# Sølvkorn 17
# Cohen's factorization theorem

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For any normed algebra $\mathscr{A}$ and a normed space $\mathscr{X}$, we say that $\mathscr{X}$ is a *left normed $\mathscr{A}$-module* if $\mathscr{X}$ is a left $\mathscr{A}$-module and there exists $\kappa \geq 1$ such that

$$\|ax\| \leq \kappa\|a\|\|x\|, \quad a \in \mathscr{A},\ x \in \mathscr{X}.$$

If $\mathscr{X}$ is also a Banach space, we say that $\mathscr{X}$ is a *left Banach $\mathscr{A}$-module*. Similarly we can define normed and Banach right $\mathscr{A}$-modules and $\mathscr{A}$-bimodules.

A net $(e_\beta)_{\beta\in B}$ of a normed algebra $\mathscr{A}$ is said to be a *left (resp. right) approximate identity for $\mathscr{A}$* if $e_\beta x \to x$ (resp. $xe_\beta \to x$) for all $x \in \mathscr{A}$. If $\mathscr{X}$ is a normed left (resp. right) $\mathscr{A}$-module we say that $(e_\beta)_{\beta\in B}$ is a *left (resp. right) approximate identity for $\mathscr{X}$* if $e_\beta x \to x$ (resp. $xe_\beta \to x$) for all $x \in \mathscr{X}$.

The following result is extremely beautiful and very surprising.

**Cohen's factorization theorem.** *Let $\mathscr{A}$ be a Banach algebra and let $\mathscr{X}$ be a left Banach $\mathscr{A}$-module (with constant $\kappa \geq 1$) and suppose that $\mathscr{A}$ has a left approximate identity $(e_\beta)_{\beta\in B}$ that is also a left approximate identity for $\mathscr{X}$, bounded by some constant $\delta \geq 1$. Then for any $x_0 \in \mathscr{X}$ and $\varepsilon > 0$, there exist $a \in \mathscr{A}$ and $x \in \overline{\mathscr{A}x_0}$ such that $x_0 = ax$, $\|a\| \leq \delta$ and $\|x - x_0\| \leq \varepsilon$. Moreover, if $(e_\beta)_{\beta\in B}$ belongs to a certain closed convex subset $C$ of $\mathscr{A}$, then $a$ can be chosen to belong to $C$ as well.*

*Proof.* Let $\mathscr{A}_1$ be the unitization of $\mathscr{A}$ with unit $e$ and norm $\|a + \lambda e\| = \|a\| + |\lambda|$; then it is easy to show that $\mathscr{X}$ is a left Banach $\mathscr{A}_1$-module (bounded by $\kappa$) if we define

$$(a + \lambda e)x = ax + \lambda x, \quad a \in \mathscr{A},\ \lambda \in \mathbb{C}\, x \in \mathscr{X}.$$

For $\gamma = \frac{1}{2\delta+1}$, define $E_\beta = (1-\gamma)e + \gamma e_\beta$ for all $\beta \in A$. Then

$$\|E_\beta x - x\| = \gamma\|e_\beta x - x\| \to 0, \quad x \in \mathscr{X},$$

so that $(E_\beta)$ is a bounded left approximate identity for $\mathscr{A}$ and $\mathscr{X}$. Moreover, we have

$$\|e - E_\beta\| = \gamma\|e - e_\beta\| = \gamma(\|e_\beta\| + 1) \leq \gamma(\delta + 1) < 1,$$

so $E_\beta$ is invertible in $\mathscr{A}_1$ and $\|E_\beta^{-1}\| \leq \Delta = \frac{1}{1-\gamma(\delta+1)}$ [4, Proposition 2.1] so now

$$\|E_\beta^{-1}x - x\| = \|E_\beta^{-1}x - E_\beta^{-1}E_\beta x\| = \|E_\beta^{-1}\|\|x - E_\beta x\| \to 0, \quad x \in \mathscr{X},$$

and thus $(E_\beta^{-1})$ is also a bounded left approximate identity for $\mathscr{A}$ and $\mathscr{X}$.

Now let $x_0 \in \mathscr{X}$ and $\varepsilon > 0$. We prove by induction that for all $n \geq 1$ there exists a sequence $(e_n)_{n\geq 1}$ of elements in $\mathscr{A}$ such that

(i) $\|e_n\| \leq \delta$ for all $n \geq 1$,
(ii) $F_n := \sum_{i=1}^n (1-\gamma)^{i-1}\gamma e_i + (1-\gamma)^n e$ is invertible for all $n \geq 1$ and
(iii) $\|F_1^{-1}x_0 - x_0\| < \frac{\varepsilon}{2}$ and $\|F_n^{-1}x_0 - F_{n-1}^{-1}x_0\| < \frac{\varepsilon}{2^n}$ if $n \geq 2$.

Indeed, by the existence of the approximate identities $(E_\beta)$ and $(E_\beta^{-1})$, we obtain the base case $n = 1$. If we have $F_1, \ldots, F_n \in \mathscr{A}_1$ satisfying (i)–(iii), then for any given $\eta_1, \eta_2 > 0$ there exists $\beta \in B$ such that $\|E_\beta^{-1}e_i - e_i\| < \eta_1$ and $\|E_\beta^{-1}x_0 - x_0\| < \eta_2$ for all $i = 1, \ldots, n$. In this case, define

$$F = \sum_{i=1}^n (1-\gamma)^{i-1}\gamma E_\beta^{-1}e_i + (1-\gamma)^n e.$$

Then

$$F - F_n = \sum_{i=1}^{n} (1-\gamma)^{i-1}\gamma(E_\beta^{-1}e_i - e_i).$$

Note that if $\eta_1 \le \|F_n^{-1}\|^{-1}$ and $A \in \mathscr{A}_1$ satisfies $\|A - F_n\| < \eta_1$, then $A$ is invertible and that

$$\|A^{-1} - F_n^{-1}\| \le \|F_n^{-1}\|\|F_n - A\|\|A^{-1}\| < 2\|F_n^{-1}\|^2\eta_1.$$

Thus, if we let $\eta_1$ be smaller than $n^{-1}\|F_n^{-1}\|^{-1}$, then $F$ will be invertible; moreover, by putting $e_{n+1} = e_\beta$ then

$$F_{n+1} = E_\beta F = \sum_{i=1}^{n+1} (1-\gamma)^{i-1}\gamma e_i + (1-\gamma)^{n+1}e$$

is invertible, and

$$\begin{aligned}\|F_{n+1}^{-1}x_0 - F_n^{-1}x_0\| &\le \|F^{-1}E_\beta^{-1}x_0 - F_n^{-1}x_0\| \\ &\le \|(F^{-1} - F_n^{-1})E_\beta^{-1}x_0\| + \|F_n^{-1}(E_\beta^{-1}x_0 - x_0)\| \\ &\le \kappa(2\|F_n^{-1}\|^2\Delta\|x_0\|\eta_1 + \|F_n^{-1}\|\eta_2).\end{aligned}$$

Therefore, choosing $\eta_1$ and $\eta_2$ small enough yields the desired properties. Now, for $n > m \ge 1$ we clearly have

$$\|F_n^{-1}x_0 - F_m^{-1}x_0\| \le \sum_{k=m+1}^{n} \frac{\varepsilon}{2^k} \le \frac{\varepsilon}{2^m},$$

so $F_n^{-1}x_0 \to x$ for some $x \in \mathscr{X}$, and $\|x - x_0\| \le \varepsilon$. Moreover, as $x_0 \in \overline{\mathscr{A}x_0}$ (because of the left approximate identity) we have $F_n^{-1}x_0 \in \overline{\mathscr{A}x_0}$ for all $n \ge 1$ by continuity, so $x \in \overline{\mathscr{A}x_0}$. If we define $a_n = \sum_{i=1}^{n}(1-\gamma)^{i-1}\gamma e_i \in \mathscr{A}$, then $(a_n)_{n\ge1}$ converges to some $a \in \mathscr{A}$ satisfying

$$\|a\| \le \gamma\delta\sum_{k=0}^{\infty}(1-\gamma)^k = \delta.$$

As $F_n = a_n + (1-\gamma)^n e$ and $(1-\gamma)^n \to 0$, we have $F_n \to a$ and $ax = \lim_{n\to\infty} F_n(F_n^{-1}x_0) = x_0$. Finally, since the $e_i$'s are all chosen to be elements of the approximate identity, note that if the $(e_\beta)$ belong to a convex closed subset $C \subseteq \mathscr{A}$, then $(1-(1-\gamma)^n)\sum_{i=1}^{n}(1-\gamma)^{i-1}\gamma e_i \in C$ for all $n \ge 1$. Hence $a \in C$. □

Let us, for the sake of abbreviation, call the factorization $x = ax_0$ of the above theorem a *Cohen factorization* of $x \in \mathscr{X}$; we will suppress the $\varepsilon$ for the most part.

**Corollary 1** (Alternate CFT)**.** *Let $\mathscr{A}$ be a Banach algebra and let $\mathscr{X}$ be a left Banach $\mathscr{A}$-module (with constant $\kappa \ge 1$) and suppose that $\mathscr{A}$ has a left approximate identity $(e_\beta)_{\beta\in B}$, bounded by some constant $\delta \ge 1$, and let $S$ denote the closed linear span of $\{ax \mid a \in \mathscr{A},\ x \in \mathscr{X}\}$. Then all $x_0 \in S$ have a Cohen factorization $x_0 = ax$ where $a \in \mathscr{A}$ and $x \in \overline{\mathscr{A}x_0} \subseteq \mathscr{X}$.*

*Proof.* The Banach space $S$ is a left Banach $\mathscr{A}$-module, and $(e_\beta)_{\beta\in B}$ is easily seen to be a left approximate identity for $S$ by a standard $\frac{\varepsilon}{3}$ argument. □

Note that the alternate CFT in fact implies the original CFT, since $S = \mathscr{X}$ if $(e_\beta)_{\beta\in B}$ is also a left approximate identity for $\mathscr{X}$. Amongst nice consequences of this fine theorem, we have the following:

**Corollary 2.** *Let $\pi\colon \mathscr{A} \to B(\mathcal{H})$ be a non-degenerate representation of a Banach $^*$-algebra $\mathscr{A}$ with a bounded left approximate identity (for instance, $\mathscr{A}$ could be a $C^*$-algebra). Then for all $\xi \in \mathcal{H}$ and $\varepsilon > 0$ there exist $a \in \mathscr{A}$ and $\eta \in \overline{\pi(\mathscr{A})\xi}$ such that $\xi = \pi(a)\eta$ and $\|\xi - \eta\| < \varepsilon$. In particular, $\mathcal{H} = \pi(\mathscr{A})\mathcal{H}$.*

*Proof.* Let $(e_\beta)_{\beta\in B}$ be the left approximate identity. Since $\pi$ is contractive, the Hilbert space $\mathcal{H}$ is made into a left Banach $\mathscr{A}$-module by defining $a\xi = \pi(a)\xi$ for all $a \in \mathscr{A}$ and $\xi \in \mathcal{H}$. As non-degeneracy of $\pi$ implies that $\mathcal{H}$ is the closed linear span of vectors of the form $\pi(a)\xi$ for $a \in \mathscr{A}$ and $\xi \in \mathcal{H}$, the result follows from the alternate CFT. □

Of course, Cohen's factorization theorem has an analogue for right Banach modules as well (we won't state it here, though). If you happen to like locally compact groups but haven't heard about the CFT, you're missing out on a lot of nice factorization properties! Just check this out:

**Corollary 3.** *Let $G$ be a locally compact group with a Haar measure $\mu$, let $f \in L^p(G)$ for $1 \leq p < \infty$, $U$ be a neighbourhood of $1 \in G$ and $\varepsilon > 0$. Then there exist a positive function $h \in L^1(G)$ and $g \in L^p(G)$ such that*

(i) $f = h * g$,
(ii) $\int h \,\mathrm{d}\mu = 1$ *and* $h(G \setminus U) = \{0\}$,
(iii) $g \in \overline{L^1(G) * f}$,
(iv) $\|g - f\|_p < \varepsilon$, *and*
(v) *if $f$ is real-valued, then $g$ can be chosen to be real-valued.*

*In particular,* $L^1(G) * L^p(G) = L^p(G)$.

*Proof.* Recall that $L^1(G)$ always has a left approximate identity for $L^p(G)$ of positive functions $e$ satisfying $\operatorname{supp}(e) \subseteq U$ and $\int e \,\mathrm{d}\mu = 1$. The subspace of such functions $e$ in $L^1(G)$ is closed and convex, since every $L^p$-convergent sequence $(e_k)$ with limit $e$ has a subsequence converging pointwise to $e$ almost everywhere [3, Corollary 12.8]. Hence the desired two functions can be obtained from CFT, and the real-valued case is obtained by noting that CFT also holds for real Banach algebras and bimodules. □

**Corollary 4.** *Let $G$ be a locally compact group with Haar measure $\mu$ and let $\mathscr{X}$ be either of the spaces $C_0(G)$ or $C_\mathrm{b}^{\mathrm{lu}}(G)$ (the space of left uniformly continuous, bounded functions on $G$). Then for all $f \in \mathscr{X}$, neighbourhoods $U$ of $1 \in G$ and $\varepsilon > 0$, there exists a positive function $h \in L^1(G)$ and $g \in \mathscr{X}$ satisfying the conditions* (i)–(v) *of Corollary 3 under the uniform norm $\|\cdot\|_u$.*

*Proof.* Note first that $C_\mathrm{b}^{\mathrm{lu}}(G)$ is a left Banach $L^1(G)$-module with the uniform norm; indeed, for all $f \in L^1(G)$ and $g \in C_\mathrm{b}^{\mathrm{lu}}(G)$, we know that $f * g$ is continuous, that $\|f * g\|_u \leq \|f\|_1 \|g\|_u$ and that

$$\|t.(f * g) - f * g\|_u = \|(t.f - f) * g\| \leq \|t.f - f\|_1 \|g\|_u \to 0$$

for $t \to 1$, so that $f * g \in C_\mathrm{b}^{\mathrm{lu}}(G)$. The proof of Corollary 3 now also holds in this case, as we can choose a left approximate identity in $L^1(G)$ for $C_\mathrm{b}^{\mathrm{lu}}(G)$ by [2, Proposition 2.42]. As for $C_0(G)$, it is a Banach $L^1(G)$-module because $f * g \in C_c(G)$ for $f, g \in C_c(G)$, so if $f \in L^1(G)$ and $g \in C_0(G)$, we take sequences $(f_n)$ and $(g_n)$ of $C_c(G)$ with $\|f_n - f\|_1 \to 0$ and $\|g_n - g\|_u \to 0$, in which case $\|f_n * g_n - f * g\|_u \to 0$. In fact we also have $C_c(G) \subseteq C_\mathrm{b}^{\mathrm{lu}}(G)$, so it follows that $C_0(G) \subseteq C_\mathrm{b}^{\mathrm{lu}}(G)$ and that the approximate identity for $C_\mathrm{b}^{\mathrm{lu}}(G)$ is also one for $C_0(G)$. □

If you can find other curious consequences of the CFT, don't hesitate to tell me about them! It seems – to me, at least – that there should be lots of possible uses for the theorem in studies of groups and operator algebras (especially approximation property-wise) that could be *extremely* interesting.

# References


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