Identities, and Collapses thereof

by R. S. Bryder

Dedicated to Kristian Jonsson Moi

ACT I

The curtain rises. The stage is filled with inanimate objects: chairs, couches, rose petals, lamps, a dead horse and a mock portrait of Queen Victoria. The lighting is dark red, the music is absent. Everything about the setting reeks of unfulfillable ambition and pretense. This is going to be a real fuck-up. Nonetheless, somewhere backstage, a violin starts to play random notes in the Phrygian scale. The Eastern overtones of the music and the mess on stage gives the play an air of direction, but suddenly the violin stops playing, and everything is back to suck. The audience is bored, not counting the ones who have been paid to sit through this: even though one, in these people, can sense a twinge of sadness from being present at this suckfest, their eyes are aglow with the thought of all the counterfeit money they will receive after the show is over and the theatre has gone bankrupt.

A MOUSE enters. The mouse is grey and furry, hasn't had a bath in days, looking slightly overwhelmed at the chairs that have started to sing bard tunes about the old days. Not so inanimate after all. A giant FOOT comes down from above and crushes the mouse to death. MARLON BRANDO enters. He is wearing a ragged tuxedo and a scarf eighty yards long, bathed in colours and motives that the reader is advised to check for him- or herself. He walks onto the middle of the stage, and the light turns green. The sound of Beethoven's Sixth is heard from a broken tape recorder backstage, making the once glorious piece sound like crap (or Stockhausen; the implication holds both ways).

> MARLON BRANDO (TALKING TO HIMSELF)
>
> Two homomorphisms of Banach algebras $\varphi_0, \varphi_1 : A \to B$ are homotopic if there exists a map $\varphi : A \to C([0,1], B)$ with $\mathrm{ev}_i \circ \varphi = \varphi_i$. (TO THE DIRECTOR:) Where's my muffin?

The DIRECTOR unleashes a flock of angry elephants from backstage, causing Brando to faint and get trampled to death. THE SPIRIT OF MARLON BRANDO rises from his corpse, looking dazed and confused. The corpse looks like a mélange of an orange and a teenage pot party in Connecticut gone horribly wrong. The Spirit is not too far behind. PAUL MCCARTNEY falls down from the ceiling. He is naked, and he is the walrus.

> PAUL MCCARTNEY
>
> Once upon a long ago, I was told that the Banach algebra B was homotopy equivalent to the space of continuous functions from the unit interval to B. Perhaps you could enlighten me on this matter.

> THE SPIRIT OF MARLON BRANDO
>
> Roses are red, and I am glue. Radioactive waste is like sum time in Aberdeen. Astonishing. Rinse the raindrops, integrate around the poles, and I'll be calling your mother telling her what affine lady she is. The horror.

> PAUL MCCARTNEY
>
> That was unexpected. But I have been waiting for centuries. Please give me an answer, tiny dancer.

Beethoven's Sixth cuts off abruptly, replaced by the piano intro to Elton John's "Tiny Dancer" repeated ad nauseam.

> THE SPIRIT OF MARLON BRANDO (TAP-DANCING)
>
> Oh well. You have given me an offer I cannot refuse. Look at my feet, white as wheat. Everybody! Rock your body. Yeah! (HE CASTS A LOOK UPWARD.) Let us define a schmapping f_0 from B to the function space by sending any x in B to the function that sends any λ from $[0,1]$ to λx. That is f_0. And another schmapping going the other way, say g_0, that sends a continuous function to its image of 1. What do you observe?

> PAUL MCCARTNEY
>
> That $g_0 \circ f_0$ is the identity on B, of course. But what does that prove?

> THE SPIRIT OF MARLON BRANDO (STILL TAP-DANCING)
>
> Something.

PAUL MCCARTNEY

Yes.

THE SPIRIT OF MARLON BRANDO (WHOSE TAP-DANCING IS BECOMING SLIGHTLY ANNOYING TO THE AUDIENCE)

... yes.

PAUL NEWMAN enters from the right, walking a pack of tigers. He looks very happy. Too happy. His head cracks open as someone from the audience yells "Cool Hand Luke". That was a very good film. He should have won the Oscar for best male lead. Rod Steiger was very good in In The Heat of the Night, but come on, Paul Newman just had way more balls than Steiger. Nonetheless, Paul Newman's head cracks open, releasing a UNICORN and a VERY DRUNK ELF from his brain. The Spirit stops tap-dancing, and the lights go crazy.

UNICORN AND VERY DRUNK ELF (IN UNISON)

We are free! Take his wallet.

McCartney, formerly of Wings and The Beatles, takes Paul Newman's wallet from his pocket. The wallet is pink, with a Hello Kitty sticker on both sides. The pack of tigers jump from the stage onto the front row of the theatre, feasting jauntily on the audience.

THE SPIRIT OF MARLON BRANDO

Off you go.

The Spirit pulls out a remote control, pressing a button, causing the unicorn and the very drunk elf to fall through a hatch, along with the dead horse (and several species of mosquitoes). He and McCartney carry on, hold on John, it is going to be alright. You are gonna win the fight. The tigers orbit around the only front row audience member left alive, singing Michael Jackson songs in the style of Chris Tucker and a clown with a lisp (without loss of generality).

THE SPIRIT OF MARLON BRANDO

Of course $g_0 \circ f_0$ is homotopic to the identity. It *is* the identity. Don't you see?

PAUL MCCARTNEY

Yes, I see it. But we are not done yet.

THE SPIRIT OF MARLON BRANDO

I will sleep soon. It is past past past my bed-t-t-time. I am starting to disintegrate.

PAUL MCCARTNEY

Hurry!

Nothing happens on stage for 25 minutes expect for McCartney checking his watch (on the left arm) and eating his own right arm. An elderly couple in the audience starts laughing at a picture of the Pope for the entire duration.

PAUL MCCARTNEY

I'll never do you no arm.

THE SPIRIT OF MARLON BRANDO (FEELING HIMSELF DISINTEGRATE)

We must carry on.

The Spirit floats to the left and pulls out a blackboard. The wallet is thrown to the audience. The tigers hover out of the theatre and into the mouth of Steven Tyler. The light turns blue. In black chalk, The Spirit writes the following:

$$G : C([0,1], B) \to C([0,1], C([0,1], B))$$
$$G(s)(t)(\lambda) = ts(\lambda) + (1-t)\lambda s(1)$$

PAUL MCCARTNEY

What is that?

THE SPIRIT OF MARLON BRANDO

That is the homotopy. Note that G of s of 0 is $f_0 \circ g_0$ of s and that G of s of 1 is the identity on the space of continuous schmappings.

PAUL MCCARTNEY

Ah! And G is well-defined?

THE SPIRIT OF MARLON BRANDO

You must be kidding me! (MUMBLES TO HIMSELF.) Rice Krispies. (NORMAL SPEAKING VOICE.) For any t, G of s of t is a continuous function of λ from $[0,1]$ to B by continuity of s and the product in B, so that part is clear. For any s, then notice that if t_n converges to t in $[0,1]$, then...

The Spirit writes on the blackboard again:

$$
\begin{aligned}
& \|G(s)(t_n) - G(s)(t)\| \\
=\ & \sup_{\lambda \in [0,1]} \|G(s)(t_n)(\lambda) - G(s)(t)(\lambda)\|_B \\
=\ & \sup_{\lambda \in [0,1]} \|(t_n - t)s(\lambda) - (t_n - t)\lambda s(1)\|_B \\
\leq\ & |t_n - t| \sup_{\lambda \in [0,1]} \|s(\lambda) - \lambda s(1)\| \to 0.
\end{aligned}
$$

THE SPIRIT OF MARLON BRANDO

And the unit interval is c-c-compact and the function inside sup is continuous.

PAUL MCCARTNEY

Is G continuous as well?

THE SPIRIT OF MARLON BRANDO

Of course it is. For a sequence of continuous schmappings, say s_n, from $[0,1]$ to B converging uniformly to s...

PAUL MCCARTNEY (INTERRUPTS)

Ah, yes. Then we must show that $G(s_n)$ converges uniformly to $G(s)$.

The music changes abruptly to a sped-up version of "Get Back" by The Beatles. This is supposed to make the remainder of the audience feel sorry about the dead mouse and the passage of time in general. McCartney goes to the blackboard, steals the black chalk from Brando's spirit fingers and writes the following:

$$
\begin{aligned}
& \|G(s_n)(t) - G(s)(t)\| \\
=\ & \sup_{\lambda \in [0,1]} \|G(s_n)(t)(\lambda) - G(s)(t)(\lambda)\|_B \\
=\ & \sup_{\lambda \in [0,1]} \|t(s_n(\lambda) - s(\lambda)) + (1-t)\lambda(s_n(1) - s(1))\|_B \\
\leq\ & t \sup_{\lambda \in [0,1]} \|s_n(\lambda) - s(\lambda)\|_B + (1-t) \sup_{\lambda \in [0,1]} |\lambda| \|s_n(1) - s(1)\|_B \\
\leq\ & t\|s_n - s\| + (1-t)\|s_n - s\| \\
=\ & \|s_n - s\|.
\end{aligned}
$$

THE SPIRIT OF MARLON BRANDO (DISINTEGRATING)

You caught on! Thank God! Thank you!

PAUL MCCARTNEY (ECSTATIC AND ACCEPTING A GRAMMY)

Yes, and that is for all t from 0 to 1. So $G(s_n)$ converges uniformly to $G(s)$ because s_n does that to s, so G is continuous, $f_0 \circ g_0$ is homotopic to the identity, and alas we are done!

VOICE OF GOD (ANGRY, RUMBLING)

That word doesn't mean what you think it means!

The Spirit collapses onto a point, Paul McCartney is homotoped into an elk, and the curtain falls. The music stops.

ENTR'ACTE

The curtain rises. WILL SMITH gets jiggy with it for a duration of 3'56" after which an OCTOPUS falls onto the stage and says "Hallelujah". The curtain falls.

ACT II

The curtain rises, inversely proportionally to the time it spends rising. The stage is empty. VASLAV NIJINSKY floats inside the theatre from the emergency exit and lands on the stage where he dances a beautiful Charleston with a teddy bear made of leather, receiving glowing reviews in tomorrow's papers. He homotopes into a cat, then into two cats and continues homotoping into 2^n cats for $n \to \infty$, as STEFAN BANACH and ALFRED TARSKI chase each other around the stage in slow motion. Suddenly, a peacock (known as Bert the Beautynator) explodes. No one knows why, not even the author. Shit happens.

Everyone leaves the stage[1]. A big spud enters. It is the OMNISCIENT POTATO. He knows his way around the world and owns two rental car places with low income. His daughter is into weeds, his son is grounded and his wife is sleeping in someone else's garden. Spotlight. The music is "Verklärte Nacht" by Arnold Schönberg with occasional outbursts of "Get Down Tonight" by KC and the Sunshine Band; it is played by Leonard Cohen and William Hung in a dirty French café in Toulouse and transmitted live through Skype. The Omniscient Potato does a little dance and points out to the (sleeping) audience.

OMNISCIENT POTATO

Lo, my friends, I have come from afar to lay upon thee a proof so well-tempered and obnoxious that thou willt spill thy apple juices onto the floor and let them seep through the cracks of this wretched world. It is a proof of beauty, proof of love, a love note from the highest of highs, the blowest of blows and I hear angels cry whence I scarcely think of it. Yo, homie. Thrice the dice and go with the flow. It is me, Cathy, I have come home to steal your functional calculus and your spectral mappings. Yo. I fancy the moonlight, the sunlight and the memories of John Belushi imitating himself. We must fence till the world ends, and then rest. I wear a thong and one on top of that, and I know that two thongs don't make a right. We steer clear of hate and blow bubbles through straws so thin that we spin a web of webs of webs without knowing. Oh the irony.

A chair is pushed to the middle of the stage by Paul McCartney. He had a lunch break originally, but was forced to do this extra job when he got caught stealing Skittles from the director's drawer.

OMNISCIENT POTATO (SITS)

We consider the space $\mathcal{X}$ of continuous functions from $[0, 1]$ to a Banach algebra B that have image zero at zero. Consider the schm... the schm... the mapping! The mapping φ from $\mathcal{X}$ into the continuous functions from $[0, 1]$ to $\mathcal{X}$ is given by φ of f of t of λ to be the image of $1 - t$ times λ under f. (HE PONDERS FOR A SHORT WHILE.) Ah! (HE PULLS OUT A SHORT KNIFE AND CARVES THE FOLLOWING INTO HIS BIG BELLY:)

$$\varphi(f)(t)(\lambda) = (1 - t)f(\lambda).$$

It is wonderous. Let me sing.

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[1]Note to self: figure out how to remove a rapidly increasing amount of cats from stage.

The Omniscient Potato sings a few improvised songs about jewelry.

OMNISCIENT POTATO

φ is well-defined, ladies and gents. For any f and t, then by continuity of f, we obtain continuity of φ of f of t. And of course, if t_n converges to t, then in the same manner as in Act I, it is seen that φ of f of t_n converges to φ of f of t.

The Omniscient Potato tap-dances furiously for a few seconds, then suddenly stops.

OMNISCIENT POTATO

φ is continuous. Why is φ continuous? It is. It is because it is that it is continuous. If it weren't, then it wouldn't be, but it is. It is what it is. Is it? Izzit? Is shit? Why is the moon round? It is continuous. Why is the universe so big? Continuous. Everything is good. I must dance, but first I may have to explain why φ is continuous. You may not have understood it the first time around. You are ignorant. I am great. I am a potato.

That is true. He is a potato.

OMNISCIENT POTATO

Let me show you.

The Omniscient Potato stares at the universe and hypnotizes ittir3.3qrt'9..r3

$$\|\varphi(f_n)(t) - \varphi(f)(t)\| = \sup_{\lambda \in [0,1]} \|f_n((1-t)\lambda) - f((1-t)\lambda)\|_B \leq \sup_{\lambda \in [0,1]} \|f_n - f\| = \|f_n - f\|$$

uniform convergence of f_n to f implies qept ppolk fa of $\varphi(f_n)$ gennnnnnnn

The author awakes in a ditch outside New Jersey. He is cold and covered in mud. The sound of fast cars causes him to cover his ears. His eyesight is poor, and he might have broken a rib, presumably from a big fall, but it is not clear to the author how or when this presumed fall did occur. All he knows is that his entire body is hurting. He tries to stand, failing a few times, but finally making it, reaching for the top of the ditch and painfully pulling himself up. He looks around. Not much else but dry land and millions of cars in a rainbow of many colours zooming past him, under a clear blue sky. He looks up, and finds out that the sky is not so clear after all. There are clouds, bright white, covering the sun. They are in shapes known to him. However before he can focus, he is walloped on the head by a flying wallet. The author passes out and regains consciousness after a short while. He notices the wallet lying beside him. It is pink and has a sticker, with the motive of a cat. He picks it up and turns it around. Another sticker, same cat. He opens it. A picture of Paul Newman as well as his credit card are contained in two of its many pockets. Inside another pocket, a receipt from Taco Bell. Inside yet another, an image of a dog jumping to catch a frisbee. Inside yet another another, he finds a note. It says

$$\varphi(f)(0)(\lambda) = f(\lambda) \rightsquigarrow \mathrm{ev}_0 \circ \varphi(f) = f,$$

$$\varphi(f)(1)(\lambda) = 0 \rightsquigarrow \mathrm{ev}_1 \circ \varphi(f) = 0,$$

$$1_{\mathcal{X}} \simeq 0.$$

The author smiles and rises slowly. Somewhere a curtain falls.

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