Yext as we shall see in the next problem.

We will now find a representative of a neutral element under this composition. Consider the m-extension of A by B given by

$$0: \quad 0 \longrightarrow B \xrightarrow{1} B \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow A \xrightarrow{1} A \longrightarrow 0.$$

It is clearly exact, and there is reason behind its name as we shall now see. Let $\mathfrak{E}$ be the extension (2). With Δ and ∇ as defined above, recall how we defined the induced homomorphisms and consider the following diagram with exact rows:

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & B \oplus B & \xrightarrow{(\mu,1)} & E_m \oplus B & \xrightarrow{\binom{\varepsilon_m}{0}} & E_{m-1} & \xrightarrow{\varepsilon_{m-1}} \cdots \xrightarrow{(\varepsilon_2,0)} & E_1 \oplus A & \xrightarrow{(\varepsilon,1)} & A \oplus A & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \nabla} & & \downarrow{\scriptstyle \omega} & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle =} & & \\
0 & \longrightarrow & B & \xrightarrow{\mu'} & (E_m \oplus B)_\nabla & \xrightarrow{d_m} & E_{m-1} & \xrightarrow{\varepsilon_{m-1}} \cdots \xrightarrow{(\varepsilon_2,0)} & E_1 \oplus A & \xrightarrow{(\varepsilon,1)} & A \oplus A & \longrightarrow & 0,
\end{array}$$

the upper row being $\mathfrak{E} \oplus 0$ and the lower row being $(\mathfrak{E} \oplus 0)_\nabla$. Consider now

$$\begin{array}{cccccccccccccc}
(\mathfrak{E} \oplus 0)_\nabla : & 0 & \longrightarrow & B & \xrightarrow{\mu'} & (E_m \oplus B)_\nabla & \xrightarrow{d_m} & E_{m-1} & \xrightarrow{\varepsilon_{m-1}} \cdots \xrightarrow{(\varepsilon_2,0)} & E_1 \oplus A & \xrightarrow{(\varepsilon,1)} & A \oplus A & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle =} & & \\
\mathfrak{E}' : & 0 & \longrightarrow & B & \xrightarrow{-\mu} & E_m & \xrightarrow{\varepsilon_m} & E_{m-1} & \xrightarrow{\varepsilon_{m-1}} \cdots \xrightarrow{(\varepsilon_2,0)} & E_1 \oplus A & \xrightarrow{(\varepsilon,1)} & A \oplus A & \longrightarrow & 0,
\end{array}$$

with $\xi : (E_m \oplus B)_\nabla \to E_m$ defined by $[((e,b),b')] \mapsto e + \mu(b - b')$. We will show that it makes the diagram commute. It is well-defined, since if $((e_1,b_1),b_1') - ((e_2,b_2),b_2') = ((\mu(b),b'),b+b')$ for $e_1, e_2 \in E_m$ and $b, b', b_1, b_1', b_2, b_2' \in B$ (using the definition of $(E_m \oplus B)_\nabla$), then $e_1 - e_2 = \mu(b)$, $b_1 - b_2 = b'$ and $b_1' - b_2' = b + b'$. Thus

$$e_1 + \mu(b_1 - b_1') - (e_2 + \mu(b_2 - b_2')) = \mu(b) + \mu(b_1 - b_2) - \mu(b_1' - b_2') = \mu(b) + \mu(b') - \mu(b + b') = 0,$$

so ξ is well-defined. For $[((e,b),b')] \in (E_m \oplus B)_\nabla$, then

$$d_m([((e,b),b')]) = \begin{pmatrix} \varepsilon_m \\ 0 \end{pmatrix}(e,b) = \varepsilon_m(e) = \varepsilon_m(e + \mu(b - b')) = \varepsilon_m \xi([((e,b),b')])$$

and $\xi\mu'(b) = \xi([((0,0),b)]) = -\mu(b)$ for all $b \in B$. The lower row is exact, and thus $(\mathfrak{E} \oplus 0)_\nabla \sim \mathfrak{E}'$, so $\nabla_*([\mathfrak{E} \oplus 0]) = [\mathfrak{E}']$. Now consider the diagram

$$\begin{array}{cccccccccccccc}
\mathfrak{E}' : & 0 & \longrightarrow & B & \xrightarrow{-\mu} & E_m & \xrightarrow{\varepsilon_m} \cdots \longrightarrow & E_2 & \xrightarrow{(\varepsilon_2,0)} & E_1 \oplus A & \xrightarrow{(\varepsilon,1)} & A \oplus A & \longrightarrow & 0 \\
 & & & \uparrow{\scriptstyle =} & & \uparrow{\scriptstyle =} & & \uparrow{\scriptstyle =} & & \uparrow & & \uparrow{\scriptstyle \Delta} & & \\
(\mathfrak{E}')^\Delta : & 0 & \longrightarrow & B & \xrightarrow{-\mu} & E_m & \xrightarrow{\varepsilon_m} \cdots \longrightarrow & E_2 & \xrightarrow{d_2} & (E_1 \oplus A)^\Delta & \xrightarrow{\varepsilon'} & A & \longrightarrow & 0,
\end{array}$$

and then the diagram

$$\begin{array}{cccccccccccccc}
\mathfrak{E} : & 0 & \longrightarrow & B & \xrightarrow{\mu} & E_m & \xrightarrow{\varepsilon_m} \cdots \longrightarrow & E_2 & \xrightarrow{\varepsilon_2} & E_1 & \xrightarrow{\varepsilon} & A & \longrightarrow & 0 \\
 & & & {\scriptstyle -1}\uparrow{\scriptstyle \cong} & & \uparrow{\scriptstyle =} & & \uparrow{\scriptstyle =} & & \uparrow{\scriptstyle \Psi} & & \uparrow{\scriptstyle =} & & \\
(\mathfrak{E}')^\Delta : & 0 & \longrightarrow & B & \xrightarrow{-\mu} & E_m & \xrightarrow{\varepsilon_m} \cdots \longrightarrow & E_2 & \xrightarrow{d_2} & (E_1 \oplus A)^\Delta & \xrightarrow{\varepsilon'} & A & \longrightarrow & 0,
\end{array}$$

with $\Psi : (E_1 \oplus A)^\Delta \to E_1$ given by $\Psi((e,a),a') = e$ for all $((e,a),a') \in (E_1 \oplus A) \oplus A$ such that $(\varepsilon(e), a) = (a', a')$. The last diagram commutes, since $\varepsilon\Psi(((e,a),a') = \varepsilon(e) = a' = \varepsilon'((e,a),a')$ for such $((e,a),a')$ and $\Psi d_2(e) = \Psi((\varepsilon_2(e),0),0) = \varepsilon_2(e)$ for all $e \in E_2$. Thus $(\mathfrak{E}')^\Delta \sim \mathfrak{E}$ since the upper row is exact, so we obtain

$$[\mathfrak{E}] + [0] = \Delta^*\nabla_*([\mathfrak{E} \oplus 0]) = \Delta^*[\mathfrak{E}'] = [\mathfrak{E}].$$

Since $\mathfrak{E} \oplus \mathfrak{E}' \sim \mathfrak{E}' \oplus \mathfrak{E}$ by using a coordinate-swapping isomorphism, we obtain $[0] + [\mathfrak{E}] = [\mathfrak{E}]$ as well, so 0 is a representative of the neutral element of the addition, and $\xi + 0 = \xi$, $\xi \in \mathrm{Yext}^m_\Lambda(A,B)$.

(b) (1), Exercise IV.9.6

Prove that the addition given above is compatible with the equivalence $\mathrm{Ext}^m_\Lambda \simeq \mathrm{Yext}^m_\Lambda$.

Let A and B be Λ-modules; then the natural isomorphism $\theta : \mathrm{Yext}^m_\Lambda(A, B) \to \mathrm{Ext}^m_\Lambda(A, B)$ is defined as follows. Let

$$\mathfrak{E} : \quad 0 \longrightarrow B \xrightarrow{\mu} E_m \xrightarrow{\varepsilon_m} \cdots \longrightarrow E_1 \xrightarrow{\varepsilon} A \longrightarrow 0$$

be an m-extension of A by B and take a projective resolution $P_\bullet$ of A. Then as $\mathrm{Ext}^m_\Lambda(A, B)$ is isomorphic to the homotopy classes of morphisms of complexes $P_\bullet \to \Sigma_m B$, we only need to find a representative morphism for $\mathfrak{E}$. Let $D_\bullet$ be the chain complex given by

$$D_\bullet : \quad \cdots \longrightarrow 0 \longrightarrow B \xrightarrow{\mu} E_m \xrightarrow{\varepsilon_m} \cdots \longrightarrow E_2 \xrightarrow{\varepsilon_2} E_1 \longrightarrow 0 \longrightarrow \cdots$$

with B in degree m and E_1 in degree 0. $\mathfrak{E}$ can be considered as a surjective quasi-isomorphism of chain complexes $\xi : D_\bullet \to A$, using exactness of $\mathfrak{E}$ in the following way:

$$\begin{tikzcd}
D_\bullet : \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & B \arrow[r, "\mu"] \arrow[d] & E_m \arrow[r, "\varepsilon_m"] \arrow[d] & \cdots \arrow[r] & E_2 \arrow[r, "\varepsilon_2"] \arrow[d] & E_1 \arrow[r] \arrow[d, "\varepsilon"] & 0 \arrow[r] & \cdots \\
A : \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & A \arrow[r, "\varepsilon"] & 0 \arrow[r] & \cdots .
\end{tikzcd}$$

In the same way, we obtain a morphism of complexes $\omega : P_\bullet \to A$. There now exists a morphism $\lambda : P_\bullet \to D_\bullet$ unique up to homotopy equivalence such that $\omega = \xi\lambda$ since the chain complexes are bounded below and $P_\bullet$ is projective. Since we also have a morphism of chain complexes $\psi : D_\bullet \to \Sigma_m B$ given by

$$\begin{tikzcd}
D_\bullet : \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & B \arrow[r, "\mu"] \arrow[d, "1"] & E_m \arrow[r, "\varepsilon_m"] \arrow[d] & \cdots \\
\Sigma_m B : \cdots \arrow[r] & 0 \arrow[r] & B \arrow[r] & 0 \arrow[r] & \cdots ,
\end{tikzcd}$$

we obtain a morphism of complexes $\psi\lambda : P_\bullet \to \Sigma_m B$. This is our representative, and we define $\theta([\mathfrak{E}]) = [\psi\lambda]$; it is well-defined and natural. The compatibility of the addition will follow once we prove for $[\mathfrak{E}], [\mathfrak{E}'] \in \mathrm{Yext}^m_\Lambda(A, B)$ that $\theta([\mathfrak{E}] + [\mathfrak{E}']) = \theta([\mathfrak{E}]) + \theta([\mathfrak{E}'])$. As θ is natural, we obtain

$$\theta([\mathfrak{E}] + [\mathfrak{E}']) = \theta([\Delta^*\nabla_*([\mathfrak{E} \oplus \mathfrak{E}'])]) = \Delta^*\nabla_*\theta([\mathfrak{E} \oplus \mathfrak{E}']),$$

using the same notation Δ^* and ∇^* for the homomorphisms induced from $\Delta : A \to A \oplus A$ and $\nabla : B \to B \oplus B$ over Ext.

Let the extension $\mathfrak{E}$, the complex $D_\bullet$ and the morphisms ξ, ω, λ and ψ given in the way above. In the case of another m-extension $\mathfrak{E}'$ of A by B (with E'_i, $i = 1, \dots, m$ being the modules in between B and A in the required exact sequence), we construct a chain complex $D'_\bullet$ analoguously along with a surjective quasi-isomorphism $\xi' : D'_\bullet \to A$, a morphism $\lambda' : P_\bullet \to D'_\bullet$ unique up to homotopy equivalence such that $\omega = \xi'\lambda'$ and a morphism $\psi' : D'_m \to \Sigma_m B$ such that the diagrams

$$\begin{tikzcd}
 & D_\bullet \arrow[r, "\psi"] \arrow[d, "\xi"] & \Sigma_m B \\
P_\bullet \arrow[ur, "\lambda"] \arrow[r, "\omega"] & A &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & D'_\bullet \arrow[r, "\psi'"] \arrow[d, "\xi'"] & \Sigma_m B \\
P_\bullet \arrow[ur, "\lambda'"] \arrow[r, "\omega"] & A &
\end{tikzcd}$$

commute. With this, we obtain $\theta([\mathfrak{E}]) = [\psi\lambda]$ and $\theta([\mathfrak{E}']) = [\psi'\lambda']$. To consider $\mathfrak{E} \oplus \mathfrak{E}'$, note that we only need take direct sums of the complexes and the morphisms, since $P_\bullet \oplus P_\bullet$ is a projective resolution of $A \oplus A$; since the diagram

$$\begin{tikzcd}
 & D_\bullet \oplus D'_\bullet \arrow[r, "\psi\oplus\psi'"] \arrow[d, "\xi\oplus\xi'"] & \Sigma_m(B \oplus B) \\
P_\bullet \oplus P_\bullet \arrow[ur, "\lambda\oplus\lambda'"] \arrow[r, "\omega\oplus\omega"] & A \oplus A &
\end{tikzcd}$$

then commutes, $\lambda \oplus \lambda'$ given by $(p, p') \mapsto (\lambda_n(p), \lambda'_n(p'))$ for all $n \in \mathbb{Z}$ and $p, p' \in P_n$ etc., we obtain $\theta([\mathfrak{E} \oplus \mathfrak{E}']) = [(\psi \oplus \psi')(\lambda \oplus \lambda')]$; note here that all morphisms making the diagram commute are homotopy-equivalent to $\lambda \oplus \lambda'$. We then get

$$\theta([\mathfrak{E}] + [\mathfrak{E}']) = \theta([\Delta^* \nabla_*([\mathfrak{E} \oplus \mathfrak{E}'])]) = \Delta^* \nabla_*[(\psi \oplus \psi')(\lambda \oplus \lambda')] = [\nabla(\psi \oplus \psi')(\lambda \oplus \lambda')\Delta].$$

The representative of the homotopy class is a morphism of complexes $P_\bullet \to \Sigma_m B$. For $n \in \mathbb{Z}$, note

$$\begin{aligned} \nabla(\psi_n \oplus \psi'_n)(\lambda_n \oplus \lambda'_n)\Delta(p) &= \nabla(\psi_n \oplus \psi'_n)(\lambda_n \oplus \lambda'_n)(p, p) \\ &= \nabla(\psi_n \oplus \psi'_n)(\lambda_n(p), \lambda'_n(p)) \\ &= \nabla(\psi_n \lambda_n(p), \psi'_n \lambda'_n(p)) \\ &= \psi_n \lambda_n(p) + \psi'_n \lambda'_n(p) \end{aligned}$$

for all $p \in P_n$, so that $\theta([\mathfrak{E}] + [\mathfrak{E}']) = [\psi\lambda + \psi'\lambda'] = [\psi\lambda] + [\psi'\lambda'] = \theta([\mathfrak{E}]) + \theta([\mathfrak{E}'])$, as wanted.

References

[1] P. J. Hilton and U. Stammbach, *A Course in Homological Algebra*, volume 4 of *Graduate Texts in Mathematics*, Springer-Verlag, second edition, 1971.