Exam 2009

Probability Theory 1 and Measure and Integration Theory

Assignment 2

Formalities

This is the second of the four compulsory assignments for the two courses *Probability Theory 1* and *Measure and Integration Theory*.

The assignment is divided into 5 problems with a total of 10 questions.

This is an exam, and the solution must be written and handed in individually **in two copies**. The solution must be equipped with the standard frontpage, which is available from the course webpage. You are only allowed to write on one side of the paper, the solution must be stapled in the upper left corner and no plastic-covers please.

The deadline for handing in the **two copies** of the solution is Monday, September 28 at **the beginning of the lecture** 13.15.

Problem 1

Let $(\mathcal{X}, \mathbb{E})$ be a measureable space.

Question 1.1. Argue that the map $H : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$H(x, y) = (x + y, x \cdot y)$$

is $\mathbb{B}_2$-$\mathbb{B}_2$-measurable. Show that if $f, g \in \mathcal{M}(\mathcal{X}, \mathbb{E})$ then the map $G : \mathcal{X} \to \mathbb{R}^2$ given by

$$G(x) = (f(x) + g(y), f(x)g(x))$$

is $\mathbb{E}$-$\mathbb{B}_2$-measurable.

Question 1.2. Show that if $f, g \in \mathcal{M}(\mathcal{X}, \mathbb{E})$ with $f(x) \neq 0$ and $g(x) \neq 0$ for all $x \in \mathcal{X}$ then the function

$$x \mapsto \frac{f(x) + g(x)}{f(x) \cdot g(x)} \tag{1.1}$$

is also in $\mathcal{M}(\mathcal{X}, \mathbb{E})$.

Question 1.3. Show that if $f, g \in \mathcal{M}(\mathcal{X}, \mathbb{E})$, if μ is a measure on $(\mathcal{X}, \mathbb{E})$, and if there are constants $a, b, c, d \in (0, \infty)$ such that $a \leq f(x) \leq b$ and $c \leq g(x) \leq d$ for all $x \in \mathcal{X}$ then

$$\frac{a+c}{bd}\mu(\mathcal{X}) \leq \int \frac{f+g}{f \cdot g} \mathrm{d}\mu \leq \frac{b+d}{ac}\mu(\mathcal{X}).$$

Argue that $(f+g)/f \cdot g$ is integrable w.r.t. μ if and only of $\mu(\mathcal{X}) < \infty$.

Problem 2

Question 2.1. Argue that the integrals

$$I_n = \int_0^\infty \left(\sqrt{x^2 + \frac{1}{n}} - x \right) \frac{1}{1+x^2} \mathrm{d}x$$

make sense for all $n \in \mathbb{N}$ and show that $0 \leq I_n < \infty$ for all $n \in \mathbb{N}$. Show that I_n converges for $n \to \infty$ and find the limit.

Problem 3

Define the function $f : \mathbb{R} \to \mathbb{R}$ by the series expansion

$$f(x) = \sum_{n=0}^{\infty} 1_{[0,1]}(x) a_n x^n$$

where we assume that $a_n \geq 0$ for all $n \geq 0$ and that the series is convergent for $x \in [0, 1)$. For $x = 1$ the series, $\sum_{n=0}^{\infty} a_n$, may be convergent or divergent towards ∞.

Question 3.1. Argue that $f \in \mathcal{M}^+(\mathbb{R}, \mathbb{B})$. Define

$$F(x) = \int_0^x f(y) \mathrm{d}y$$

for $x \in [0, 1]$ and show that

$$F(x) = \sum_{n=1}^{\infty} \frac{a_{n-1}}{n} x^n.$$

Question 3.2. Show that

$$\log\left(\frac{1}{1-x}\right) = \sum_{n=1}^{\infty} \frac{1}{n} x^n$$

for $x \in [0, 1)$.

Hint: Recall the identity

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

valid for $|x| < 1$.

Question 3.3. Show that

$$\lim_{m\to\infty} F\left(1 - \frac{1}{m}\right) = F(1).$$

Use this to show that the harmonic series does not converge, that is,

$$\sum_{n=1}^{\infty} \frac{1}{n} = \infty. \tag{3.2}$$

Problem 4

Question 4.1. Show that

$$J_n(y) = \frac{1}{\pi}\int_0^{\pi} \cos(y\sin(x) - nx)\mathrm{d}x$$

gives a well defined, continuous function $J_n : \mathbb{R} \to \mathbb{R}$ for $n \in \mathbb{N}_0$.

Question 4.2. Show that J_n is differentiable and that

$$J_1'(0) = \frac{1}{2}, \quad J_n'(0) = 0, \text{ for } n \neq 1.$$

Remark: The functions J_n are known as Bessel functions and they have several alternative representations. They pop up in various applications and most notably they solve Bessel's second order differential equation

$$y^2 f'' + yf' + (y^2 - n^2)f = 0.$$

They play an important role in the study of partial differential equations with applications in physics and in stochastic processes.

Problem 5

Let $f \in \mathcal{M}^+(\mathbb{R}, \mathbb{B})$. Define for $t \geq 0$

$$\varphi(t) = \int_0^\infty f(x) e^{-tx} \mathrm{d}x,$$

which is a well defined function $\varphi : [0, \infty) \to [0, \infty]$ since the sections $x \mapsto 1_{(0,\infty)}(x) f(x) e^{-tx}$ are all $\mathcal{M}^+$-functions. You can use that without further arguments.

Question 5.1. Show that if f is integrable then φ is differentiable on $(0, \infty)$ and

$$\lim_{n \to \infty} \varphi' \left(n^{-1} \right) = - \int_0^\infty x f(x) \mathrm{d}x.$$