Introduction to K-theory

Assignment 4

Rasmus Sylvester Bryder

March 9th, 2012

Show that $GL_n(\mathbb{C})$ is path-connected and that $GL_n(\mathbb{R})$ has two path-components.

In the case of $n = 1$, $GL_1(\mathbb{C})$ is in fact just $\mathbb{C} \setminus \{0\}$ which is clearly path-connected (e.g. for any two points in $\mathbb{C}$ different from zero, choose a path consisting of a vertical and horizontal line segment, not necessarily in that order). $GL_1(\mathbb{R})$ is $\mathbb{R} \setminus \{0\}$ which is made up of the two disjoint non-empty open sets $(-\infty, 0)$ and $(0, \infty)$, implying that $GL_1(\mathbb{R})$ is disconnected and thus not path-connected; since $\mathbb{R} \setminus \{0\}$ is locally path-connected and each of the intervals is connected, these are exactly the path-components of $GL_1(\mathbb{R})$. $GL_n(\mathbb{R})$ is not connected for any $n \in \mathbb{N}$, since the continuous map $\det : GL_n(\mathbb{R}) \to \mathbb{R} \setminus \{0\}$ has disconnected image, and therefore $GL_n(\mathbb{R})$ has at least two path components.

As the case $n = 1$ has now been covered, we then turn to the (slightly more exciting) case of $n \geq 2$ in $GL_n(\mathbb{C})$. Let A be an invertible $n \times n$ matrix with complex entries. Using the Jordan normal form, there exists an invertible $n \times n$ matrix C and a Jordan block matrix B given by

$$B = \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_k \end{pmatrix} \text{ with } J_i = \begin{pmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{pmatrix}$$

for $i = 1, \ldots, k$ with $k \leq n$, such that $A = CBC^{-1}$. B is an upper triangular matrix with non-zero determinant as the diagonal consists of the eigenvalues of A, none of which are 0 (as A is invertible).

Let $\sigma_i = B_{ii}$, $i = 1, \ldots, n$, be the diagonal entries of B in their standard order, and define n paths $\gamma_i : [0, 1] \to \mathbb{C}$ with $\gamma_i(0) = \sigma_i$ and $\gamma_i(1) = 1$, $i = 1, \ldots, n$, none of the paths passing through 0.

Defining $B(t)$, $t \in [0, 1]$, to be the $n \times n$ matrix obtained from B by multiplying all the entries above the diagonal by $1 - t$ and having diagonal entries $B(t)_{ii} = \gamma_i(t)$ for $i = 1, \ldots, n$, $B(t)$ is clearly continuous as its entries are continuous functions. Furthermore, since the diagonal entries of $B(t)$ are non-zero for all $t \in [0, 1]$, we obtain that $\det B(t) \neq 0$ for all $t \in [0, 1]$, as it is upper triangular.

Defining $A(t) = CB(t)C^{-1}$ for $t \in [0, 1]$, we therefore obtain a path of invertible matrices from $A(0) = A$ to $A(1) = CB(1)C^{-1} = CI_nC^{-1} = I_n$. As all points of $GL_n(\mathbb{C})$ can be connected to the identity matrix by a path, we conclude that $GL_n(\mathbb{C})$ is path-connected.

❦ ❦ ❦

In the case of $GL_n(\mathbb{R})$ for $n \geq 2$, let $GL_n(\mathbb{R})^+$ denote the subspace of invertible $n \times n$ matrices with positive determinant, and similarly $GL_n(\mathbb{R})^-$ the ones with negative determinant. Defining a matrix M by

$$M = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix},$$

we obtain a homeomorphism $\varphi : GL_n(\mathbb{R})^+ \to GL_n(\mathbb{R})^-$ given by $\varphi(A) = MA$; it is well-defined as

$$\det MA = \det M \det A = -\det A < 0$$

for all $A \in GL_n(\mathbb{R})^+$, and it is bijective and continuous with its inverse φ^{-1} given in the same way. If we show that $GL_n(\mathbb{R})^+$ is path-connected, then for any matrices A and B with negative determinant, there is a path from $\varphi^{-1}(A)$ to $\varphi^{-1}(B)$, and composing this path with φ yields a path from A to B. Thus $GL_n(\mathbb{R})^-$ is path-connected as well, and it then follows that $GL_n(\mathbb{R})$ has at most two path-components, and therefore exactly two.

Therefore, let A be an invertible $n \times n$ matrix with real entries and positive determinant. We will construct paths connecting A to the identity matrix.

By using row operation matrices, we can "change" A into something a little easier to work with. For $i, j \in \{1, \dots, n\}$, $i \neq j$, then by defining $F_{ij}(\lambda)$ to be the identity matrix with λ at entry (i,j) one can observe that $F_{ij}(\lambda)A$ is the matrix obtained from A by adding λ times the j'th row to the i'th row. By multiplying A with matrices of the form $F_{ij}(t\lambda)$, $t \in [0,1]$, one can obtain a path from A to an upper triangular matrix A'; indeed, one can just start by applying $F_{ij}(t\lambda)$ such that the first column of the resultant matrix has its bottom $n-1$ entries equal to 0 for $t = 1$ (the entries of the first column cannot all be zero), then the second column to have its bottom $n-2$ entries equal to 0 and so on. This path is contained in $GL_n(\mathbb{R})$, as the $F_{ij}(\lambda)$ are all triangular matrices with determinant 1, and therefore $\det A' = \det A > 0$.

Let $\sigma_i = A'_{ii}$, $i = 1, \dots, n$, be the diagonal entries of A' in their standard order. We must have that only an even number of the σ_i are negative. Define n paths $\gamma_i : [0,1] \to \mathbb{R}$, $i = 1, \dots, n$, as follows: if $\sigma_i > 0$, let $\gamma_i(t) = (1-t)\sigma_i + t$, and if $\sigma_i < 0$, let $\gamma_i(t) = (1-t)\sigma_i - t$. For the i such that σ_i is positive, $\gamma_i(t) > 0$ for all $t \in [0,1]$, and likewise for the i such that σ_i is negative, $\gamma_i(t) < 0$ for all $t \in [0,1]$.

Defining $A'(t)$, $t \in [0,1]$, to be the $n \times n$ matrix obtained from A' by multiplying all the entries above the diagonal by $1-t$ and having diagonal entries $A'(t)_{ii} = \gamma_i(t)$ for $i = 1, \dots, n$, $A'(t)$ is clearly continuous as its entries are continuous functions. For $t \in [0,1]$ then $\gamma_i(t) < 0$ for only an even number of i and we obtain $\det A'(t) = \prod_{i=1}^n \gamma_i(t) > 0$, as $A'(t)$ is upper triangular. Therefore $A'(t)$ is a path of invertible matrices from A' to $A'(1)$, a diagonal matrix containing only 1's and -1's, the amount of -1's being an even number.

Finally, we construct a path from $A'(1)$ to I_n. Let ρ_i be the entry of $A'(1)$ at the place (i,i). If $\rho_i = \rho_j = -1$ for $1 \leq i < j \leq n$, then define a rotation matrix

$$R_{ij}(t) = \begin{array}{cc} \begin{array}{ccccc} \phantom{I_{k_1}} & i & \phantom{I_{k_2}} & j & \phantom{I_{k_3}} \\ & \downarrow & & \downarrow & \end{array} & \\ \begin{pmatrix} I_{k_1} & & & & \\ & \cos(t\pi) & & \sin(t\pi) & \\ & & I_{k_2} & & \\ & -\sin(t\pi) & & \cos(t\pi) & \\ & & & & I_{k_3} \end{pmatrix} & \begin{array}{c} \\ \leftarrow i \\ \\ \leftarrow j \\ \\ \end{array} \end{array}$$

for $t \in [0,1]$ with the I_{k_p} denoting identity matrices. Since $\det R_{ij}(t) = \cos^2(t\pi) + \sin^2(t\pi) = 1$ for all $t \in [0,1]$, the map $t \mapsto R_{ij}(t)A'(1)$, $t \in [0,1]$ is a path of invertible matrices from $A'(1)$ to $A'(1)$ with the -1's at the places (i,i) and (j,j) changed to 1's. Since the number of -1's in $A'(1)$ is even, we can continue multiplying by these rotation matrices until we finally obtain I_n.

By gluing these paths together, we obtain a path from A to I_n; hence all invertible matrices with real entries can be connected by a path to I_n and we conclude that $GL_n(\mathbb{R})^+$ is path-connected.