Sølvkorn 15
The continuous functional calculus for arbitrary C^*-algebras

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This nugget will continue where (3) left off, regarding the properties of the continuous functional calculus. We will try to construct a valid extension of this calculus to any arbitrary C^*-algebras, and not just unital ones as covered in the above source.

In order to find a way to discuss the continuous functional calculus for a non-unital C^*-algebra, then to even consider it we cannot escape from the fact that we need to be able to work with a unit. This of course implies that the unitization has to be part of the discussion, and once we have brought it up we will try to wring loose of the claws that this extra structure has to grab us with. We know beforehand that this is no suicide mission, so we will succeed (no need to wet the bed here).

1 The unitization

Throughout this section, $\mathcal{A}$ denotes a non-unital C^*-algebra.

In order to construct the unitization for $\mathcal{A}$, we define a map $L_a\colon \mathcal{A} \to \mathcal{A}$ for any $a \in \mathcal{A}$ given by $L_a b = ab$ for $b \in \mathcal{A}$. This map is clearly well-defined, linear and bounded with $\|L_a\| \leq \|a\|$. Hence L_a belongs to the Banach algebra $B(\mathcal{A})$ for all $a \in \mathcal{A}$. Let $\tilde{\mathcal{A}}$ be the subset of $B(\mathcal{A})$ given by

$$\tilde{\mathcal{A}} = \{L_a + \lambda \mathbb{1} \,|\, a \in \mathcal{A},\ \lambda \in \mathbb{C}\},$$

where $\mathbb{1}$ denotes the identity operator $\mathcal{A} \to \mathcal{A}$, and define a map $\Omega\colon \mathcal{A} \to \tilde{\mathcal{A}}$ by $\Omega(a) = L_a$.

Lemma 1. *The map Ω is an isometric algebra homomorphism such that $\Omega(a) \neq \mathbb{1}$ for all $a \in \mathcal{A}$. $\tilde{\mathcal{A}}$ is a subalgebra of $B(\mathcal{A})$.*

Proof. For $a, b, c \in \mathcal{A}$ and $\lambda \in \mathbb{C}$, we have

(i) $\Omega(a+b)(c) = L_{a+b}c = (a+b)c = ac + bc = L_a c + L_b c = (L_a + L_b)c = (\Omega(a) + \Omega(b))(c)$,
(ii) $\Omega(\lambda a)(c) = L_{\lambda a}c = (\lambda a)c = \lambda(ac) = (\lambda L_a)c = (\lambda\Omega(a))(c)$,
(iii) $\Omega(ab)(c) = L_{ab}c = (ab)c = a(bc) = L_a(L_b c) = \Omega(a)(\Omega(b)(c)) = (\Omega(a)\Omega(b))(c)$.

To see that Ω is an isometry, note that $\|\Omega(a)\| \leq \|a\|$ and

$$\|a\|^2 = \|a^*\|^2 = \|aa^*\| = \|\Omega(a)(a^*)\| \leq \|\Omega(a)\|\|a^*\| = \|\Omega(a)\|\|a\|.$$

If there were an $a \in \mathcal{A}$ such that $\Omega(a) = \mathbb{1}$, then for any $b \in \mathcal{A}$ we would have

$$ab = \Omega(a)(b) = \mathbb{1}b = b.$$

Hence a is a left unit for $\mathcal{A}$. The above equality also implies $ab^* = b^*$ or $ba^* = b$ for all $b \in \mathcal{A}$, so a^* is a right unit for $\mathcal{A}$. Thus $a = aa^* = a^*$, implying that a is a unit, contradicting the assumption that $\mathcal{A}$ is non-unital. The final statement now follows from the fact that Ω and the map $\mathbb{C} \to \tilde{\mathcal{A}}$ given by $\lambda \mapsto \lambda\mathbb{1}$ are algebra homomorphisms. □

Corollary 2. *The map $\mathcal{A} \times \mathbb{C} \to \tilde{\mathcal{A}}$ given by $(a, \lambda) \mapsto L_a + \lambda\mathbb{1}$ is a linear isomorphism.*

Proof. Surjectivity and linearity is clear. If $a_1, a_2 \in \mathcal{A}$ and $\lambda_1, \lambda_2 \in \mathbb{C}$ satisfy $L_{a_1} + \lambda_1\mathbb{1} = L_{a_2} + \lambda_2\mathbb{1}$, then $L_{a_1 - a_2} = (\lambda_2 - \lambda_1)\mathbb{1}$. If $\eta = \lambda_2 - \lambda_1$ were a non-zero number, we would have $L_{\eta^{-1}(a_1 - a_2)} = \mathbb{1}$, contradicting the above lemma. Hence $\lambda_1 = \lambda_2$, so

$$\|a_1 - a_2\| = \|L_{a_1 - a_2}\| = \|(\lambda_2 - \lambda_1)\mathbb{1}\| = 0$$

and hence $a_1 = a_2$. □

The above corollary in turn yields that the map $^*\colon \tilde{\mathcal{A}} \to \tilde{\mathcal{A}}$ given by $(L_a + \lambda \mathbb{1})^* = L_{a^*} + \overline{\lambda}\mathbb{1}$ is well-defined, and one easily shows that it satisfies the properties of an involution. Hence $\tilde{\mathcal{A}}$ becomes a normed *-algebra.

Proposition 3. *For all $s \in \tilde{\mathcal{A}}$ we have $\|s^*s\| = \|s\|^2$.*

Proof. Let $a, x \in \mathcal{A}$ and $\lambda \in \mathbb{C}$. For any $b \in \mathcal{A}$ we have

$$(x^*a^* + \overline{\lambda}x^*)b = x^*(a^*b + \overline{\lambda}b) = x^*(L_a + \lambda\mathbb{1})^*b. \tag{†}$$

This in turn implies

$$\begin{aligned}
\|(L_a + \lambda\mathbb{1})x\|^2 &= \|ax + \lambda x\|^2 \\
&= \|(ax + \lambda x)^*(ax + \lambda x)\| \\
&= \|(x^*a^* + \overline{\lambda}x^*)(ax + \lambda x)\| \\
&= \|x^*(L_a + \lambda\mathbb{1})^*(L_a + \lambda\mathbb{1})x\| \\
&\le \|x^*\|\|(L_a + \lambda\mathbb{1})^*(L_a + \lambda\mathbb{1})x\| \le \|(L_a + \lambda\mathbb{1})^*(L_a + \lambda\mathbb{1})\|\|x\|^2,
\end{aligned}$$

using (†) at the fourth equality. Therefore

$$\|L_a + \lambda\mathbb{1}\|^2 \le \|(L_a + \lambda\mathbb{1})^*(L_a + \lambda\mathbb{1})\|.$$

The above inequality implies

$$\|L_a + \lambda\mathbb{1}\|^2 \le \|(L_a + \lambda\mathbb{1})^*\|\|L_a + \lambda\mathbb{1}\|,$$

so $\|L_a + \lambda\mathbb{1}\| \le \|(L_a + \lambda\mathbb{1})^*\| = \|L_{a^*} + \overline{\lambda}\mathbb{1}\|$. Replacing a by a^* and λ by $\overline{\lambda}$, we see that

$$\|L_a + \lambda\mathbb{1}\| = \|(L_a + \lambda\mathbb{1})^*\|.$$

This finally tells us that

$$\|L_a + \lambda\mathbb{1}\|^2 \le \|(L_a + \lambda\mathbb{1})^*(L_a + \lambda\mathbb{1})\| \le \|(L_a + \lambda\mathbb{1})^*\|\|L_a + \lambda\mathbb{1}\| = \|L_a + \lambda\mathbb{1}\|^2,$$

completing the proof. □

Proposition 4. *Let $\mathfrak{X}$ be a Banach space with closed subspaces $\mathfrak{Y}$ and $\mathfrak{Z}$. If $\mathfrak{Z}$ is finite-dimensional, then $\mathfrak{Y} + \mathfrak{Z}$ is a closed subspace of $\mathfrak{X}$.*

Proof. Recall that the quotient space $\mathfrak{X}/\mathfrak{Y}$ is a Banach space and that the quotient map $\pi\colon \mathfrak{X} \to \mathfrak{X}/\mathfrak{Y}$ is a linear contraction. Then $\pi(\mathfrak{Z})$ is a finite-dimensional subspace of $\mathfrak{X}/\mathfrak{Y}$, so it must be closed. Therefore $\pi^{-1}(\pi(\mathfrak{Z}))$ is closed as well by continuity, but

$$\begin{aligned}
x \in \pi^{-1}(\pi(\mathfrak{Z})) &\Leftrightarrow \pi(x) = \pi(z) \text{ for some } z \in \mathfrak{Z} \\
&\Leftrightarrow x - z = y \text{ for some } z \in \mathfrak{Z} \text{ and } y \in \mathfrak{Y} \\
&\Leftrightarrow x \in \mathfrak{Y} + \mathfrak{Z}.
\end{aligned}$$

Hence $\mathfrak{Y} + \mathfrak{Z}$ is closed. □

Corollary 5. *$\tilde{\mathcal{A}}$ is a closed subset of $B(\mathcal{A})$, making it a unital C^*-algebra.*

Proof. Applying Proposition 4 to $\mathfrak{X} = B(\mathcal{A})$, $\mathfrak{Y} = \Omega(\mathcal{A})$ and $\mathfrak{Z} = \mathbb{C}\mathbb{1}$ yields that $\tilde{\mathcal{A}}$ is closed. Hence $\tilde{\mathcal{A}}$ is a Banach *-algebra satisfying the C^*-identity (Proposition 3), so it is a C^*-algebra with unit $\mathbb{1}$. □

By Corollary 2, there is a linear isomorphism $\mathcal{A} \times \mathbb{C} \to \tilde{\mathcal{A}}$. Using this isomorphism, we can define a multiplication, involution and norm on $\mathcal{A} \times \mathbb{C}$ such that it becomes a unital C^*-algebra in which $\mathcal{A}$ is isometrically embedded by means of the map $\mathcal{A} \to \mathcal{A} \times \mathbb{C}$ given by $a \mapsto (a, 0)$. The multiplicative unit of $\mathcal{A} \times \mathbb{C}$ is the element $(0, 1)$. The C^*-algebra $\mathcal{A} \times \mathbb{C}$ is called the *unitization* of $\mathcal{A}$, and to honour the subset of $B(\mathcal{A})$ that made it possible, we will denote it by $\tilde{\mathcal{A}}$. The *-algebra operations in $\tilde{\mathcal{A}}$ are therefore

(i) $\mu_1(a_1, \lambda_1) + \mu_2(a_2, \lambda_2) = (\mu_1 a_1 + \mu_2 a_2, \mu_1\lambda_1 + \mu_2\lambda_2)$,
(ii) $(a_1, \lambda_1)(a_2, \lambda_2) = (a_1a_2 + \lambda_1 a_2 + \lambda_2 a_1, \lambda_1\lambda_2)$ and
(iii) $(a_1, \lambda_1)^* = (a_1^*, \overline{\lambda_1})$

for $a_1, a_2 \in \mathcal{A}$ and $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{C}$. The norm is given by

$$\|(a, \lambda)\| = \sup\{\|ax + \lambda x\| \mid x \in \mathcal{A},\ \|x\| \le 1\}.$$

2 The continuous functional calculus, part I

We first take some time to construct the continuous functional calculus for unital C^*-algebras. If $\mathcal{A}$ is a unital C^*-algebra, the spectrum $\sigma(a)$ of an element $a \in \mathcal{A}$ consists of all the $\lambda \in \mathbb{C}$ such that $\lambda 1_{\mathcal{A}} - a$ is not invertible. The space of non-zero multiplicative linear functionals $\mathcal{A} \to \mathbb{C}$ (also called *characters*) is denoted by $\Delta(\mathcal{A})$. Any such is bounded with norm 1 and hence $\Delta(\mathcal{A})$ is a weak*-compact subset of $\mathcal{A}^*$ by Alaoglu's theorem.

The *Gelfand transform* for the unital C^*-algebra $\mathcal{A}$ is the contractive algebra homomorphism $\Gamma\colon \mathcal{A} \to C(\Delta(\mathcal{A}))$ given by

$$\Gamma(a)(\varphi) = \varphi(a), \quad a \in \mathcal{A},\ \varphi \in \Delta(\mathcal{A}).$$

If $\mathcal{A}$ is also commutative, we have that the range of $\Gamma(a)$ for any $a \in \mathcal{A}$ is in fact $\sigma(a)$. If $\mathcal{A}$ is a commutative unital C^*-algebra, Γ is an isometric *-isomorphism.

Theorem 6. *Let $\mathcal{A}$ be a unital C^*-algebra and let $a \in \mathcal{A}$ be normal. Then $C^*(1_{\mathcal{A}}, a)$ is commutative, and the character space $\mathfrak{M}$ of $C^*(1_{\mathcal{A}}, a)$ equipped with the weak* topology is homeomorphic to $\sigma(a)$ by the map $\Psi\colon \varphi \mapsto \varphi(a)$. Therefore the Gelfand transform becomes an isometric *-isomorphism $\Gamma\colon C^*(1_{\mathcal{A}}, a) \to C(\sigma(a))$.*

Proof. Omitted. See (3, Theorem 10.2). □

Definition 7. *If $\mathcal{A}$ is a unital C^*-algebra and $a \in \mathcal{A}$ is normal, the inverse unital *-homomorphism $\Gamma^{-1}\colon C(\sigma(a)) \to C^*(1_{\mathcal{A}}, a)$ is called the* continuous functional calculus *for a. For $f \in C(\sigma(a))$, we define $f(a) = \Gamma^{-1}(f)$.*

Theorem 8 (Properties of the continuous functional calculus for unital C^*-algebras)**.** *Let $\mathcal{A}$ be a unital C^*-algebra and let $a \in \mathcal{A}$. Then for all $\lambda, \mu \in \mathbb{C}$ and $f, g \in C(\sigma(a))$ we have*

(i) $(\lambda f + \mu g)(a) = \lambda f(a) + \mu g(a)$.
(ii) $(fg)(a) = f(a)g(a)$.
(iii) $\overline{f}(a) = f(a)^*$.
(iv) *If 1 denotes the identity map $\sigma(a) \to \sigma(a)$, then $1(a) = a$.*
(v) *If 1 denotes the constant function $z \mapsto 1$ for $z \in \sigma(a)$, then $1(a) = 1_{\mathcal{A}}$.*
(vi) *If $P\colon z \mapsto p(z, \overline{z})$ is a complex polynomial in z and $\overline{z}$ with no constant term, then $P(a) = p(a, a^*)$.*
(vii) *If Ω is a subset of $\mathbb{C}$ such that $\sigma(a) \subseteq \Omega$, then*

$$\|f(a)\| = \sup_{z\in\sigma(a)} |f(z)| \leq \sup_{z\in\Omega} |f(z)|.$$

(viii) $\sigma(f(a)) = f(\sigma(a))$.
(ix) *If $h \in C(f(\sigma(a)))$, then $(h \circ f)(a) = h(f(a))$.*
(x) *If Φ is a unital *-homomorphism of $\mathcal{A}$ into another unital C^*-algebra $\mathcal{B}$, then $\Phi(f(a)) = f(\Phi(a))$.*

Proof. (i), (ii) and (iii) and (v) follow directly from the fact that the continuous functional calculus is a unital *-homomorphism. As for (iv), note that $\Gamma(a)$ is the identity map on $\sigma(a)$; any $z \in \sigma(a)$ corresponds uniquely to a character $\varphi \in \Delta(C^*(1_{\mathcal{A}}, a))$ such that $\varphi(a) = \Psi(\varphi) = z$. Under the identification of Theorem 6, we have

$$\Gamma(a)(z) = \Gamma(a)(\varphi) = \varphi(a) = z.$$

(vi) then follows from (i)-(v). (vii) is the content of (3, Theorem 8.1).

Since Γ^{-1} is a *-isomorphism, $\sigma(f) = \sigma(f(a))$. $f - \lambda$ is invertible if and only if $f(z) - \lambda \neq 0$ for all $z \in \sigma(a)$; hence $\lambda \in \sigma(f)$ if and only if $f(z) = \lambda$ for some $z \in \sigma(a)$ or $\lambda \in f(\sigma(a))$, so we conclude

$$\sigma(f(a)) = \sigma(f) = f(\sigma(a)),$$

and hence (viii).

To prove (ix), note that $g \circ f \in C(\sigma(a))$, so $(g \circ f)(a)$ is well-defined, and that $f(a) = \Gamma^{-1}(f)$ is normal in $\mathcal{A}$ since f is. Hence $g(f(a)) \in \mathcal{A}$ is well-defined, as $g \in C(\sigma(f(a))$ by (viii). Take a sequence

$(q_n)_{n\geq 1}$ of complex polynomials in z and $\overline{z}$ such that $q_n \to g$ uniformly on $f(\sigma(a)) = \sigma(f(a))$ by Stone-Weierstrass' theorem. Then $q_n \circ f \to g \circ f$ uniformly on $\sigma(a)$, so

$$q_n(f(a)) = (q_n \circ f)(a) \to (g \circ f)(a)$$

in norm. To comprehend the first equality above, note that if $q_n(z) = \sum \lambda_{ij} z^i \overline{z}^j$ then we have

$$(q_n \circ f)(z) = \sum \lambda_{ij} f(z)^i \overline{f}(z)^j.$$

The continuous functional calculus is a *-homomorphism and thus maps this function to

$$(q_n \circ f)(a) = \sum \lambda_{ij} f(a)^i (f(a)^*)^j = q_n(f(a)).$$

Noting that $q_n(f(a)) \to g(f(a))$ in norm as well, (ix) follows.

For (x), note that $b = \psi(a)$ is normal. Assume first that $\mathcal{B}$ is unital. As $\sigma(b)) \subseteq \sigma(a)$, we have $f \in C(\sigma(b)$, so $f(b)$ is a well-defined element of $C^*(1_{\mathcal{B}}, b) \subseteq \mathcal{B}$. Let $(p_n)_{n\geq 1}$ be a sequence of complex polynomials in z and $\overline{z}$ such that $p_n \to f$ uniformly on $\sigma(a)$. Then $p_n(a) \to f(a)$ and $p_n(b) \to f(b)$ in norm, so we have

$$\psi(f(a)) = \psi\left(\lim_{n\to\infty} p_n(a)\right) = \lim_{n\to\infty} \psi(p_n(a)) = \lim_{n\to\infty} p_n(b) = f(b),$$

since ψ is a *-homomorphism and hence continuous. We save the non-unital case for later, once we prove the next theorem. □

We will see later that $\mathcal{B}$ in (x) does not have to be unital, as long as $f(0) = 0$.

3 The continuous functional calculus, part II

In this section, assume that $\mathcal{A}$ is a non-unital C^*-algebra and that $a \in \mathcal{A}$ is a normal element. Let $\iota\mathcal{A}$ be the image of $\mathcal{A}$ in $\tilde{\mathcal{A}}$ under the isometric inclusion *-homomorphism. The *spectrum* $\sigma(a)$ of $a \in \mathcal{A}$ is defined to be the spectrum of $\tilde{a} \in \tilde{\mathcal{A}}$.

Proposition 9. *For any normal element a of a non-unital C^*-algebra, we have*

(i) *a is self-adjoint if and only if $\sigma(a) \subseteq \mathbb{R}$.*
(ii) *a is positive if and only if $\sigma(a) \subseteq \mathbb{R}_+$.*
(iii) *a is a projection if and only if $\sigma(a) \subseteq \{0, 1\}$.*
(iv) *$\|a\| = \sup_{z\in\sigma(a)} |z|$.*

Proof. We have that a is self-adjoint iff $\tilde{a}$ is self-adjoint iff $\sigma(a) = \sigma(\tilde{a}) \subseteq \mathbb{R}$ and that a is a projection iff $\tilde{a}$ is a projection iff $\sigma(a) \subseteq \{0, 1\}$ by (3, Theorem 10.4); hence (i) and (iii). (iv) follows from (3, Theorem 8.1). If a is positive, then $\tilde{a}$ is positive and $\sigma(a) = \sigma(\tilde{a}) \subseteq \mathbb{R}_+$ by (3, Theorem 11.5). We save the converse for later, i.e. that a is positive if $\tilde{a}$ is positive. □

The above proposition ensures that all we know about spectra of special elements in a C^*-algebra still holds whether the C^*-algebra in question has a unit or not.

Proposition 10. *For any *-homomorphism $\varphi\colon \mathcal{A} \to \mathcal{B}$ of arbitrary C^*-algebras and any $a \in \mathcal{A}$, we have*

$$\{0\} \cup \sigma(\varphi(a)) \subseteq \{0\} \cup \sigma(a).$$

Proof. Assume first that $\mathcal{A}$ and $\mathcal{B}$ are unital. If $\lambda \neq 0$ and $a - \lambda 1_{\mathcal{A}}$ is invertible with inverse x, then

$$(\varphi(a) - \lambda 1_{\mathcal{B}})\left(\varphi(x) + \frac{1}{\lambda}\varphi(1_{\mathcal{A}}) - \frac{1}{\lambda}1_{\mathcal{B}}\right) = 1_{\mathcal{B}},$$

so the result holds. (Note that if $\varphi(1_{\mathcal{A}}) = 1_{\mathcal{B}}$), then $\sigma(\varphi(a)) \subseteq \sigma(a)$.)

Assume that $\mathcal{A}$ is unital and that $\mathcal{B}$ isn't. If $\lambda \neq 0$ and $a - \lambda 1_{\mathcal{A}}$ is invertible with inverse x, then

$$(\varphi(a), -\lambda)\left(\varphi(x) + \frac{1}{\lambda}\varphi(1_{\mathcal{A}}), -\frac{1}{\lambda}\right) = (0, 1),$$

so $(\varphi(a), -\lambda) = \widetilde{\varphi(a)} - \lambda 1_{\tilde{\mathcal{B}}}$ is invertible in $\tilde{\mathcal{B}}$.

Assume penultimately that $\mathcal{A}$ is non-unital and that $\mathcal{B}$ is. If $\lambda \neq 0$ and $(a, -\lambda)$ is invertible in $\tilde{\mathcal{A}}$ with inverse (x, μ), then $ax - \lambda x + \mu a = 0$ and $-\lambda\mu = 1$, so that

$$(\varphi(a) - \lambda 1_{\mathcal{B}})\,(\varphi(x) + \mu 1_{\mathcal{B}}) = 1_{\mathcal{B}},$$

so $\varphi(a) =$ is invertible in $\tilde{\mathcal{B}}$.

Lastly, assume that $\mathcal{A}$ and $\mathcal{B}$ are both non-unital. If $\lambda \neq 0$ and $(a, -\lambda)$ is invertible in $\tilde{\mathcal{A}}$ with inverse (x, μ), then

$$(\varphi(a), -\lambda)(\varphi(x), \mu) = (0, 1),$$

completing the proof. □

Corollary 11. *Any *-homomorphism $\varphi\colon \mathcal{A} \to \mathcal{B}$ of arbitrary C^*-algebras is contractive.*

Proof. We have

$$\|\varphi(x)\|^2 = \|\varphi(x^*x)\| = \sup_{z\in\sigma(\varphi(x^*x))} |z| \leq \sup_{z\in\sigma(x^*x)\cup\{0\}} |z| = \sup_{z\in\sigma(x^*x)} |z| = \|x^*x\| = \|x\|^2$$

by Propositions 9 and 10. □

Let $\pi\colon \tilde{\mathcal{A}} \to \mathbb{C}$ be the unital *-homomorphism given by $(a, \lambda) \mapsto \lambda$, and note that $\ker\pi = \iota\mathcal{A}$. If $a \in \mathcal{A}$ is normal and $f \in C(\sigma(a))$, then $0 \in \sigma(a)$, so $f(0)$ is well-defined. By Theorem 8 we have

$$\pi(f(\tilde{a})) = f(\pi(\tilde{a})) = f(0).$$

Hence $f(\tilde{a})$ belongs to $\iota\mathcal{A}$ if and only if $f(0) = 0$. The set $C_z(\sigma(a)))$ of $f \in C(\sigma(a))$ that satisfy $f(0) = 0$ is a C^*-subalgebra of $C(\sigma(a))$, so we obtain the following result:

Corollary 12. *If a is a normal element of a non-unital C^*-algebra $\mathcal{A}$, there is an injective *-homomorphism $C_z(\sigma(a)) \to A$ such that the identity map $1\colon \sigma(a) \to \sigma(a)$ is mapped to a itself.*

Proof. The injective *-homomorphism $C_z(\sigma(a)) \to C(\sigma(a)) \to C^*(1_{\tilde{\mathcal{A}}}, \tilde{a})$ maps into $\iota\mathcal{A}$. Compose with the *-isomorphism $\iota\mathcal{A} \to \mathcal{A}$ to obtain the desired homomorphism. □

Definition 13. *If $a \in \mathcal{A}$ is normal and $f \in C_z(\sigma(a))$, we define $f(a) \in \mathcal{A}$ to be the image of f under the above *-homomorphism. The *-homomorphism itself is called the* continuous functional calculus *for a.*

Note that

$$\widetilde{f(a)} = (f(a), 0) = f(\tilde{a})$$

by how the *-homomorphism is defined.

Theorem 14 (Properties of the continuous functional calculus for non-unital C^*-algebras)**.** *Let $\mathcal{A}$ be a non-unital C^*-algebra. If $a \in \mathcal{A}$ is normal, then for all $\lambda, \mu \in \mathbb{C}$ and $f, g \in C_z(\sigma(a))$ we have*

(i) $(\lambda f + \mu g)(a) = \lambda f(a) + \mu g(a)$.
(ii) $(fg)(a) = f(a)g(a)$.
(iii) $\overline{f}(a) = f(a)^*$.
(iv) *If 1 denotes the identity map $\sigma(a) \to \sigma(a)$, then $1(a) = a$.*
(v) *If $P\colon z \mapsto p(z, \tilde{z})$ is a complex polynomial in z and $\overline{z}$ with no constant term, then $P(a) = p(a, a^*)$.*
(vi) *If Ω is a subset of $\mathbb{C}$ such that $\sigma(a) \subseteq \Omega$, then*

$$\|f(a)\| = \sup_{z\in\sigma(a)} |f(z)| \leq \sup_{z\in\Omega} |f(z)|.$$

(vii) $\sigma(f(a)) = f(\sigma(a))$.
(viii) *If $h \in C_z(f(\sigma(a)))$, then $(h \circ f)(a) = h(f(a))$.*
(ix) *If Φ is a *-homomorphism of $\mathcal{A}$ into another C^*-algebra $\mathcal{B}$, then $\Phi(f(a)) = f(\Phi(a))$.*

Proof. (i)-(iii) follows from the continuous functional calculus being a *-homomorphism. (iv) is immediate from the definition, and (v) follows accordingly. To see (vi), note that Theorem 8(vii) yields

$$\|f(a)\| = \|f(\tilde{a})\| = \sup_{z\in\sigma(\tilde{a})} |f(z)| = \sup_{z\in\sigma(a)} |f(z)|.$$

(vii) is easy, as Theorem 8(viii) yields

$$\sigma(f(a)) = \sigma(\widetilde{f(a)}) = \sigma(f(\tilde{a})) = f(\sigma(\tilde{a})) = f(\sigma(a)).$$

To prove (viii), Theorem 8(ix) yields

$$\widetilde{(g\circ f)(a)} = (g\circ f)(\tilde{a}) = g(f(\tilde{a}) = g(\widetilde{f(a)}) = \widetilde{g(f(a))}.$$

For (ix), assume first that $\mathcal{B}$ is unital and define a *-homomorphism $\tilde{\Phi}\colon \tilde{\mathcal{A}} \to \mathcal{B}$ by $\tilde{\Phi}(a,\lambda) = \Phi(a) + \lambda 1_{\mathcal{B}}$. Theorem 8(x) now yields

$$\Phi(f(a)) = \tilde{\Phi}(\widetilde{f(a)}) = \tilde{\Phi}(f(\tilde{a})) \overset{8(x)}{=} f(\tilde{\Phi}(\tilde{a})) = f(\Phi(a)).$$

If $\mathcal{B}$ is non-unital, we instead define a *-homomorphism $\tilde{\Phi}\colon \tilde{\mathcal{A}} \to \tilde{\mathcal{B}}$ by $\tilde{\Phi}(a,\lambda) = (\Phi(a),\lambda)$. Then

$$\widetilde{\Phi(f(a))} = \tilde{\Phi}(\widetilde{f(a)}) = \tilde{\Phi}(f(\tilde{a})) \overset{8(x)}{=} f(\tilde{\Phi}(\tilde{a})) = f(\widetilde{\Phi(a)}) = \widetilde{f(\Phi(a))},$$

and hence we obtain the wanted equality. □

Thus all the usual tricks that one may perform in order to construct specific "nice" elements of a C^*-algebra still work.

Last part of the proof of Proposition 9. Since $\sigma(a) = \sigma(\tilde{a}) \subseteq \mathbb{R}_+$, let $f(t) = \sqrt{t}$ for $t \in \sigma(a)$. As $f(0) = 0$, we have $f \in C_z(\sigma(a))$ and $f(a) \in \mathcal{A}$. Since $f(a)$ is self-adjoint as $\overline{f} = f$ and $f(t)^2 = 1(t)$ for all $t \in \sigma(\tilde{a})$ we conclude that $f(a)^* f(a) = 1(a) = a$, i.e. that a is positive. □

As promised, we provide a proof of Theorem 8(x) in the case where $\mathcal{B}$ is not unital.

Theorem 15. *Let $\mathcal{A}$ be a unital C^*-algebra, and let $a \in \mathcal{A}$ be normal. If $\mathcal{B}$ is a non-unital C^*-algebra and $\Phi\colon \mathcal{A} \to \mathcal{B}$ is a *-homomorphism, then*

$$\Phi(f(a)) = f(\Phi(a))$$

for all $f \in C_z(\sigma(a) \cup \{0\})$.

Proof. Let $f \in C_z(\sigma(a)\cup\{0\})$ and take a sequence of complex polynomials $(p_n)_{n\geq 1}$ in z and $\overline{z}$ such that $p_n \to f$ uniformly on $\sigma(a)\cup\{0\}$. Define polynomials $q_n(z) = p_n(z) - p_n(0)$ and note that for all $z \in \sigma(a)\cup\{0\}$, we have

$$|q_n(z) - f(z)| \leq |p_n(z) - f(z)| + |f(0) - p_n(0)| \leq 2 \sup_{z\in\sigma(a)\cup\{0\}} \|p_n(z) - f(z)\|,$$

so that $q_n \to f$ uniformly on $\sigma(a)\cup\{0\}$. We now have $q_n(\Phi(a)) = \Phi(q_n(a)) \to \Phi(f(a))$ by continuity of Φ. Since the continuous functional calculus for non-unital C^*-algebras is continuous, then because $\Phi(a)$ is normal and $\sigma(\Phi(a)) \subseteq \sigma(a)\cup\{0\}$, we have $q_n(\Phi(a)) \to f(\Phi(a))$. The result follows. □

4 For your eyes only, only for you

We provide a summary of what we have proved so far, enabling us to use the continuous functional calculus for any C^*-algebra, unital or not.

Theorem 16 (Properties of the continuous functional calculus for arbitrary C^*-algebras)**.** *Let $\mathcal{A}$ be an arbitrary C^*-algebra, let $a \in \mathcal{A}$ be normal and define a *-subalgebra C_z of $C(\sigma(a))$ given by*

$$C_z = \{f \in C(\sigma(a)) \mid \exists g \in C(\sigma(a)\cup\{0\}) : g|_{\sigma(a)} = f \text{ and } g(0) = 0\}.$$

Then there exists a *-homomorphism $C_z \to \mathcal{A}$, $f \mapsto f(a)$ such that for all $\lambda,\mu \in \mathbb{C}$ and $f, g \in C_z$, we have

(i) $(\lambda f + \mu g)(a) = \lambda f(a) + \mu g(a)$.
(ii) $(fg)(a) = f(a)g(a)$.
(iii) $\overline{f}(a) = f(a)^*$.
(iv) *If* 1 *denotes the identity map* $\sigma(a) \to \sigma(a)$, *then* $1(a) = a$.
(v) *If* $P\colon z \mapsto p(z, \tilde{z})$ *is a complex polynomial in* z *and* $\overline{z}$ *with no constant term, then* $P(a) = p(a, a^*)$.
(vi) *If* Ω *is a subset of* $\mathbb{C}$ *such that* $\sigma(a) \subseteq \Omega$, *then*

$$\|f(a)\| = \sup_{z\in\sigma(a)} |f(z)| \leq \sup_{z\in\Omega} |f(z)|.$$

(vii) $\sigma(f(a)) = f(\sigma(a))$.
(viii) *If* $h \in C(f(\sigma(a)))$ *has an continuous extension to* $f(\sigma(a)) \cup \{0\}$ *such that* $h(0) = 0$, *then* $(h \circ f)(a) = h(f(a))$.
(ix) *If* Φ *is a* **-homomorphism of* $\mathcal{A}$ *into another* C^**-algebra* $\mathcal{B}$, *then* $\Phi(f(a)) = f(\Phi(a))$.

Proof. This follows from Theorems 8, 14 and 15. □

5 An example

We provide some nice applications to emphasize that we do not need to know whether a C^*-algebra is unital or not in order to use the continuous functional calculus.

Proposition 17. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be arbitrary* C^**-algebras and let* $\varphi\colon \mathcal{A} \to \mathcal{B}$ *be a* **-homomorphism. Then if* $b \in \varphi(\mathcal{A})$ *and* b *is self-adjoint, there exists a self-adjoint element* $a \in \mathcal{A}$ *such that* $\varphi(a) = b$ *and* $\|a\| = \|b\|$.

Proof. Take $x \in \mathcal{A}$ such that $\varphi(x) = b$ and define $y = \frac{1}{2}(x + x^*)$. Then y is self-adjoint and

$$\varphi(y) = \frac{1}{2}(\varphi(x) + \varphi(x)^*) = \frac{1}{2}(b + b) = b.$$

Define $f\colon \mathbb{R} \to \mathbb{R}$ by

$$f(z) = \begin{cases} -\|b\| & z \leq -\|b\| \\ z & -\|b\| \leq z \leq \|b\| \\ \|b\| & z \geq \|b\| \end{cases}$$

Clearly f is continuous and $f(0) = 0$. We have that $\sigma(b) \subseteq \mathbb{R}$ and that $f|_{\sigma(b)}$ is the identity map on $\sigma(b)$, so we conclude $f(b) = b$. Likewise $\sigma(y) \subseteq \mathbb{R}$, so we can define $a = f(y) \in \mathcal{A}$. Since $\sigma(a) = \sigma(f(y)) = f(\sigma(y)) \subseteq \mathbb{R}$, a is self-adjoint and

$$\|a\| = \sup_{z\in\sigma(y)} |f(z)| \leq \|b\|.$$

As

$$\varphi(a) = \varphi(f(y)) = f(\varphi(y)) = f(b) = b$$

and $\|b\| = \|\varphi(a)\| \leq \|a\|$, we obtain the desired result. □

Theorem 18. *Let* $\varphi\colon \mathcal{A} \to \mathcal{B}$ *be a* **-homomorphism of arbitrary* C^**-algebras. Then the image* $\varphi(\mathcal{A})$ *is a* C^**-subalgebra of* $\mathcal{B}$.

Proof. The proof of (3, Theorem 11.1) adjusts easily to the general functional calculus, since all functions used in the proof map 0 to 0. □

Corollary 19. *If* $\varphi\colon \mathcal{A} \to \mathcal{B}$ *is an injective* **-homomorphism of arbitrary* C^**-algebras, then it is an isometry.*

Proof. By the above theorem, $\varphi(\mathcal{A})$ is a C^*-algebra and the map $\varphi^{-1}\colon \varphi(\mathcal{A}) \to \mathcal{A}$ is a *-homomorphism. Use Corollary 11 on φ and φ^{-1}. □

Theorem 20. *Any self-adjoint element* x *of any* C^**-algebra* $\mathcal{A}$ *is a difference of positive elements* x^+ *and* x^- *in* $\mathcal{A}$ *such that* $x^+x^- = x^-x^+ = 0$ *and* $\|x\| = \max\{\|x^+\|, \|x^-\|\}$.

Proof. The proof of (3, Theorem 11.2) adjusts easily. □

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