Corrections

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The proof of Theorem 2.27 (in the revision) has some mistakes in it, and it was pointed out during my bachelor's defense that the notation was very confusing. Hence this attempt at fixing that... almost nine years later. Yeah, I have nothing better to do.

Theorem 1. *Let G be a locally compact group with a σ-finite left Haar measure μ and assume that G is amenable as a discrete group. Then the Banach algebra $L^1(G) = L^1(G, \mu)$ is amenable.*

In the original setting, we looked at $\mathbb{R}$ with the Lebesgue measure. For a characterization of locally compact groups that are amenable as discrete groups, we refer to [1].

Proof. Let $\mathfrak{A}$ be the vector space $\ell^1(G) \oplus L^1(G)$ with coordinate-wise addition and scalar multiplication. As

$$\int \sum_{s\in G} |\varphi(s) f(s^{-1}t)| \,\mathrm{d}\mu(t) \leq \sum_{s\in G} |\varphi(s)| \int |f(st)| \,\mathrm{d}\mu(t) \leq \|f\|_1 \|\varphi\|_1,$$

due to left invariance of μ, we see that it makes sense to define

$$\varphi * f = \sum_{s\in G} \varphi(s) f \circ \tau_s \in L^1(G)$$

for $\varphi \in \ell^1(G)$ and $f \in L^1(G)$. By Tonelli's theorem one for instance sees that

$$\begin{aligned}(\varphi_1 * \varphi_2) * f &= \sum_{s\in G}\sum_{t\in G} \varphi_1(t)\varphi_2(t^{-1}s) f \circ \tau_s \\ &= \sum_{t\in G} \varphi_1(t) \left(\sum_{s\in G} \varphi_2(s) f \circ \tau_s \right) \circ \tau_t \\ &= \sum_{t\in G} \varphi_1(t) (\varphi_2 * f) \circ \tau_t = \varphi_1 * (\varphi_2 * f).\end{aligned}$$

Similarly, define $f * \varphi = \sum_{s\in G} \varphi_1(s^{-1}) f \circ \tau_s$ for $\varphi \in \ell^1(G)$ and $f \in L^1(G)$. By means of these two definitions, one may now define multiplication on $\mathfrak{A}$ by

$$(\varphi_1, f_1) * (\varphi_2, f_2) = (\varphi_1 * \varphi_2, f_1 * f_2 + \varphi_1 * f_2 + f_1 * \varphi_2).$$

Equipping $\mathfrak{A}$ with the 1-norm $\|(\varphi, f)\| := \|\varphi\|_1 + \|f\|_1$ for $\varphi \in \ell^1(G)$ and $f \in L^1(G)$, the space $\mathfrak{A}$ becomes a Banach algebra with the identity element $1_{\mathfrak{A}} = (\mathbf{e}, 0)$ and a closed ideal of elements of the form $(0, f)$ for $f \in L^1(G)$; we will identify f with the tuple $(0, f)$ in the following.

Remember that there exists an isometric isomorphism $L^\infty(G) = L^\infty(G, \mu) \to L^1(G)'$, $\psi \mapsto \overline{\psi}$, given by

$$\overline{\psi}(f) = \int_G f\psi \,\mathrm{d}\mu, \quad \psi \in L^\infty(G),\ f \in L^1(G).$$

Notice that for any $f_1, f_2 \in L^1(G)$, $\psi \in L^\infty(G)$,

$$\overline{\psi}(f_1 * f_2) = \int_G \psi(t)(f_1 * f_2)(t) \,\mathrm{d}\mu(t) = \int_G \int_G \psi(t) f_1(s) f_2(s^{-1}t) \,\mathrm{d}\mu(s) \,\mathrm{d}\mu(t).$$

Because of absolute integrability, since by Tonelli's theorem,

$$\int_G \int_G \left|\psi(t) f_1(s) f_2(s^{-1}t)\right| \,\mathrm{d}\mu(s) \,\mathrm{d}\mu(t) \leq \|\psi\|_\infty \int_G |f(s)| \int_G \left|f_2(s^{-1}t)\right| \,\mathrm{d}\mu(t) \,\mathrm{d}\mu(s) \leq \|\psi\|_\infty \|f_1\|_1 \|f_2\|_1,$$

we can then change the order of integration by Fubini's theorem, such that

$$\overline{\psi}(f_1 * f_2) = \int_G f_1(s) \int_G \psi(t) f_2(s^{-1}t)\,\mathrm{d}\mu(t)\,\mathrm{d}\mu(s) = \int_G f_1(s)\overline{\psi}(f_2 \circ \tau_s)\,\mathrm{d}\mu(s) = \int_G f_1(s)\overline{\psi}(\mathbf{s} * f_2)\,\mathrm{d}\mu(s).$$

Now let $\mathfrak{X}$ be a neo-unital Banach $L^1(G)$-bimodule and let $D \in \mathscr{Z}^1(L^1(G), \mathfrak{X}')$. $\mathfrak{X}$ is made into a Banach $\mathfrak{A}$-bimodule canonically by Lemma 1.11 (revision) and we may extend D to $\mathfrak{A}$; notice that $1_{\mathfrak{A}} x = x$ for all $x \in \mathfrak{X}$.

Fix $x \in \mathfrak{X}$ and $F \in \mathfrak{X}'$. The function $L^1(G) \to \mathbb{C}$ defined by $f \mapsto F(fx)$ is linear and bounded; thus there exists $\Psi \in L^\infty(G)$ such that $\overline{\Psi}(f) = F(fx)$ for all $f \in L^1(G)$. For $f_1, f_2 \in L^1(G)$, then

$$F(f_1(f_2 x)) = F((f_1 * f_2)x) = \overline{\Psi}(f_1 * f_2) = \int_G f_1(s)\overline{\Psi}(\mathbf{s} * f_2)\,\mathrm{d}\mu(s) = \int_G f_1(s) F((\mathbf{s} * f_2)x)\,\mathrm{d}\mu(s),$$

and since x and F were arbitrary, this holds for all $x \in \mathfrak{X}$ and $F \in \mathfrak{X}'$. Because $\mathfrak{X}$ is neo-unital, all $y \in \mathfrak{X}$ can be written $y = f_2 x = (0, f_2)x$ for some $f_2 \in L^1(G)$, $x \in \mathfrak{X}$, and so

$$F(fy) = \int_G f(s) F((\mathbf{s} * f_2)x)\,\mathrm{d}\mu(s) = \int_G f(s) F((\mathbf{s}, 0)y)\,\mathrm{d}\mu(s), \quad f \in L^1(G),\ y \in \mathfrak{X},\ F \in \mathfrak{X}'.$$

We show analogously that

$$F(yf) = \int_G f(s) F(y(\mathbf{s}, 0))\,\mathrm{d}\mu(s), \quad f \in L^1(G),\ y \in \mathfrak{X},\ F \in \mathfrak{X}'.$$

Using the notation $x \mapsto \hat{x}$ of the embedding of $\mathfrak{X}$ in the second dual $\mathfrak{X}''$, we have

$$D(f)(x) = \hat{x}(D(f)), \quad x \in \mathfrak{X},\ f \in L^1(G).$$

Defining $D^*(y)(f) := y(D(f))$ for $y \in \mathfrak{X}''$ and $f \in L^1(G)$ defines the dual mapping $D^* \colon \mathfrak{X}'' \to L^1(G)'$, so that for a fixed $x \in \mathfrak{X}$ and arbitrary $f_1, f_2 \in L^1(G)$, there exists $\Psi \in L^\infty(G)$ such that

$$D^*(\hat{x})(f_1 * f_2) = \overline{\Psi}(f_1 * f_2) = \int_G f_1(s)\overline{\Psi}(\mathbf{s} * f_2)\,\mathrm{d}\mu(s) = \int_G f_1(s) D^*(\hat{x})(\mathbf{s} * f_2)\,\mathrm{d}\mu(s),$$

and therefore

$$D(f_1 * f_2)(x) = D^*(\hat{x})(f_1 * f_2) = \int_G f_1(s) D^*(\hat{x})(\mathbf{s} * f_2)\,\mathrm{d}\mu(s) = \int_G f_1(s) D(\mathbf{s} * f_2)(x)\,\mathrm{d}\mu(s)$$

for all $x \in \mathfrak{X}$, $f_1, f_2 \in L^1(G)$. Now, let $(e_n)_{n\in\mathbb{N}}$ be a bounded approximate identity for $L^1(G)$. Then

$$D(f)(x) = \lim_{n\to\infty} D(f * e_n)(x) = \lim_{n\to\infty} \int_G f(s) D(\mathbf{s} * e_n)(x)\,\mathrm{d}\mu(s)$$

for all $n \in \mathbb{N}$ and $f \in L^1(G)$, by continuity of D. We want to evaluate the above limit, and this we may do by using Lebesgue's dominated convergence theorem. Fix $x \in \mathfrak{X}$, and let $\Psi \in L^\infty(G)$ such that $D^*(\hat{x})(f) = \overline{\Psi}(f)$ for all $f \in L^1(G)$. For $n \in \mathbb{N}$ and fixed $f \in L^1(G)$, consider the function

$$s \mapsto f(s) D(\mathbf{s} * e_n)(x) = f(s) D^*(\hat{x})(\mathbf{s} * e_n) = \int_G f(s)\Psi(t) e_n(s^{-1}t)\,\mathrm{d}t.$$

Every section $t \mapsto f(s)\Psi(t)e_n(s^{-1}t)$ for $s \in G$ is integrable and $(s,t) \mapsto f(s)\Psi(t)e_n(s^{-1}t)$ is measurable because of measurability of projections and f, Ψ and e_n, so the above function is measurable, and moreover, bounded. The extension to $\mathfrak{A}$ of the derivation D satisfies

$$D(\alpha)(x) = \lim_{n\to\infty} D(\alpha * e_n)(x), \quad \alpha \in L^1(G)\mathfrak{A},\ x \in \mathfrak{X}.$$

Hence $f(s)D(\mathbf{s}, 0)(x) = \lim_{n\to\infty} f(s) D(\mathbf{s} * e_n)(x)$. Finally, since there is $S > 0$ such that $\|e_n\|_1 \le S$ for all $n \in \mathbb{N}$, the mappings $s \mapsto |f(s)D(\mathbf{s} * e_n)(x)|$ have the integrable upper bound $S\|\Psi\|_\infty f$ for all $n \in \mathbb{N}$, so

$$D(f)(x) = \lim_{n\to\infty} \int_G f(s) D(\mathbf{s} * e_n)(x)\,\mathrm{d}\mu(s) = \int_G f(s) D(\mathbf{s}, 0)(x)\,\mathrm{d}\mu(s)$$

by Lebesgue's dominated convergence theorem.

Finally, notice that $\mathfrak{X}$ can be treated as a Banach $\ell^1(G)$-bimodule, and so the restriction of D to $\ell^1(G)$ is a bounded $\mathfrak{X}'$-derivation on $\ell^1(G)$ and thus an inner derivation (since G is amenable as a discrete group). Thus there exists $\mathfrak{g} \in \mathfrak{X}'$ so that $D(\mathbf{s}, 0) = (\mathbf{s}, 0)\mathfrak{g} - \mathfrak{g}(\mathbf{s}, 0)$ for all $s \in G$. For any $f \in L^1(G)$, then we have

$$\begin{aligned}
D(f)(x) &= \int_G f(s)(\mathbf{s}\mathfrak{g} - \mathfrak{g}\mathbf{s})(x)\,\mathrm{d}\mu(s) \\
&= \int_G f(s)\left(\mathfrak{g}(x(\mathbf{s},0)) - \mathfrak{g}((\mathbf{s},0)x)\right)\,\mathrm{d}\mu(s) \\
&= \int_G f(s)\mathfrak{g}(x(\mathbf{s},0))\,\mathrm{d}\mu(s) - \int_G f(s)\mathfrak{g}((\mathbf{s},0)(x))\,\mathrm{d}\mu(s) \\
&= \mathfrak{g}(xf) - \mathfrak{g}(fx) \\
&= (f\mathfrak{g} - \mathfrak{g}f)(x)
\end{aligned}$$

Thus D is inner as a derivation on $L^1(G)$, so by Theorem 1.10 (revision) $L^1(G)$ is amenable. □

References

[1] C. Chou. *Locally compact groups which are amenable as discrete groups*, Proceeds of the American Mathematical Society, Volume 76, No. 1, 1979.