Algebra 3, 2010

Compulsory exercises, third set
Tredie sæt obligatoriske opgaver

16. March 2010

The students in the course have to submit a written solution to the following exercises in the mailbox of the instructor (Martin Wedel Jacobsen) on tuesday 23. March at the latest at the start of the exercise classes. The written solutions (in Danish or English) should be carefully written with references to quoted results from the notes and the exercises.

Exercise C3.1: Consider the polynomial $g(x) = x^3 + 2x + 2$ over $\mathbb{Q}$.

(1) Show that $g(x)$ is irreducible in $\mathbb{Q}[x]$.

(2) Show that $g(x)$ has exactly one real root.

(3) Let M be the splitting field for $g(x)$ over $\mathbb{Q}$. Show that $M/\mathbb{Q}$ is normal and determine the Galois group $\mathrm{Gal}(M/\mathbb{Q})$.

Exercise C3.2: Let M be the splitting field for the polynomial $f(x) = (x^3 - 2)(x^4 - 2)$ (of degree 7) over $\mathbb{Q}$.

(1) Show that M contains splitting fields L_1, L_2 for the polynomials $f_1(x) = (x^3 - 2)$ and $f_2(x) = (x^4 - 2)$ over $\mathbb{Q}$.

(2) Show that $M = L_1 L_2$ is the compositum of L_1 and L_2 and that $L_1 \cap L_2 = \mathbb{Q}$.

(3) Compute the degree $[M : \mathbb{Q}]$ and determine the Galois group $\mathrm{Gal}(M/\mathbb{Q})$. (Hint: You may use without proof that the Galois groups $\mathrm{Gal}(L_1/\mathbb{Q})$ and $\mathrm{Gal}(L_2/\mathbb{Q})$ are known from the notes and exercises.)

(4) Is $\varphi\mathrm{Gal}(M/\mathbb{Q})$ a transitive subgroup of S_7? (Here the map φ is as described in Theorem 3.40).