Sølvkorn 13
Cauchy completions

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Searching for notes on how to complete inner product spaces, I came up short. Hence this li'l nugget.

1 Metric spaces

First a relevant definition.

Definition 1. *Let M be a non-empty set. A* pseudometric *on M is a function $d : M \times M \to \mathbb{R}$ such that*

1. $d(x, y) \geq 0$ *and* $d(x, x) = 0$,
2. $d(x, y) = d(y, x)$ *and*
3. $d(x, y) \leq d(x, z) + d(z, y)$

for all $x, y, z \in M$. If d is a pseudometric on M, then (M, d) is called a pseudometric space*.*

Of course metrics are pseudometrics. The only thing that pseudometrics cannot guarantee is that $x = y$ whenever $d(x, y) = 0$ for $x, y \in M$, and in some cases we might want a pseudometric space to allow a construction of a metric space from what is given. In fact this is always allowed. Here is how we overcome the problem.

Lemma 2. *The relation on a pseudometric space (M, d) defined by $x \sim y$ iff $d(x, y) = 0$ is an equivalence relation.*

Proof. Clearly, $x \sim x$ from the definition. If $x \sim y$, then $d(y, x) = d(x, y) = 0$ so $y \sim x$. Finally, if $x \sim y$ and $y \sim z$, then $0 \leq d(x, z) \leq d(x, y) + d(y, z) = 0$, so $d(x, z) = 0$ and $x \sim z$. □

Definition 3. *The relation on the pseudometric space (M, d) as defined in Lemma 2 is called the* metric identification*.*

Here comes the construction. If (M, d) is a pseudometric space, then define $M_\sim$ to be the set of equivalence classes under the metric identification and define a map $d_\sim : M_\sim \times M_\sim \to \mathbb{R}$ by $d_\sim([x], [y]) = d(x, y)$ for $x, y \in M$.

Theorem 4. *$d_\sim$ is well-defined and a metric on $M_\sim$, making $(M_\sim, d_\sim)$ a metric space.*

Proof. Assume that $x \sim x'$ and $y \sim y'$. Then since $x' \sim x$ and $y' \sim y$, we obtain

$$d(x, y) \leq d(x, x') + d(x', y') + d(y', y) = d(x', y')$$

and

$$d(x', y') \leq d(x', x) + d(x, y) + d(y, y') = d(x, y),$$

so that $d(x, y) = d(x', y')$; hence $d_\sim$ is well-defined. For $x, y, z \in M$, note that

1. $d_\sim([x], [y]) = d(x, y) \leq d(x, z) + d(z, y) = d_\sim([x], [z]) + d_\sim([z], [y])$,
2. $d_\sim([x], [y]) = d(x, y) = d(y, x) = d_\sim([y], [x])$ and
3. $d_\sim([x], [y]) = d(x, y) \geq 0$.

Finally, assume that $d_\sim([x], [y]) = 0$. Then $d(x, y) = 0$ so $x \sim y$ and $[x] = [y]$. Since M is non-empty, $M_\sim$ is non-empty, so $d_\sim$ is a metric on $M_\sim$. □

This construction proves quite helpful in cases when we do not immediately have a metric space at hand.

In the following, let (M, d) be a metric space with metric d and let $\mathcal{F}_C(M)$ denote the set of Cauchy sequences in M. The title of this note may lead one into thinking that it can make (M, d) complete, but to do so, it needs to be a metric space itself. To be on the safe side, we will define a pseudometric on $\mathcal{F}_C(M)$. Here we essentially just take the most obvious construction we can think of and prove that it works. Define a function $D : \mathcal{F}_C(M) \times \mathcal{F}_C(M) \to \mathbb{R}$ by

$$D(x, y) = \lim_{n\to\infty} d(x_n, y_n)$$

for Cauchy sequences $x = (x_n)_{n=1}^{\infty}$, $y = (y_n)_{n=1}^{\infty}$ with $x_n, y_n \in M$ for all $n \in \mathbb{N}$. Jackpot!

Theorem 5. *D is a pseudometric on $\mathcal{F}_C(M)$.*

Proof. First and foremost, D is well-defined. Indeed, let $x = (x_n)$, $y = (y_n) \in \mathcal{F}_C(M)$ and $\varepsilon > 0$. We claim that the sequence $(d(x_n, y_n))$ in $\mathbb{R}$ (equipped with the usual metric) is Cauchy. Indeed, there exists $N \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon/2$ og $d(y_n, y_m) < \varepsilon/2$ for $m, n \geq N$, x and y being Cauchy sequences (we pick one for each sequence and let our N be the largest of the two). Then for $m, n \geq N$ we have

$$\begin{aligned} d(x_m, y_m) - d(x_n, y_n) &\leq d(x_m, x_n) + d(x_n, y_m) - d(x_n, y_n) \\ &\leq d(x_m, x_n) + d(y_n, y_m) < \varepsilon/2 + \varepsilon/2 = \varepsilon, \end{aligned}$$

since

$$d(x_n, y_m) \leq d(x_n, y_n) + d(y_n, y_m).$$

Similarly,

$$\begin{aligned} d(x_n, y_n) - d(x_m, y_m) &\leq d(x_n, x_m) + d(x_m, y_n) - d(x_m, y_m) \\ &\leq d(x_n, x_m) + d(y_m, y_n) < \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Hence $|d(x_n, y_n) - d(x_m, y_m)| < \varepsilon$ for $m, n \geq N$. Since $(d(x_n, y_n))$ is Cauchy in $\mathbb{R}$, it converges, proving that D is well-defined.

It is then routine to check that D is a pseudo-metric. For Cauchy sequences $x = (x_n)$, $y = (y_n)$ and $z = (z_n)$ in $\mathcal{F}_C(M)$ we have $d(x_n, y_n) \geq 0$ for all $n \in \mathbb{N}$; hence

$$D(x, y) = \lim_{n\to\infty} d(x_n, y_n) \geq 0.$$

Since $d(x_n, x_n) = 0$ for all $n \in \mathbb{N}$ we also have $D(x, x) = \lim_{n\to\infty} 0 = 0$. Also, $d(x_n, y_n) = d(y_n, x_n)$ for all $n \in \mathbb{N}$, so

$$D(x, y) = \lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(y_n, x_n) = D(y, x).$$

Finally, since $d(x_n, y_n) \leq d(x_n, z_n) + d(z_n, y_n)$ for all $n \in \mathbb{N}$, we have

$$\begin{aligned} D(x, y) &= \lim_{n\to\infty} d(x_n, y_n) \leq \lim_{n\to\infty} (d(x_n, z_n) + d(z_n, y_n)) \\ &= \lim_{n\to\infty} d(x_n, z_n) + \lim_{n\to\infty} d(z_n, y_n) = D(x, z) + D(z, y). \end{aligned}$$

Since $\mathcal{F}_C(M)$ is non-empty (M is non-empty so there exists a constant sequence in M which is Cauchy), D is a pseudo-metric. □

Three things are worth noting. We cannot be sure that D is a metric; if M contains two distinct elements a and b, then letting $x_n = a$ and $y_{n+1} = a$ for all $n \in \mathbb{N}$ along with $y_1 = b$, then (x_n) and (y_n) are Cauchy sequences and $D((x_n), (y_n)) = 0$ but $(x_n) \neq (y_n)$.

The proof above also does not use that d is actually a metric. Hence the construction can be used for pseudometric spaces as well.

The proof above also uses completeness of $\mathbb{R}$ to conclude that the pseudometric is well-defined; this is the only extraneous fact we are going to take for granted. Hence we cannot use this construction to prove that the completion of $\mathbb{Q}$ is actually what we know as $\mathbb{R}$; nonetheless, this can be proved in a similar way using Cauchy sequences in $\mathbb{Q}$. We will omit the proof for now.

Theorems 5 and 4 then yield the following corollary:

Corollary 6. *$(\mathcal{F}_C(M), D)$ yields a metric space $(\mathcal{F}_C(M)_\sim, D_\sim)$ with $\mathcal{F}_C(M)_\sim$ denoting the set of equivalence classes under the relation $x \sim y$ iff $D(x,y) = 0$ and $D_\sim([x],[y]) = D(x,y)$ for $x, y \in \mathcal{F}_C(M)$.*

By now, our construction seems far away from what we started with. We shall prove that it is not so. Not at all, in fact.

Lemma 7. *Defining $\rho : (M,d) \to (\mathcal{F}_C(M)_\sim, D_\sim)$ by $\rho(x) = [\hat{x}]$, $\hat{x}$ denoting the sequence in M with x as all its terms, ρ is an isometry and $\rho(M)$ is dense in $\mathcal{F}_C(M)_\sim$.*

Proof. ρ is clearly an isometry; for all $x, y \in M$, we have

$$D_\sim(\rho(x), \rho(y)) = D(\hat{x}, \hat{y}) = \lim_{k\to\infty} d(x,y) = d(x,y).$$

Let $[x] \in \mathcal{F}_C(M)_\sim$, $x = (x_n)_{n=1}^\infty \in \mathcal{F}_C(M)$. Since x is a Cauchy sequence, there exists $N \in \mathbb{N}$ such that $d(x_n, x_m) < \frac{\varepsilon}{2}$ for $m, n \geq N$. Hence for all $k \geq N$, $d(x_k, x_N) < \frac{\varepsilon}{2}$, so

$$D_\sim([x], \rho(x_N)) = D(x, \widehat{x_N}) = \lim_{k\to\infty} d(x_k, x_N) \leq \frac{\varepsilon}{2} < \varepsilon.$$

Hence $[x] \in \overline{\rho(M)}$, $\rho(M)$ is dense in $\mathcal{F}_C(M)_\sim$. □

Hence $\mathcal{F}_C(M)_\sim$ contains a copy of M, and now the face of God shows itself.

Theorem 8. *The metric space $(\mathcal{F}_C(M)_\sim, D_\sim)$ is complete.*

Before we prove this, some preparation is in order.

Lemma 9. *If a metric space (M,d) contains a dense subset A for which it holds that all Cauchy sequences in A converge in M, then (M,d) is complete.*

Proof. Let (x_n) be a Cauchy sequence in M, let $\varepsilon > 0$ and for each $n \in \mathbb{N}$ pick $a_n \in A$ such that $d(x_n, a_n) < \frac{\varepsilon}{3}$. Since (x_n) is a Cauchy sequence, there exists $N \in \mathbb{N}$ such that $d(x_n, x_m) < \frac{\varepsilon}{3}$ for all $m, n \geq N$. Hence

$$d(a_n, a_m) \leq d(a_n, x_n) + d(x_n, x_m) + d(x_m, a_m) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varepsilon$$

for $n, m \geq N$. Hence (a_n) is a Cauchy sequence in A; let $x = \lim_{n\to\infty} a_n$. Let $N' \in \mathbb{N}$ such that $d(a_n, x) < \frac{\varepsilon}{3}$ for all $n \geq N'$; then

$$d(x_n, x) \leq d(x_n, a_n) + d(a_n, x) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varepsilon$$

for all $n \geq N'$. Hence x_n converges to x in M, so (M,d) is complete. □

We are ready.

Proof of Theorem 8. By Lemmas 7 and 9, it suffices to prove that all Cauchy sequences in $\rho(M)$ converge in $\mathcal{F}_C(M)_\sim$. Let $([x^n])$ be a Cauchy sequence in $\rho(M)$, i.e. $[x^n] = \rho(x_n)$ for some $x_n \in M$. Since

$$d(x_n, x_m) = \lim_{k\to\infty} d(x_n, x_m) = D(\widehat{x_n}, \widehat{x_m}) = D_\sim(\rho(x_n), \rho(x_m))$$

for all $n, m \in \mathbb{N}$, $x = (x_n)$ is a Cauchy sequence in M. Consider now $[x] \in \mathcal{F}_C(M)_\sim$ and let $\varepsilon > 0$. Pick $N \in \mathbb{N}$ such that $d(x_n, x_m) < \frac{\varepsilon}{2}$ for $n, m \geq N$; then

$$D_\sim([x^n], [x]) = D(\widehat{x_n}, x) = \lim_{k\to\infty} d(x_n, x_k) \leq \frac{\varepsilon}{2} < \varepsilon$$

for all $n \geq N$, so $[x^n]$ converges to $[x]$ in $\mathcal{F}_C(M)_\sim$. Hence $(\mathcal{F}_C(M)_\sim, D_\sim)$ is a complete metric space. □

Hence we can embed any metric space (M,d) isometrically in a complete metric space (M', d') such that the original metric space has dense image. This procedure has a name:

Definition 10. *Let (M,d) be a metric space. If there is a complete metric space (M',d') and an isometry $\rho' : (M,d) \to (M',d')$ satisfying $\overline{\rho'(M)} = M'$, then the triple (M',d',ρ') is called a* Cauchy completion *of (M,d).*

Hence we obtain the following corollary from Lemma 7 and Theorem 8:

Corollary 11. *Any metric space has a Cauchy completion.*

Now, one may naturally ask whether it is possible for a metric space (M,d) with one Cauchy completion (M',d') to find another Cauchy completion of (M,d) that is distinguishable from (M',d') as a metric space; for instance, it might happen that one completion might be a "larger" metric space than another completion. In fact, the answer is no, as the following theorem states.

Theorem 12. *Let (M,d) be a metric space, and let (M',d',ρ') be a Cauchy completion of (M,d). If (M'',d'',ρ'') is another Cauchy completion of (M,d), then there exists a bijective isometry $\tau : (M',d') \to (M'',d'')$ such that $\tau \circ \rho' = \rho''$.*

Using the theorem, the bijection and isometry τ tells us that (M',d') and (M'',d'') are essentially the same metric space. The question is, how do we prove this? The answer comes in the following seemingly extraneous theorem.

Theorem 13. *Let (M,d) be a metric space and let $A \subseteq M$. Furthermore, let (M',d') be a complete metric space. If $f : (A,d) \to (M',d')$ is a uniformly continuous map, then f extends uniquely to a uniformly continuous map $\tilde{f} : (\overline{A},d) \to (M',d')$.*

Recall that a map of metric spaces $f : (M,d) \to (M',d')$ is *uniformly continuous* if there for all $\varepsilon > 0$ exists a $\delta > 0$ such that $d(x,y) < \delta$ implies $d'(f(x),f(y)) < \varepsilon$ for all $x,y \in M$. Uniformly continuous maps are of course continuous; it is clear from well-known counter-examples that the converse does not hold. Clearly, all isometries of metric spaces are uniformly continuous.

Before we prove Theorem 13, two lemmas are in order.

Lemma 14. *Let (M,d) and (M',d') be metric spaces and let $f : (M,d) \to (M',d')$ be uniformly continuous. If (x_n) is a Cauchy sequence of M, then $(f(x_n))$ is a Cauchy sequence in M'.*

Proof. Let $\varepsilon > 0$. By uniform continuity of f, there is $\delta > 0$ such that $d(x,y) < \delta$ implies $d'(f(x),f(y)) < \varepsilon$ for all $x,y \in M$. Because (x_n) is Cauchy, there exists $N \in \mathbb{N}$ such that $d(x_n,x_m) < \delta$ for all $n,m \geq N$. Hence $n,m \geq N$ implies $d'(f(x_n),f(x_m)) < \varepsilon$. □

Lemma 15. *Let (M,d) be a metric space. Then the metric d is continuous.*

Proof. Let $x,y \in M$ and let (x_n) and (y_n) be sequences in M such that $x_n \to x$ and $y_n \to y$. Then

$$d(x_n,y_n) - d(x,y) \leq d(x_n,x) + d(x,y) + d(y,y_n) - d(x,y) = d(x_n,x) + d(y,y_n)$$

and

$$d(x,y) - d(x_n,y_n) \leq d(x,y) - d(x_n,x) - d(x,y) - d(y,y_n) \leq d(x_n,x) + d(y,y_n),$$

yielding $|d(x_n,y_n) - d(x,y)| \leq d(x_n,x) + d(y,y_n)$. Letting $n \to \infty$ yields

$$d(x,y) = \lim_{n\to\infty} d(x_n,y_n),$$

so d is continuous. □

Proof of Theorem 13. Uniqueness of the extension is clear: if $\tilde{f}_1$ and $\tilde{f}_2$ are continuous extensions of $f : A \to M'$ to $\overline{A}$, then $\tilde{f}_1|_A = f|_A = \tilde{f}_2|_A$. For any $x \in \overline{A}$, then picking a sequence $(x_n) \subseteq A$ such that $x_n \to x$, then by continuity of $\tilde{f}_1$ and $\tilde{f}_2$, we obtain

$$\tilde{f}_1(x) = \lim_{n\to\infty} \tilde{f}_1(x_n) = \lim_{n\to\infty} \tilde{f}_2(x_n) = \tilde{f}_2(x).$$

Note that we did not use the assumptions of uniform continuity and completeness.

Now that uniqueness has been taken care of, how can we define $\tilde{f}$? Since we used in the uniqueness part that the closure of A can be characterized in terms of sequences in A that converge, and we want $\tilde{f}$ to be continuous, we may try to define

$$\tilde{f}(x) = \lim_{n\to\infty} f(x_n)$$

where (x_n) is a sequence of A converging to x, but it is not at all clear that it is well-defined: if there is a limit at all, is it then independent of the choice of sequence? Is there a limit? We will prove that the answer is yes to both questions.

Let $x \in \overline{A}$ and let (x_n) and (y_n) be sequences in A that converge to x. First and foremost, (x_n) is Cauchy, so $(f(x_n))$ is Cauchy by Lemma 14 and hence converges. To prove that the limit is independent of the choice of sequence, define $z_{2n-1} = x_n$ and $z_{2n} = y_n$ for $n \in \mathbb{N}$. Then $z_n \to x$: for $\varepsilon > 0$, pick $N \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$ and $d(y_n, x) < \varepsilon$ for $n \geq N$. Then

$$\begin{aligned} d(z_n, x) &= \begin{cases} d(x_{\frac{n+1}{2}}, x) & \text{for } n \text{ odd} \\ d(y_{\frac{n}{2}}, x) & \text{for } n \text{ even} \end{cases} \\ &\leq \varepsilon \end{aligned}$$

for $n \geq 2N$. Since (z_n) is Cauchy, $(f(z_n))$ is Cauchy by Lemma 14 and hence converges to some $y \in M'$ by the assumption of M' being complete. Since $(f(x_n))$ and $(f(y_n))$ are subsequences of $(f(z_n))$, they must both converge to y. This proves that

1. for any sequence (x_n) in A converging to $x \in \overline{A}$, $(f(x_n))$ converges, and
2. for any two sequences (x_n) and (y_n) in A converging to $x \in \overline{A}$, then

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(y_n).$$

Hence $\tilde{f}$ is well-defined. It is also clearly an extension of f: for any $x \in A$, then letting $x_n = x$ for all $n \in \mathbb{N}$, we see that

$$\tilde{f}(x) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(x) = f(x),$$

so $\tilde{f}|_A = f$.

It only remains to prove that $\tilde{f}$ is uniformly continuous. Let $\varepsilon > 0$ and pick $\delta > 0$ such that $d(x, y) < \delta$ implies $d'(f(x), f(y)) < \frac{\varepsilon}{2}$ for all $x, y \in A$. Assume that $x, y \in \overline{A}$ satisfy $d(x, y) < \delta$. Picking sequences (x_n) and (y_n) in A such that $x_n \to x$ and $y_n \to y$, then $d(x_n, y_n) \to d(x, y)$ by Lemma 15, so there exists $N \in \mathbb{N}$ such that

$$|d(x_n, y_n) - d(x, y)| < \delta - d(x, y)$$

for $n \geq N$, implying $d(x_n, y_n) < \delta$ for $n \geq N$. Hence $d'(f(x_n), f(y_n)) < \frac{\varepsilon}{2}$ for $n \geq N$, so

$$d'(\tilde{f}(x), \tilde{f}(y)) = \lim_{n\to\infty} d'(f(x_n), f(y_n)) \leq \frac{\varepsilon}{2} < \varepsilon$$

by Lemma 15. Hence $\tilde{f}$ is uniformly continuous. □

We also prove the case where f is an isometry. What could possibly happen, he said snickeringly.

Theorem 16. *Let (M, d) be a metric space and let $A \subseteq M$. Furthermore, let (M', d') be a complete metric space. If $f : (A, d) \to (M', d')$ is an isometry, then f extends uniquely to an isometry $\tilde{f} : (\overline{A}, d) \to (M', d')$.*

Proof. We define $\tilde{f}$ as in the proof of Theorem 13; uniqueness and well-definedness is then clear and $\tilde{f}$ is a continuous extension of f. Let $x, y \in \overline{A}$ and let (x_n) and (y_n) be sequences in A such that $x_n \to x$ and $y_n \to y$. Then

$$d'(\tilde{f}(x), \tilde{f}(y)) = \lim_{n\to\infty} d'(f(x_n), f(y_n)) = \lim_{n\to\infty} d(x_n, y_n) = d(x, y),$$

so $\tilde{f}$ is an isometry. □

We can now prove Theorem 12.

Proof of Theorem 12. Note that ρ' is a bijection onto its image and allows for an isometry $\rho'^{-1} : \rho'(M) \to M$. As

$$\rho'' \circ \rho'^{-1} : (\rho'(M), d') \to (M'', d'')$$

is an isometric map and isometries are uniformly continuous, it follows from Theorem 16 that $\rho'' \circ \rho'^{-1}$ extends to an isometry $\tau : (M', d') \to (M'', d'')$ satisfying $\tau(x) = \rho'' \circ \rho'^{-1}(x)$ or $\tau(\rho(x)) = \rho''(x)$ for all $x \in M$, since $\overline{\rho'(M)} = M'$. Analoguously, we obtain an isometry $\rho''^{-1} : \rho''(M) \to M$, yielding an isometry

$$\rho' \circ \rho''^{-1} : (\rho''(M), d'') \to (M', d'),$$

and finally, using Theorem 16, an isometry $\psi : (M'', d'') \to (M', d')$ satisfying $\psi(\rho''(x)) = \rho'(x)$ for all $x \in M$.

By substitution, $\psi \circ \tau \circ \rho' = \rho'$ and $\tau \circ \psi \circ \rho'' = \rho''$. Since $\rho'(M)$ is dense in M' and $\rho''(M)$ likewise in M'', then by the maps in the equations being continuous, we obtain that $\psi \circ \tau$ is the identity on M' and $\tau \circ \psi$ is the identity on M''. Hence τ is a bijection and an isometry, with $\tau^{-1} = \psi$. □

Hence Corollary 11 and Theorem 12 combine into the following theorem:

Theorem 17. *Any metric space (M, d) has a Cauchy completion (M', d', ρ'), i.e. there exists a complete metric space (M', d') and an isometry $\rho' : (M, d) \to (M', d')$ with dense image. Furthermore, the Cauchy completion is unique up to isometry, i.e. if (M'', d'', ρ'') is another Cauchy completion of (M, d), then there is a bijective isometry $(M', d') \to (M'', d'')$.*

Whenever we speak of *completing* a metric space, it is the Cauchy completion of the metric space that we use.

2 Normed spaces

When passing from metric spaces to normed spaces, one might ask if the Cauchy completion actually possesses the structure of a vector space so that it might become a Banach space. One should be optimistic.

Theorem 18. *Let $(\mathfrak{X}, \|\cdot\|)$ be a normed space over $\mathbb{F}$, $\mathbb{F}$ denoting $\mathbb{R}$ or $\mathbb{C}$, and consider the Cauchy completion $(\mathcal{F}_C(\mathfrak{X})_\sim, D_\sim, \rho)$ from Corollary 6 and Lemma 7. Then $\mathcal{F}_C(\mathfrak{X})_\sim$ can be made into a vector space over $\mathbb{F}$ such that ρ is linear and $D_\sim$ is induced by a norm on $\mathcal{F}_C(\mathfrak{X})_\sim$. Under this norm ρ is an isometry, and hence the completion becomes a Banach space containing the dense subspace $\rho(\mathfrak{X})$ that is isometrically isomorphic to $\mathfrak{X}$.*

Proof. We will first define natural vector space operations on the set $\mathcal{F}_C(\mathfrak{X})$. For all Cauchy sequences $x = (x_n)_{n=1}^\infty$ and $y = (y_n)_{n=1}^\infty$ in $\mathfrak{X}$ as well as $\lambda \in \mathbb{F}$, then

$$\|(x_n + y_n) - (x_m + y_m)\| \leq \|x_n - x_m\| + \|y_n - y_m\|$$

and

$$\|\lambda x_n - \lambda x_m\| = |\lambda| \|x_n - x_m\|,$$

proving that the sequences $(x_n + y_n)$ and (λx_n) are Cauchy. Hence, we can define vector space operations on $\mathcal{F}_C(\mathfrak{X})$ by

$$x + y := (x_n + y_n)_{n=1}^\infty, \quad \lambda x := (\lambda x_n)_{n=1}^\infty$$

for $x = (x_n)_{n=1}^\infty$, $y = (y_n)_{n=1}^\infty \in \mathcal{F}_C(\mathfrak{X})$ and $\lambda \in \mathbb{F}$. That $\mathcal{F}_C(\mathfrak{X})$ satisfies the vector space axioms follows from the fact that $\mathfrak{X}$ is a vector space, and the neutral element is the zero sequence. The pseudometric D on $\mathcal{F}_C(\mathfrak{X})$ from Theorem 5 is now defined by

$$D(x, y) = \lim_{n\to\infty} \|x_n - y_n\|, \quad x = (x_n),\ y = (y_n) \in \mathcal{F}_C(\mathfrak{X}).$$

Let $x = (x_n)$, $x' = (x'_n)$, $y = (y_n)$, $y =' (y'_n)$ and $\lambda \in \mathbb{F}$, assume that $x \sim x'$ and $y \sim y'$, i.e.

$$\lim_{n\to\infty} \|x_n - x'_n\| = \lim_{n\to\infty} \|y_n - y'_n\| = 0.$$

Then

$$D(x + y, x' + y') = \lim_{n\to\infty} \|(x_n + y_n) - (x'_n - y'_n)\| \leq \lim_{n\to\infty} \|x_n - x'_n\| + \lim_{n\to\infty} \|y_n - y'_n\| = 0$$

and

$$D(\lambda x, \lambda x') = \lim_{n\to\infty} \|\lambda x_n - \lambda x'_n\| = |\lambda| \lim_{n\to\infty} \|x_n - x'_n\| = 0,$$

proving that $x+y \sim x'+y'$ and $\lambda x \sim \lambda x'$. Hence we can define vector space operations on $\mathcal{F}_C(\mathfrak{X})_\sim$ by

$$[x]+[y] := [x+y], \quad \lambda[x] := [\lambda x], \quad x,y \in \mathcal{F}_C(\mathfrak{X}),\ \lambda \in \mathbb{F}.$$

That $\mathcal{F}_C(\mathfrak{X})_\sim$ is a vector space now follows from the fact that $\mathcal{F}_C(\mathfrak{X})$ is, and the neutral element is the equivalence class of the zero sequence.

ρ is linear; indeed, it is clear that for all $x,y \in \mathfrak{X}$ and $\lambda \in \mathbb{C}$ we have

$$\rho(x+y) = \left[\widehat{x+y}\right] = [\hat{x}+\hat{y}] = [\hat{x}]+[\hat{y}] = \rho(x)+\rho(y)$$

and

$$\rho(\lambda x) = \left[\widehat{\lambda x}\right] = [\lambda\hat{x}] = \lambda[\hat{x}] = \lambda\rho(x).$$

Hence $\rho(\mathfrak{X})$ is a subspace of $\mathcal{F}_C(\mathfrak{X})_\sim$ and dense by Lemma 7, and $\mathfrak{X}$ is isomorphic to $\rho(\mathfrak{X})$ since ρ is surjective onto $\rho(\mathfrak{X})$ and ρ is injective by construction. Once we construct the norm on $\mathcal{F}_C(\mathfrak{X})_\sim$, we shall see that $\rho(\mathfrak{X})$ is isometrically isomorphic to $\mathfrak{X}$.

We define a norm $\mathcal{F}_C(\mathfrak{X})_\sim \to \mathbb{R}$ by first defining a map $f : \rho(\mathfrak{X}) \to \mathbb{R}$ by $f(\rho(x)) = \|x\|$. It is well-defined, since if $\rho(x) = \rho(y)$ for $x,y \in \mathfrak{X}$, then $\hat{x} \sim \hat{y}$ implying

$$\|x-y\| = \lim_{n\to\infty} \|x-y\| = D(\hat{x},\hat{y}) = 0$$

so that $x = y$ and hence $\|x\| = \|y\|$. In fact, f is uniformly continuous; it follows from the fact that for all $x,y \in \mathfrak{X}$,

$$|f(\rho(x)) - f(\rho(y))| = |\|x\| - \|y\|| \leq \|x-y\| = D(\hat{x},\hat{y}) = D_\sim(\rho(x),\rho(y)).$$

Since $\rho(\mathfrak{X})$ is dense in $\mathcal{F}_C(\mathfrak{X})_\sim$ and $\mathbb{R}$ is complete, then Theorem 13 yields a unique uniformly continuous extension $\tilde{f} : \mathcal{F}_C(\mathfrak{X})_\sim \to \mathbb{R}$ of f.

We claim that $\tilde{f}$ is the norm we seek, and the proof of this requires some very symbol-heavy work. Let $x = (x_n),\ y = (y_n) \in \mathcal{F}_C(\mathfrak{X})$ and $\lambda \in \mathbb{F}$. Since $\rho(\mathfrak{X})$ is dense in $\mathcal{F}_C(\mathfrak{X})_\sim$, there exist sequences $(x'_n),(y'_n) \in \mathfrak{X}$ such that $\rho(x'_n) \to [x]$ and $\rho(y'_n) \to [y]$. Note that the proof of Theorem 13 yields that the sequences $(f(\rho(x'_n)) = (\|x'_n\|)$ and $(f(\rho(y'_n)) = (\|y'_n\|)$ converge. As

$$\begin{aligned} D_\sim(\rho(x'_n+y'_n),[x+y]) &= \lim_{k\to\infty} \|(x'_n+y'_n)-(x_k+y_k)\| \\ &\leq \lim_{k\to\infty} \|x'_n - x_k\| + \lim_{k\to\infty} \|y'_n - y_k\| \\ &= D_\sim(\rho(x'_n),[x]) + D_\sim(\rho(y'_n),[y]) \to 0 \end{aligned}$$

and

$$\begin{aligned} D_\sim(\lambda\rho(x'_n),\lambda[x]) &= D_\sim(\rho(\lambda x'_n),[\lambda x]) \\ &= \lim_{k\to\infty} \|\lambda x'_n - \lambda x_k\| \\ &= |\lambda| \lim_{k\to\infty} \|x'_n - x_k\| \\ &= |\lambda| D_\sim(\rho(x'_n),[x]) \to 0, \end{aligned}$$

we have $\rho(x'_n+y'_n) \to [x+y]$ and $\rho(\lambda x'_n) \to [\lambda x]$ (hence $(\|x'_n+y'_n\|)$ and $(|\lambda|\|x'_n\|)$ converge as well). Therefore

$$\begin{aligned} \tilde{f}([x]+[y]) &= \tilde{f}([x+y]) \\ &= \lim_{n\to\infty} f(\rho(x'_n+y'_n)) \\ &= \lim_{n\to\infty} \|x'_n+y'_n\| \\ &\leq \lim_{n\to\infty} \|x'_n\| + \lim_{n\to\infty} \|y'_n\| \\ &= \lim_{n\to\infty} f(\rho(x'_n)) + \lim_{n\to\infty} f(\rho(y'_n)) \\ &= \tilde{f}([x]) + \tilde{f}([y]) \end{aligned}$$

and

$$\tilde{f}(\lambda[x]) = \tilde{f}([\lambda x]) = \lim_{n\to\infty} f(\rho(\lambda x'_n)) = \lim_{n\to\infty} \|\lambda x'_n\| = |\lambda| \lim_{n\to\infty} \|x'_n\| = |\lambda| \lim_{n\to\infty} f(\rho(x'_n)) = |\lambda|\tilde{f}([x]).$$

Of course, $\tilde{f}([x]) = \lim_{n\to\infty} f(\rho(x'_n)) \geq 0$. Finally, if $\tilde{f}([x]) = 0$, then

$$\lim_{n\to\infty} D_\sim(\rho(x'_n), \rho(0)) = \lim_{n\to\infty} \|x'_n\| = \lim_{n\to\infty} f(\rho(x'_n)) = \tilde{f}([x]) = 0.$$

Since

$$D_\sim(\rho(0), [x]) \leq D_\sim(\rho(0), \rho(x'_n)) + D_\sim(\rho(x'_n), [x])$$

for all $n \in \mathbb{N}$, then letting $n \to \infty$, we obtain

$$D_\sim(\rho(0), [x]) \leq \lim_{n\to\infty} D_\sim(\rho(x'_n), [x]) = 0.$$

Hence $[x] = \rho(0) = [\hat{0}]$, so $\tilde{f}$ is a norm. Hence $\rho(\mathfrak{X})$ is isometrically isomorphic to $\mathfrak{X}$ under this norm, since $\tilde{f}(\rho(x)) = f(\rho(x)) = \|x\|$ for all $x \in \mathfrak{X}$. Finally,

$$\begin{aligned}
\tilde{f}([x]-[y]) &= \tilde{f}([x-y]) \\
&= \lim_{n\to\infty} f(\rho(x'_n - y'_n)) \\
&= \lim_{n\to\infty} \|x'_n - y'_n\| \\
&= \lim_{n\to\infty} \lim_{k\to\infty} \|x'_n - y'_n\| \\
&= \lim_{n\to\infty} D(\widehat{x'_n}, \widehat{y'_n}) \\
&= \lim_{n\to\infty} D_\sim(\rho(x'_n), \rho(y'_n)) \\
&= D_\sim([x],[y]),
\end{aligned}$$

by $D_\sim$ being continuous (Lemma 15). Hence the metric induced by $\tilde{f}$ is actually $D_\sim$, so $(\mathcal{F}_C(\mathfrak{X})_\sim, \tilde{f})$ is a Banach space. □

From this theorem, we reach the following conclusion:

Theorem 19. *Let $(\mathfrak{X}, \|\cdot\|)$ be a normed space over $\mathbb{F}$, $\mathbb{F}$ denoting $\mathbb{R}$ or $\mathbb{C}$. Then:*

(i) There exists a Banach space $(\mathfrak{X}', \|\cdot\|')$ over $\mathbb{F}$ and a linear isometry $\rho : (\mathfrak{X}, \|\cdot\|) \to (\mathfrak{X}', \|\cdot\|')$ with dense image in $\mathfrak{X}'$.

(ii) Any Cauchy completion (M', d', ρ') of $(\mathfrak{X}, \|\cdot\|)$ as a metric space can be equipped with an $\mathbb{F}$-vector space structure and a norm that induces the metric d', so that the completion becomes a Banach space, and ρ' becomes a linear isometry of normed spaces. Furthermore, $\rho'(\mathfrak{X})$ is dense in the completion, and $\mathfrak{X}$ and $\rho'(\mathfrak{X})$ are isometrically isomorphic as normed spaces.

(iii) Let $(\mathfrak{X}', \|\cdot\|')$ be a Banach space over $\mathbb{F}$ such that there is a linear isometry $\rho' : (\mathfrak{X}, \|\cdot\|) \to (\mathfrak{X}', \|\cdot\|')$ with dense image in $\mathfrak{X}'$. If $(\mathfrak{X}'', \|\cdot\|'')$ is another Banach space over $\mathbb{F}$ and there is a linear isometry $\rho'' : (\mathfrak{X}, \|\cdot\|) \to (\mathfrak{X}'', \|\cdot\|'')$ with dense image in $\mathfrak{X}''$, then there exists an isometric isomorphism of Banach spaces $\psi : (\mathfrak{X}', \|\cdot\|) \to (\mathfrak{X}'', \|\cdot\|'')$ such that $\psi \circ \rho' = \rho''$.

Proof. The Banach space $(\mathcal{F}_C(\mathfrak{X})_\sim, \tilde{f})$ as constructed in Theorem 18 as well as the linear isometry $\rho : (\mathfrak{X}, \|\cdot\|) \to (\mathcal{F}_C(\mathfrak{X})_\sim, \tilde{f})$ yields *(i)*.

We will now use the properties of the Banach space $(\mathcal{F}_C(\mathfrak{X})_\sim, \tilde{f})$ and the linear isometry ρ to prove *(ii)*. If (M', d', ρ') is a Cauchy completion of $(\mathfrak{X}, \|\cdot\|)$ as a metric space, then since $\tilde{f}$ induces the metric $D_\sim$ and $(\mathcal{F}_C(\mathfrak{X})_\sim, D_\sim, \rho)$ is a Cauchy completion of $(\mathfrak{X}, \|\cdot\|)$, there exists an bijective isometry $\tau : (M', d') \to (\mathcal{F}_C(\mathfrak{X})_\sim, D_\sim)$ with $\tau \circ \rho' = \rho$ by Theorem 12. We give M' a vector space structure by *making* τ linear. Hence we define

$$x + y := \tau^{-1}(\tau(x) + \tau(y)), \quad \lambda x := \tau^{-1}(\lambda\tau(x))$$

for $x, y \in M'$ and $\lambda \in \mathbb{F}$. These operations are well-defined since τ is bijective, and one easily proves the vector space axioms by noting that the definitions imply

$$\tau(x+y) = \tau(x) + \tau(y), \quad \tau(\lambda x) = \lambda\tau(x)$$

for $x, y \in M'$ and $\lambda \in \mathbb{F}$. The zero element in M' is hence $\tau^{-1}(0)$. By defining $\|x\|' = \tilde{f}(\tau(x))$ for all $x \in M'$, then $\|\cdot\|'$ is easily seen to be a norm on M'; non-negativity, the triangle equality and the scalar law clearly hold, and if $\|x\|' = 0$, then $\tilde{f}(\tau(x)) = 0$, so $\tau(x) = 0$ and $x = \tau^{-1}(0)$. Finally, $\|\cdot\|'$ induces d'; indeed,

$$\|x - y\|' = \tilde{f}(\tau(x - y)) = \tilde{f}(\tau(x) - \tau(y)) = D_\sim(\tau(x), \tau(y)) = d'(x, y)$$

since τ is an isometry. Hence $(M', \|\cdot\|')$ is a Banach space.

Finally, ρ' is linear: indeed because ρ and τ^{-1} are linear, then for all $x, y \in \mathfrak{X}$ and $\lambda \in \mathbb{F}$,

$$\rho'(x + y) = \tau^{-1}(\rho(x + y)) = \tau^{-1}(\rho(x)) + \tau^{-1}(\rho(y)) = \rho'(x) + \rho'(y)$$

and $\rho'(\lambda x) = \lambda\rho'(x)$ similarly. Since $\|\rho'(x)\|' = \tilde{f}(\tau(\rho'(x))) = \tilde{f}(\rho(x)) = \|x\|$ for all $x \in \mathfrak{X}$, ρ' is an isometry of normed spaces. Hence $\rho'(\mathfrak{X})$ is isometrically isomorphic to $\mathfrak{X}$ and is dense in the metric space (M', d') by assumption. This proves *(ii)*.

If the Banach space $(\mathfrak{X}'', \|\cdot\|'')$ and the linear isometry ρ'' satisfy the conditions in *(iii)*, then considering $(\mathfrak{X}, \|\cdot\|)$ and $(\mathfrak{X}'', \|\cdot\|'')$ as metric spaces with metrics d and d'', ρ'' is an isometry of metric spaces and since $\overline{\rho''(\mathfrak{X})} = \mathfrak{X}''$, $(\mathfrak{X}'', d'')$ is a Cauchy completion of $(\mathfrak{X}, d)$. Hence by Theorem 12, there exists a bijective isometry of metric spaces $\psi : (\mathfrak{X}', d') \to (\mathfrak{X}'', d'')$ with $\psi \circ \rho' = \rho''$, d' denoting the metric induced by the norm $\|\cdot\|'$ on $\mathfrak{X}'$. ψ is linear: for $x', y' \in \mathfrak{X}'$ and $\lambda \in \mathbb{F}$, then $\rho'(x_n) \to x'$ and $\rho'(y_n) \to y'$ for appropriately chosen sequences $(x_n), (y_n) \subseteq \mathfrak{X}$. Then

$$\psi(x' + y') = \lim_{n\to\infty} \rho''(x_n + y_n) = \lim_{n\to\infty} (\rho''(x_n) + \rho''(y_n)) = \psi(x') + \psi(y')$$

and $\psi(\lambda x') = \lambda\psi(x')$ similarly, simply because ρ'' is linear and ψ is continuous. Finally, for any $x' \in \mathfrak{X}'$, then $\rho'(x_n) \to x'$ for a sequence $(x_n) \in \mathfrak{X}$, so that

$$\|\psi(x')\|'' = \lim_{n\to\infty} \|\rho''(x_n)\|'' = \lim_{n\to\infty} \|x_n\| = \lim_{n\to\infty} \|\rho'(x_n)\|' = \|x'\|'$$

by continuity of ρ' and ψ, so we conclude that ψ is an isometric isomorphism of Banach spaces, hence *(iii)*. □

For a normed space $(\mathfrak{X}, \|\cdot\|)$, a Banach space $(\mathfrak{X}', \|\cdot\|')$ over $\mathbb{F}$ and a linear isometry $\rho : (\mathfrak{X}, \|\cdot\|) \to (\mathfrak{X}', \|\cdot\|')$ with dense image in $\mathfrak{X}'$ may be called a *Banach completion* of $(\mathfrak{X}, \|\cdot\|)$. By the above theorem, it always exists, and any Cauchy completion of $(\mathfrak{X}, \|\cdot\|)$ can be made into a Banach completion. Thus when completing a normed space, we can obtain a Banach completion, and it will be unique up to isometric isomorphism, as any two Banach completions are isometrically isomorphic. Hence it makes sense to speak of *the* Banach completion of a normed space.

3 Inner product spaces

We may now consider inner product spaces and ask the same questions. Since inner product spaces are also normed spaces, we can use the theorems for normed spaces. Recall that the metric of an inner product space is induced by the norm given by the inner product.

Theorem 20. *Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{F}$, $\mathbb{F}$ denoting $\mathbb{R}$ or $\mathbb{C}$, and consider a Banach completion $(\mathcal{H}', \|\cdot\|', \rho')$ of $(\mathcal{H}, \langle\cdot,\cdot\rangle)$. Then $\mathcal{H}'$ can be made into an inner product space over $\mathbb{F}$ such that $\|\cdot\|'$ is induced by the inner product. Hence the Banach completion becomes a Hilbert space containing a dense subspace that is isometrically isomorphic to $\mathcal{H}$.*

Proof. We need only find the inner product and prove that it induces the norm on $\|\cdot\|'$; the rest follows from the properties of the Banach completion proved in Theorem 19.

The question is, how do we define this inner product? The most obvious way to do it is by defining $\tau : \mathcal{H}' \times \mathcal{H}' \to \mathbb{R}$ by

$$\tau(x, y) = \lim_{n\to\infty} \langle x_n, y_n \rangle$$

for $x, y \in \mathcal{H}'$ where (x_n) and (y_n) are sequences in $\mathcal{H}$ such that $\rho'(x_n) \to x$ and $\rho'(y_n) \to y$. It is not clear that this is well-defined, however. We will show this.

First of all, the limit does exist. Indeed, since

$$\begin{aligned} & |\langle x_n, y_n\rangle - \langle x_m, y_m\rangle| \\ = \;& |\langle x_n, y_n - y_m\rangle + \langle x_n - x_m, y_m\rangle| \\ \leq \;& \|x_n\|\|y_n - y_m\| + \|x_n - x_m\|\|y_m\| \\ = \;& \|\rho'(x_n)\|'\|\rho'(y_n) - \rho'(y_m)\|' + \|\rho'(x_n) - \rho'(x_m)\|'\|\rho'(y_m)\|' \end{aligned}$$

for $m, n \in \mathbb{N}$, by ρ' being a linear isometry, then because the sequences $(\|\rho'(x_n)\|')$ and $(\|\rho'(y_m)\|')$ are bounded it follows that the sequence $(\langle x_n, y_n\rangle)$ is Cauchy and thus has a limit in $\mathbb{F}$. Next, the limit does not depend on the choice of sequence: for another two sequences (x'_n) and (y'_n) in $\mathcal{H}$ such that $\rho'(x'_n) \to x$ and $\rho'(y'_n) \to y$, then $\rho'(x_n - x'_n) \to 0$ and $\rho'(y_n - y'_n) \to 0$, and thus

$$\begin{aligned} & |\langle x_n, y_n\rangle - \langle x'_n, y'_n\rangle| \\ = \;& |\langle x_n, y_n - y'_n\rangle + \langle x_n - x'_n, y'_n\rangle| \\ \leq \;& \|x_n\|\|y_n - y'_n\| + \|x_n - x'_n\|\|y'_n\| \\ = \;& \|\rho'(x_n)\|'\|\rho'(y_n - y'_n)\|' + \|\rho'(x_n - x'_n)\|'\|\rho'(y'_n)\|', \end{aligned}$$

proving that $|\langle x_n, y_n\rangle - \langle x'_n, y'_n\rangle| \to 0$, so the limits of $(\langle x_n, y_n\rangle)$ and $(\langle x'_n, y'_n\rangle)$ are the same. Hence τ is well-defined.

We now prove that τ is actually an inner product on $\mathcal{H}'$. For $x, y, z \in \mathcal{H}'$, pick sequences (x_n), (y_n) and (z_n) in $\mathcal{H}$ such that $\rho'(x_n) \to x$, $\rho'(y_n) \to y$ and $\rho'(z_n) \to z$. Then $\rho'(x_n + y_n) \to x + y$, so

$$\tau(x + y, z) = \lim_{n\to\infty} \langle x_n + y_n, z_n\rangle = \lim_{n\to\infty} \langle x_n, z_n\rangle + \lim_{n\to\infty} \langle y_n, z_n\rangle = \tau(x, z) + \tau(y, z).$$

For $\lambda \in \mathbb{F}$, note that $\rho'(\lambda x_n) \to \lambda x$, so

$$\tau(\lambda x, z) = \lim_{n\to\infty} \langle \lambda x_n, z_n\rangle = \lambda \lim_{n\to\infty} \langle x_n, z_n\rangle = \lambda\tau(x, z).$$

Additionally,

$$\tau(x, z) = \lim_{n\to\infty} \langle x_n, z_n\rangle = \lim_{n\to\infty} \overline{\langle z_n, x_n\rangle} = \overline{\lim_{n\to\infty} \langle z_n, x_n\rangle} = \overline{\tau(z, x)}$$

by continuity of complex conjugation. Note that τ actually induces the norm on $\mathcal{H}'$; indeed

$$\tau(x, x) = \lim_{n\to\infty} \langle x_n, x_n\rangle = \lim_{n\to\infty} \|x_n\|^2 = \lim_{n\to\infty} \|\rho'(x_n)\|'^2 = \left\|\lim_{n\to\infty} \rho'(x_n)\right\|'^2 = \|x'\|'^2$$

by continuity of $\|\cdot\|'$ and the product in $\mathbb{F}$. Hence $\tau(x, x) = 0$ implies $\|x\|' = 0$ and thus $x = 0$. Thus τ is an inner product on $\mathcal{H}'$ that also induces the norm on $\mathcal{H}'$. Since $\mathcal{H}'$ is complete with respect to this norm, it becomes a Hilbert space. □

Before we go on, two lemmas are in order:

Lemma 21. *Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be an inner product space. If $x_n \to x$ and $y_n \to y$ in $\mathcal{H}$, then $\langle x_n, y_n\rangle \to \langle x, y\rangle$.*

Proof. Let $\varepsilon > 0$ and pick $N_1 \in \mathbb{N}$ such that $\|x_n - x\| < 1$ for $n \geq N_1$. Then $\|y_n\| < \|y\| + 1$ for $n \geq N_1$. Pick $N_2 \in \mathbb{N}$ such that $\|x_n - x\| < \varepsilon/2(\|y\|+1)$ for $n \geq N_2$ and $N_3 \in \mathbb{N}$ such that $\|y_n - y\| < \varepsilon/2(\|x\|+1)$ for $n \geq N_3$. Then for $n \geq \max\{N_1, N_2, N_3\}$, Cauchy-Schwarz' inequality yields

$$\begin{aligned} |\langle x_n, y_n\rangle - \langle x, y\rangle| &= |\langle x_n - x, y_n\rangle + \langle x, y_n - y\rangle| \\ &\leq \|x_n - x\|\|y_n\| + \|x\|\|y_n - y\| \\ &< \frac{\varepsilon}{2(\|y\| + 1)}(\|y\| + 1) + \|x\|\frac{\varepsilon}{2(\|x\| + 1)} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

so $\langle x_n, y_n\rangle \to \langle x, y\rangle$. □

Lemma 22. *If $\psi : (\mathcal{H}, \langle\cdot,\cdot\rangle) \to (\mathcal{H}', \langle\cdot,\cdot\rangle')$ is a linear map satisfying $\|\psi(x)\|' = \|x\|$ for all $x \in \mathcal{H}$, then $\langle\psi(x), \psi(y)\rangle' = \langle x, y\rangle$ for all $x, y \in \mathcal{H}$.*

Proof. This follows from the polarisation identity. □

We summarize our results in the manner of Theorem 19:

Theorem 23. *Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{F}$, $\mathbb{F}$ denoting $\mathbb{R}$ or $\mathbb{C}$. Then:*

(i) There is a Hilbert space $(\mathcal{H}', \langle\cdot,\cdot\rangle')$ over $\mathbb{F}$ and a linear map $\rho : (\mathcal{H}, \langle\cdot,\cdot\rangle) \to (\mathcal{H}', \langle\cdot,\cdot\rangle')$ with dense image in $\mathcal{H}'$ such that $\langle x, y\rangle = \langle \rho(x), \rho(y)\rangle$ for all $x, y \in \mathcal{H}$.

(ii) Any Cauchy completion (M', d', ρ') of $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ as a metric space can be equipped with an $\mathbb{F}$-vector space structure and an inner product that induces the metric d' so that the completion becomes a Hilbert space and ρ' becomes a linear map that preserves inner products. Furthermore, $\rho'(\mathcal{H})$ is dense in the completion, and $\mathcal{H}$ and $\rho'(\mathcal{H})$ are unitary as inner product spaces.

(iii) Let $(\mathcal{H}', \langle\cdot,\cdot\rangle')$ be a Hilbert space over $\mathbb{F}$ with a linear isometry $\rho' : (\mathcal{H}, \langle\cdot,\cdot\rangle) \to (\mathcal{H}', \langle\cdot,\cdot\rangle')$ with dense image in $\mathcal{H}'$. If $(\mathcal{H}'', \langle\cdot,\cdot\rangle'')$ is another Hilbert space over $\mathbb{F}$ with a linear isometry $\rho'' : (\mathcal{H}, \langle\cdot,\cdot\rangle) \to (\mathcal{H}'', \langle\cdot,\cdot\rangle'')$ with dense image in $\mathcal{H}''$, then there exists a unitary operator $\psi : (\mathcal{H}', \langle\cdot,\cdot\rangle') \to (\mathcal{H}'', \langle\cdot,\cdot\rangle'')$ such that $\psi \circ \rho' = \rho''$.

Proof. *(i)* follows from Theorem 20 and Lemma 22, since there is a Banach completion of $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ as a normed space by Theorem 19(i).

If (M', d', ρ') is a Cauchy completion of $(\mathcal{H}, \langle\cdot,\cdot\rangle)$, then per Theorem 19, we can equip it with an $\mathbb{F}$-vector space structure and a norm $\|\cdot\|'$ such that $(M', \|\cdot\|')$ becomes a Banach space and $\rho' : (\mathcal{H}, \langle\cdot,\cdot\rangle) \to (M', \|\cdot\|')$ is a linear isometry of normed spaces. Additionally, $\rho'(\mathcal{H})$ and $\mathcal{H}$ are isometrically isomorphic normed spaces and $\rho'(\mathcal{H})$ is dense in M'. Theorem 20 then yields that we can equip M' with an inner product that induces $\|\cdot\|'$, making it a Hilbert space. Lemma 22 yields that ρ' preserves inner products, and hence we obtain *(ii)*.

Finally, if $(\mathcal{H}'', \langle\cdot,\cdot\rangle'')$ is a Hilbert space satisfying the conditions of *(iii)*, then $(\mathcal{H}'', \|\cdot\|'')$ is a Banach space where $\|\cdot\|''$ is the norm induced by $\langle\cdot,\cdot\rangle''$. Hence Theorem 19 yields an isometric isomorphism $\psi : (\mathcal{H}', \|\cdot\|') \to (\mathcal{H}'', \|\cdot\|'')$ (where $\|\cdot\|'$ is the norm induced by $\langle\cdot,\cdot\rangle'$) such that $\psi \circ \rho = \rho''$. For $x, y \in \mathcal{H}'$, there are sequences (x_n) and (y_n) of $\mathcal{H}$ such that $\rho(x_n) \to x$ and $\rho(y_n) \to y$, yielding

$$\begin{aligned}
\langle \psi(x), \psi(y)\rangle'' &= \lim_{n\to\infty} \langle \psi(\rho(x_n)), \psi(\rho(y_n))\rangle'' \\
&= \lim_{n\to\infty} \langle \rho''(x_n), \rho''(y_n)\rangle'' \\
&= \lim_{n\to\infty} \langle x_n, y_n\rangle \\
&= \lim_{n\to\infty} \langle \rho(x_n), \rho(y_n)\rangle' \\
&= \langle x, y\rangle'
\end{aligned}$$

by Lemmas 21 and 22. Hence ψ is unitary. □

In the same manner as in Section 2, then for an inner product space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$, a Hilbert space $(\mathcal{H}', \langle\cdot,\cdot\rangle')$ over $\mathbb{F}$ and a linear map $\rho' : (\mathcal{H}, \langle\cdot,\cdot\rangle) \to (\mathcal{H}', \langle\cdot,\cdot\rangle')$ that preserves inner products and has dense image in $\mathcal{H}'$ may be called a *Hilbert completion* of an inner product space (in fact, ρ' need only be a linear isometry by Lemma 22). By the above theorem, it always exists, and any Cauchy completion of $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ can be made into a Hilbert completion. Therefore, by completing an inner product space we can obtain a Hilbert space. Finally, since any two Hilbert completions are unitary, it makes sense to speak of *the* Hilbert completion of $(\mathcal{H}, \langle\cdot,\cdot\rangle)$; all Hilbert space properties of two Hilbert completions of the same inner product space are the same.