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Graduate project in mathematics

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Injective and semidiscrete von Neumann algebras

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Abstract

This graduate project concerns the concepts of injectivity and semidiscreteness for von Neumann algebras. A von Neumann algebra $\mathscr{M}$ is injective if it holds for any C^*-algebra $\mathcal{A}$ with a C^*-subalgebra $\mathcal{B}$ that a completely positive map $\mathcal{B} \to \mathscr{M}$ extends to a completely positive map $\mathcal{A} \to \mathscr{M}$, and $\mathscr{M}$ is semidiscrete if the identity map on $\mathscr{M}$ can be approximated ultraweakly by normal and completely positive maps of finite rank. The main theorem of the project states that these notions are in fact equivalent. To prove this theorem it is necessary to determine if certain von Neumann algebras inherit injectivity or semidiscreteness from others, and the proof also requires knowledge about Hilbert-Schmidt operators and continuous crossed products, the first of which will be dealt with thoroughly. Equally important is that the project provides the needed theoretical background for defining injectivity and semidiscreteness. To this end we will develop the relevant theory of tensor products of C^*-algebras and completely positive maps from scratch, as well as define and find properties of the ultraweak and ultrastrong operator topologies on the space of bounded linear operators on Hilbert spaces. In the process, we are also able to establish the notion of a predual of a von Neumann algebra, namely a Banach space whose dual can be identified with the von Neumann algebra, and the enveloping von Neumann algebra of a C^*-algebra $\mathcal{A}$ which can be identified with the double dual space $\mathcal{A}^{**}$ of the C^*-algebra in question.

Resumé

Dette fagprojekt omhandler egenskaberne injektivitet og semidiskrethed for von Neumann-algebraer. En von Neumann-algebra $\mathscr{M}$ er injektiv hvis der gælder for enhver C^*-algebra $\mathcal{A}$ indeholdende en C^*-delalgebra $\mathcal{B}$ at en fuldstændig positiv afbildning $\mathcal{B} \to \mathscr{M}$ kan udvides til en fuldstændig positiv afbildning $\mathcal{A} \to \mathscr{M}$, og $\mathscr{M}$ er semidiskret hvis identitetsafbildningen $\mathscr{M} \to \mathscr{M}$ kan tilnærmes ultrasvagt af normale, fuldstændig positive afbildninger med endeligdimensionalt billede. Hovedsætningen i dette projekt siger, at disse egenskaber faktisk er ækvivalente. For at bevise denne sætning kræves en række resultater om arvelighed af disse egenskaber, og beviset benytter også viden om Hilbert-Schmidt-operatorer og såkaldte kontinuerte krydsprodukter, hvoraf begrebet om Hilbert-Schmidt-operatorer vil blive uddybet helt og aldeles. Ikke mindst vil projektet også give den nødvendige teoretiske baggrund for at kunne definere injektivitet og semidiskrethed. Dette indebærer, at vi opbygger noget af teorien for tensorprodukter af C^*-algebraer og fuldstændig positive afbildninger fra bunden, samt definerer og finder egenskaber for den ultrasvage og ultrastærke operatortopologi på rum af begrænsede lineære operatorer over Hilbert-rum. I processen vil vi etablere begrebet om prædualet af en von Neumann algebra, navnlig et Banach-rum hvis duale rum kan identificeres med den oprindelige von Neumann algebra, samt den universelle von Neumann algebra for en C^*-algebra $\mathcal{A}$, som kan identificeres med det dobbeltduale rum $\mathcal{A}^{**}$ for C^*-algebraen.

Contents

PROLOGUE

Let me start out by saying that this project is not what it was originally supposed to be. Before I started writing, I was firmly convinced that by the time I had handed the project in I would have proved equivalence of not just the concepts injectivity and semidiscreteness, but also amenability and hyperfiniteness on top of that. Alas, that was not to be. I found out pretty quickly that I wasn't emotionally capable of writing about something that I did not understand down to the smallest detail, simply because every time I did, I would have a guilty conscience about doing it. I could compare it to trying to step into the middle of a busy conversation and try to join in – often one can't help but fail.

So instead this project became something else. Indeed I had to take some steps back and lower my ambitions, but once I accepted my limitations everything work-related felt quite a bit better just because I had a real chance of finding out what was going on. I probably do not need to tell you that the theory of von Neumann algebra often takes you places you did not expect (I for one would not expect the fact that $(\mathbb{R}, +)$ is an amenable group to have anything to do with the proof of the aforementioned equivalence, but what do I know), and I have not wasted my opportunity to find out what has really been going on beneath the surface.

That last statement might actually explain the length of this project: I have attempted to explain anything that *could* be explained. Some readers may find the length to be extreme overkill, and I don't blame them: the standard length of a graduate project is probably somewhere in the neighbourhood of 40 or 50 pages, at least in Copenhagen. However, I cannot state enough that the only intent of the project has been *for myself to learn something*, and if at least some of the things I have put in here are correct then I think I haven't failed in the least. (The fact that this project is also to be judged by my advisor and an external censor is, after all, more than anything an opportunity to learn, even though I hope that I haven't made some *really* big mistakes throughout.)

Nonetheless, the fact that I have sought throughout to understand everything fully implies big ambitions, and I have had a couple of big brain meltdowns and at least one emotional breakdown during the writing period. I only hope for future graduate project authors that they are not as sensitive as I have been, that they have friends as good as mine and that they listen to a lot of fantastic music.

This might be a good place to quickly run through what the project covers:

- When I write something, I prefer that all the required tools are laid on a table beforehand, and this project is no exception to that preference. Hence the first 10 pages or so are devoted to introducing all the needed concepts for C^*-algebras, von Neumann algebras, Hilbert spaces and positive linear functionals, including the GNS construction.

- As we hit upon the first chapter we shift gears and develop the theory of tensor products: we first cover vector spaces, then Hilbert spaces and finally *-algebras. Matrix algebras are defined and analyzed, allowing for an almost smooth transition into the world of tensor products of von Neumann algebras. We finally give a description of tensor products of C^*-algebra with a view toward algebraic states, culminating with the equivalence of the so-called minimal norm and a somewhat peculiar norm to be needed later.

- We next hit upon my perhaps favourite part of the project, Chapter 2. As a project-within-a-project concerning the ultraweak and ultrastrong operator topology on the space of bounded linear operators over Hilbert spaces, it is a real smorgasbord of concepts such as preduals, central supports, reduced von Neumann algebras, normal maps, σ-finiteness and the enveloping von Neumann algebra, along with more powerful versions of theorems essential for basic von Neumann algebra theory, namely von Neumann's density and bicommutant theorems. The chapter also contains a section about the Jordan decomposition of any bounded linear functional on a C^*-algebra.

- In the third chapter, we introduce the concepts of positive and completely positive maps between C^*-algebras or duals of C^*-algebras and explain them by means of relevant examples and theorems. The big result of this chapter is of course Stinespring's representation theorem, a generalization of the GNS construction. Using this theorem one can find out a lot about certain maps of von Neumann algebras, only for the greater good of the project.

- The fourth chapter concerns injectivity. The highlight here is Tomiyama's theorem, used to prove a von Neumann algebra condition equivalent to injectivity. With this criterion in hand, one can determine various hereditary properties of the concept. The chapter closes out with an introduction to continuous crossed products and amenable locally compact groups.

- The fifth and final chapter naturally concerns semidiscreteness. By means of the predual, we establish a couple of conditions equivalent to semidiscreteness and proceed along the lines of the previous chapter to prove hereditary properties of that concept. The final section is devoted to the proof that semidiscreteness is in fact equivalent to injectivity, using a lot of the theory we have been developing in all of the previous chapters.

- The project also contains two appendices: the first contains a lot of important results for Banach spaces and C^*-algebras, and the second develops the theory of trace class operators and Hilbert-Schmidt operators (it is needed in Chapter 5).

Let me also stress that finding great literature isn't necessarily as easy as one would think. Most of this project is based on the work done in [28] which has unfortunately not been the best of sources. Much time has been spent merely deciphering the proofs therein, either because there were mistakes, or because things that really needed a proper explanation did not get one. Hopefully my work will undo that wrongdoing.

One final piece of advice that is easy to remember but may be hard to follow at times: remember to sleep. My circadian rhythm has never been more messed up than when writing this project, and boy, have I suffered. Enjoy reading!

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Thanks to my parents Carsten and Judith and my sister Julie for taking care of me and for providing a great place to rest and write undisturbed, to my two bands for being such great friends and for keeping afloat my other main interest – music – and to the people I live with at Rigshospitalets Kollegium for being so great and nice to me and for feeding me once in a while with something other than pizza and microwave burgers. Finally, thanks to Anne Petersen, Katrine Frovin Gravesen and Anna Berger for just being there.

La boîte à joujoux

We will begin our tour of von Neumann algebra territory by introducing the most relevant concepts along with the notation that will be used for it; a box of tools (or toys) is perhaps the most fitting description of this introductory chapter. References will be given throughout, as we will not spend time proving *all* the statements we will give. Since we will be working with Hilbert spaces to no end throughout this project, let us also say now so that nobody forgets:

$\mathcal{H}$ denotes an arbitrary Hilbert space unless otherwise stated.

The inner product on $\mathcal{H}$ is denoted by $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ or just $\langle\cdot,\cdot\rangle$ if the Hilbert space is clear from the context, and $B(\mathcal{H})$ denotes the C^*-algebra of bounded linear operators on $\mathcal{H}$. We also introduce the follow convention immediately: for any normed space $\mathfrak{X}$ and $r>0$, $(\mathfrak{X})_r$ denotes the set $\{x\in\mathfrak{X}\,|\,\|x\|\le r\}$. The symbol n will usually denote a positive integer unless otherwise stated.

C^*-algebras

Here we will summarize the most important things one should know about C^*-algebras before reading the main parts of the project. Some of the results are more important or useful than others; the main idea is just to give the reader a short course on the varieties of elements one can find in C^*-algebras, as well as essential types of maps over them.

In the following, let $\mathcal{A}$ be a C^*-algebra. If $\mathcal{A}$ is unital, the *identity* or *unit* of $\mathcal{A}$ will be denoted by $1_{\mathcal{A}}$; in this case, then for any $a\in\mathcal{A}$ the *spectrum* $\sigma(a)$ is the non-empty compact subset of $\mathbb{C}$ consisting of all $\lambda\in\mathbb{C}$ such that $\lambda 1_{\mathcal{A}}-a$ is not invertible. We will *always* try to be as specific as possible concerning whether the C^*-algebras put under the microscope in this project have an identity or not.

- If $\mathcal{A}$ is non-unital, then the *unitization* of $\mathcal{A}$ is denoted by $\tilde{\mathcal{A}}$. As a set, $\tilde{\mathcal{A}}$ consists of all tuples (a,λ) with $a\in\mathcal{A}$ and $\lambda\in\mathbb{C}$, with coordinatewise addition and scalar multiplication, and the product given by
$$(a,\lambda)\cdot(b,\mu)=(ab+\mu a+\lambda b,\lambda\mu),\quad a,b\in\mathcal{A},\ \lambda,\mu\in\mathbb{C}.$$
These compositions then yield an identity $1_{\tilde{\mathcal{A}}}=(0,1)$. The involution and norm in $\tilde{\mathcal{A}}$ are given by
$$(a,\lambda)^*=(a^*,\overline{\lambda}),\quad \|(a,\lambda)\|=\sup\{\|ax+\lambda x\|\,|\,x\in\mathcal{A},\ \|x\|\le 1\},\quad a\in\mathcal{A},\ \lambda\in\mathbb{C}.$$
Note here that $\|(a,0)\|=\|a\|$ for all $a\in\mathcal{A}$, so that the inclusion $\mathcal{A}\to\tilde{\mathcal{A}}$ is an isometric *-homomorphism. The spectrum of an element a in a non-unital C^*-algebra is defined to be the spectrum of $(a,0)$ in the unitization.

- $\mathcal{A}_{\mathrm{sa}}$ denotes the set of *self-adjoint* elements of a C^*-algebra $\mathcal{A}$, i.e. elements $a\in\mathcal{A}$ such that $a=a^*$. Every element $a\in\mathcal{A}$ satisfies $a=a_1+ia_2$ where $a_1=\frac{1}{2}(a+a^*)\in\mathcal{A}_{\mathrm{sa}}$ and $a_2=\frac{1}{2i}(a-a^*)\in\mathcal{A}_{\mathrm{sa}}$. Note in this case that we also have $\|a_1\|\le\|a\|$ and $\|a_2\|\le\|a\|$. If $\mathcal{A}$ is unital, then for any $a\in\mathcal{A}_{\mathrm{sa}}$, $\sigma(a)\subseteq\mathbb{R}$ [31, Proposition 8.2]. Any self-adjoint element $a\in\mathcal{A}$ is *normal*, i.e. it satisfies the identity $a^*a=aa^*$. A subset $\mathscr{S}$ of a C^*-algebra $\mathcal{A}$ is *self-adjoint* if $x\in\mathscr{S}$ implies $x^*\in\mathscr{S}$, and we will write $\mathscr{S}=\mathscr{S}^*$ if this is the case.

- $\mathcal{A}_+$ denotes the set of *positive* elements of a C^*-algebra $\mathcal{A}$, i.e. elements $a\in\mathcal{A}$ for which there exists $b\in\mathcal{A}$ such that $a=b^*b$. It is well-known for unital C^*-algebras – see e.g. [31, Theorem 11.5] – that $a\in\mathcal{A}$ is positive if and only if $a\in\mathcal{A}_{\mathrm{sa}}$ and
$$\sigma(a)\subseteq\mathbb{R}_+=\{\lambda\in\mathbb{R}\,|\,\lambda\ge 0\}.$$

If $\mathcal{A} = B(\mathcal{H})$, then an element $T \in \mathcal{A}$ is positive if and only if $\langle T\xi, \xi\rangle \geq 0$ for all $\xi \in \mathcal{H}$ [31, Theorem 12.5]. It is possible to define an order relation on $\mathcal{A}_{\text{sa}}$ by defining $a \leq b$ if $b - a \in \mathcal{A}_+$ (see Proposition 0.6).

- It is a non-trivial result that any C^*-algebra has an *approximate identity*, i.e. there exists a net $(e_\alpha)_{\alpha\in A}$ in $\mathcal{A}$ such that $\|e_\alpha\| \leq 1$ and $e_\alpha \geq 0$ for all $\alpha \in A$ and $\|e_\alpha x - x\| \to 0$ and $\|xe_\alpha - x\| \to 0$ for all $x \in \mathcal{A}$ [31, Corollary 15.4].

- If $\mathcal{A}$ is a C^*-algebra, then $p \in \mathcal{A}$ is a *projection* if it satisfies $p^2 = p = p^*$. Projections in $B(\mathcal{H})$ have very nice properties: if $P \in B(\mathcal{H})$ is a projection, then $P(\mathcal{H})$ is a closed subspace of $\mathcal{H}$ and $\xi - P\xi \in P(\mathcal{H})^\perp$ for all $\xi \in \mathcal{H}$. If $\mathfrak{X}$ is a closed subspace of $\mathcal{H}$, then any $\xi \in \mathcal{H}$ can be decomposed uniquely as a sum of elements $\xi_1 \in \mathfrak{X}$ and $\xi_2 \in \mathfrak{X}^\perp$, $\mathfrak{X}^\perp$ denoting the orthogonal complement of $\mathfrak{X}$ [30, Theorem 4.24]. Defining $P\xi = \xi_1$, one obtains a map $P\colon \mathcal{H} \to \mathcal{H}$ which is in fact a bounded linear operator on $\mathcal{H}$ and a projection in $B(\mathcal{H})$, called the *orthogonal projection* onto $\mathfrak{X}$. For any two projections $P, Q \in B(\mathcal{H})$, we have equivalent conditions

$$P \leq Q \Leftrightarrow QP = P \Leftrightarrow QP = PQ = P \Leftrightarrow P(\mathcal{H}) \subseteq Q(\mathcal{H}).$$

- A *unitary* of a unital C^*-algebra $\mathcal{A}$ is an element $u \in \mathcal{A}$ that satisfies $u^*u = uu^* = 1_\mathcal{A}$ or $u^{-1} = u^*$. The set of unitaries of $\mathcal{A}$, denoted $\mathcal{U}(\mathcal{A})$, is a multiplicative group. If $\mathcal{A} = B(\mathcal{H})$, we will write $\mathcal{U}(\mathcal{H}) = \mathcal{U}(B(\mathcal{H}))$.

- For *-algebras $\mathcal{A}$ and $\mathcal{B}$, a map $\varphi\colon \mathcal{A} \to \mathcal{B}$ is a **-homomorphism* if it is linear and satisfies

$$\varphi(ab) = \varphi(ab), \quad \varphi(a^*) = \varphi(a)^*, \quad a, b \in \mathcal{A}.$$

If $\mathcal{A}$ and $\mathcal{B}$ are unital, a *-homomorphism is *unital* if it maps $1_\mathcal{A}$ to $1_\mathcal{B}$. If a *-homomorphism is bijective, it is called a **-isomorphism*. By [24, Proposition 5.2], any *-homomorphism of C^*-algebras is contractive, and by [24, Corollary 5.4], any *injective* *-homomorphism of C^*-algebras is an isometry and hence maps C^*-subalgebras to C^*-subalgebras. A unital *-homomorphism maps C^*-algebras to C^*-algebras [31, Theorem 11.1].

- A *representation* of a *-algebra $\mathcal{A}$ is a *-homomorphism $\mathcal{A} \to B(\mathcal{H})$ where $\mathcal{H}$ is some Hilbert space. A representation is called *faithful* if it is injective.

- The *continuous functional calculus* for normal elements of a unital C^*-algebra $\mathcal{A}$ is in general an immensely useful tool for constructing new operators with certain properties. For any normal element $a \in \mathcal{A}$, there is a *-isomorphism $C(\sigma(a)) \to \mathcal{B}$ where $\mathcal{B}$ is the C^*-subalgebra of $\mathcal{A}$ generated by a and $1_\mathcal{A}$, and the image in $\mathcal{B}$ of $f \in C(\sigma(a))$ under this map is denoted by $f(a)$. The essential properties of this *-isomorphism are briefly mentioned in [31, Theorem 10.3]. It is also possible to work with the continuous functional calculus for non-unital C^*-algebras: if $\mathcal{A}$ is non-unital, $a \in \mathcal{A}$ is normal and $f \in C(\sigma(a))$ satisfies $f(0) = 0$, then f defines an element $f(a) \in \mathcal{A}$ as f can be approximated uniformly by polynomials without constant term [24, p. 19]. We will come back to this whenever it will be needed in the main parts of the project.

  If $\mathcal{A}$ is unital, then perhaps the most intriguing application of the continuous functional calculus is the construction of unique square roots of positive elements a, i.e. a unique element $b \in \mathcal{A}$ such that $b^2 = a$. In this case we denote b by $a^{1/2}$, and we define $|a| \in \mathcal{A}$ for any $a \in \mathcal{A}$ by $|a| = (a^*a)^{1/2}$.

  It is worth mentioning that the continuous functional calculus also yields some useful inequalities. For instance,

$$-\|a\|1_\mathcal{A} \leq a \leq \|a\|1_\mathcal{A}, \quad a \in \mathcal{A}_{\text{sa}}.$$

  If $-b \leq a \leq b$ for $a, b \in \mathcal{A}_{\text{sa}}$, then one can show that $\|a\| \leq \|b\|$ by using the preceding result. For all $a, b \in \mathcal{A}_{\text{sa}}$, then $a \leq b$ implies $c^*ac \leq c^*bc$ for all $c \in \mathcal{A}$ just by using the definition of the order relation. This along with the first result yields $b^*a^*ab \leq \|a\|^2 b^*b$ for all $a, b \in \mathcal{A}$.

- In the C^*-algebra $B(\mathcal{H})$ we will often be working with *partial isometries*. $U \in B(\mathcal{H})$ is a partial isometry if its restriction to $(\ker V)^\perp$ is an isometry. One can prove that $U \in B(\mathcal{H})$ is a partial isometry if and only if U^*U is a projection [31, Proposition 12.6], in which case U^*U projects onto $\ker U$. Also, U is a partial isometry if and only if U^* is a partial isometry [31, Corollary 12.7]. Finally, partial isometries are insanely useful for decomposing arbitrary operators in $B(\mathcal{H})$. The so-called *polar decomposition* of any operator $T \in B(\mathcal{H})$ yields a partial isometry $U \in B(\mathcal{H})$ such that $T = U|T|$. U can be chosen to be the orthogonal projection onto the closure of the image of $|T|$ [31, Theorem 12.8].

❖ For any subset $\mathscr{S}$ of a C^*-algebra $\mathcal{A}$, the *commutant* $\mathscr{S}'$ is defined as follows:

$$\mathscr{S}' = \{a \in \mathcal{A} \,|\, ab = ba \text{ for all } b \in \mathscr{S}\},$$

i.e. $\mathscr{S}'$ consists of all elements of $\mathcal{A}$ that commute with elements of $\mathscr{S}$. It is easily seen that $\mathscr{S}'$ is a Banach subalgebra of $\mathcal{A}$, and if $\mathscr{S}$ is self-adjoint, then $\mathscr{S}'$ is a C^*-subalgebra of $\mathcal{A}$.

Positive linear functionals on C^*-algebras

For a *-algebra $\mathcal{A}$, a linear functional $\varphi\colon \mathcal{A} \to \mathbb{C}$ is *positive* if $\varphi(x^*x) \geq 0$ for all $x \in \mathcal{A}$. It can be proved that any positive linear functional on any C^*-algebra (even a non-unital one) is bounded; see [24, Proposition 9.12].

Given a *-algebra $\mathcal{A}$, then for $\varphi_1, \varphi_2 \in \mathcal{A}^*$, we will write $\varphi_1 \leq \varphi_2$ if $\varphi_2 - \varphi_1$ is a positive linear functional on $\mathcal{A}$; we say the φ_2 *dominates* φ_1.

For any Banach *-algebra $\mathcal{A}$, $S(\mathcal{A})$ denotes the space of *states* of $\mathcal{A}$, i.e. the set of all positive linear functionals $\varphi\colon \mathcal{A} \to \mathbb{C}$ with $\|\varphi\| = 1$. If $\mathcal{A}$ is a unital C^*-algebra, a linear functional $\varphi\colon \mathcal{A} \to \mathbb{C}$ is positive if and only if it is bounded with $\|\varphi\| = \varphi(1_{\mathcal{A}})$, the proof of which can be found in [31, Theorem 13.5] (one implication is the consequence of Proposition 0.3); hence $\varphi \in S(\mathcal{A})$ if and only if $\|\varphi\| = \varphi(1_{\mathcal{A}}) = 1$ or φ is positive and $\varphi(1_{\mathcal{A}}) = 1$. A state $\varphi \in S(\mathcal{A})$ is *faithful* if $\varphi(a^*a) > 0$ for all nonzero $a \in \mathcal{A}$ and *tracial* if it satisfies $\varphi(ab) = \varphi(ba)$ for all $a, b \in \mathcal{A}$.

We shall often need the following results for estimation purposes.

Proposition 0.1. *Let $\mathcal{A}$ be a *-algebra and let φ be a positive linear functional on $\mathcal{A}$. Then*

(i) $\varphi(b^*a) = \overline{\varphi(a^*b)}$ *for all $a, b \in \mathcal{A}$.*
(ii) *(The Cauchy-Schwarz inequality)* $|\varphi(b^*a)|^2 \leq \varphi(a^*a)\varphi(b^*b)$ *for all $a, b \in \mathcal{A}$.*

Proof. For (i), we have

$$\varphi(a^*a) + \varphi(b^*a) + \varphi(a^*b) + \varphi(b^*b) = \varphi((a+b)^*(a+b)) \geq 0,$$

so $\varphi(b^*a) + \varphi(a^*b) \in \mathbb{R}$, and hence $\mathrm{Im}\varphi(b^*a) = -\mathrm{Im}\varphi(a^*b)$. As $\mathrm{Re}z = \mathrm{Im}iz$ for all $z \in \mathbb{C}$, we have

$$\mathrm{Re}\varphi(b^*a) = \mathrm{Im}\varphi(b^*(ia)) = -\mathrm{Im}\varphi((ia)^*b) = -\mathrm{Im}(-i\varphi(a^*b)) = \mathrm{Im}(i\varphi(a^*b)) = \mathrm{Re}\varphi(a^*b)$$

and hence $\varphi(b^*a) = \overline{\varphi(a^*b)}$.

To prove (ii), note that for $\lambda \in \mathbb{C}$, we have

$$0 \leq \varphi((a-\lambda b)^*(a-\lambda b)) = \varphi(a^*a) - \overline{\lambda}\varphi(b^*a) - \lambda\varphi(a^*b) + |\lambda|^2\varphi(b^*b).$$

Assuming first that $\varphi(b^*b) > 0$, then by setting $\lambda = \varphi(b^*a)\varphi(b^*b)^{-1}$, we obtain

$$\varphi(a^*a) - \frac{|\varphi(b^*a)|^2}{\varphi(b^*b)} \geq 0$$

by using (i). By rearranging terms, we obtain the wanted inequality. If $\varphi(b^*b) = 0$, let $n \geq 1$ be a positive integer and put $\lambda = n\varphi(b^*a)$. This implies $\varphi(a^*a) - 2n|\varphi(b^*a)|^2 \geq 0$, so $2n|\varphi(b^*a)|^2 \leq \varphi(a^*a)$ for all $n \geq 1$. This implies that $|\varphi(b^*a)|^2 = 0$, and hence the inequality also holds in this case. □

The following theorem lays down an intimate connection between C^*-algebras and their states. The properties of the *Gelfand transform* are essential to the proof; see [31, Chapter 5].

Theorem 0.2. *Let $\mathcal{A}$ be a C^*-algebra and let $a \in \mathcal{A}$ be normal. Then there exists a state $\varphi \in S(\mathcal{A})$ such that $|\varphi(a)| = \|a\|$. In particular, if $a \in \mathcal{A}$ is any element and $\varphi(a) = 0$ for all $\varphi \in S(\mathcal{A})$, then $a = 0$.*

Proof. We can assume that $a \neq 0$. Assume that $\mathcal{A}$ is non-unital first. Let $\mathcal{B}$ be the C^*-subalgebra of the unitization $\tilde{\mathcal{A}}$ of $\mathcal{A}$ generated by the unit $1_{\tilde{\mathcal{A}}}$ and $(a, 0)$. $\mathcal{B}$ is commutative and hence the Gelfand transform $\Gamma\colon \mathcal{B} \to C(\Delta(\mathcal{B}))$ is a *-isomorphism [31, Theorem 10.3], $\Delta(\mathcal{B}) \subseteq \mathcal{B}^*$ denoting the weak* compact Hausdorff space of multiplicative linear functionals on $\mathcal{B}$. Since $\Delta(\mathcal{B})$ is weak* compact, there exists a multiplicative linear functional $\varphi_2 \in \Delta(\mathcal{B})$ such that

$$|\Gamma((a,0))(\varphi_2)| = \|\Gamma((a,0))\|_\infty = \sup\{|\Gamma((a,0))(\psi)| \mid \psi \in \Delta(\mathcal{B})\},$$

hence yielding

$$\|a\| = \|(a,0)\| = \|\Gamma((a,0))\|_\infty = |\Gamma((a,0))(\varphi_2)| = |\varphi_2((a,0))|.$$

By the Hahn-Banach theorem [13, Theorem 5.7] there exists a bounded linear functional φ_1 on $\tilde{\mathcal{A}}$ such that $\varphi_1|_{\mathcal{B}} = \varphi_2$ and $\|\varphi_1\| = \|\varphi_2\| = 1$. Since $\varphi_1(1_{\tilde{\mathcal{A}}}) = \varphi_2(1_{\tilde{\mathcal{A}}}) = 1$, it follows that φ_1 is positive and $|\varphi_1((a,0))| = \|a\|$. Let φ denote the restriction to $\mathcal{A}$ (i.e. $\varphi(a) = \varphi_1((a,0))$ for all $a \in \mathcal{A}$). Since $\|\varphi\| \leq \|\varphi_1\| = 1$ and $|\varphi(\|a\|^{-1}a)| = \|a\|^{-1}|\varphi(a)| = 1$, it follows that $\|\varphi\| = 1$. Since φ is also positive, the first statement follows in the non-unital case. If $\mathcal{A}$ is unital, the proof above applies without the need to pass to unitizations.

From the first statement, it now follows that if $\varphi(a) = 0$ for a normal element $a \in \mathcal{A}$ and all states $\varphi \in S(\mathcal{A})$, then $a = 0$. For any $a \in \mathcal{A}$, write $a = a_1 + ia_2$, where $a_1, a_2 \in \mathcal{A}_{\mathrm{sa}}$. If $\varphi(a) = 0$, then $\varphi(a_1) = \varphi(a_2) = 0$ for all $\varphi \in S(\mathcal{A})$, so $a_1 = a_2 = 0$, and hence $a = 0$. □

We will also need the next result in the first two chapters.

Proposition 0.3. *If $\mathcal{A}$ is a C^*-algebra and $\varphi\colon \mathcal{A} \to \mathbb{C}$ is a positive linear functional on $\mathcal{A}$, then $\|\varphi\| = \lim_{\alpha\in A} \varphi(e_\alpha)$ for any approximate identity $(e_\alpha)_{\alpha\in A}$ in $\mathcal{A}$.*

Proof. Let $(e_\alpha)_{\alpha\in A}$ be an approximate identity in $\mathcal{A}$. As $(\varphi(e_\alpha))_{\alpha\in A}$ is a bounded increasing net in $\mathbb{R}_+$, it follows that the net has a limit $\lambda \leq \|\varphi\|$ which is also its least upper bound. Let $f_\alpha \in \mathcal{A}$ be such that $f_\alpha^* f_\alpha = e_\alpha$ for all $\alpha \in \mathcal{A}$. Then for each $a \in (\mathcal{A})_1$, the Cauchy-Schwarz inequality yields

$$|\varphi(e_\alpha a)|^2 = |\varphi(f_\alpha^* f_\alpha a)|^2 \leq \varphi(a f_\alpha^* f_\alpha a^*)\varphi(f_\alpha^* f_\alpha) = \varphi(a e_\alpha a^*)\varphi(e_\alpha) \leq \|\varphi\|\lambda.$$

Taking the limit over A and then the supremum over all $a \in (\mathcal{A})_1$ yields $\|\varphi\|^2 \leq \|\varphi\|\lambda$ from which $\|\varphi\| \leq \lambda$ follows, completing the proof. □

The direct sum of Hilbert spaces

Let $(\mathcal{H}_i)_{i\in I}$ be a family of Hilbert spaces. The *direct sum* of the Hilbert spaces $(\mathcal{H}_i)_{i\in I}$ is the subset of the cartesian product $\prod_{i\in I} \mathcal{H}_i$ consisting of all families of elements $(\xi_i)_{i\in I}$ with $\xi_i \in \mathcal{H}_i$ that are *square-summable*, i.e. we have $\sum_{i\in I} \|\xi_i\|^2 < \infty$, and it is denoted $\bigoplus_{i\in I} \mathcal{H}_i$.

To the knowledge and reading experience of the author, many textbooks skip a proof that $\bigoplus_{i\in I} \mathcal{H}_i$ is actually a Hilbert space, so we will put one here. Now at least some might have a reason to read this project...

Proposition 0.4. *With coordinatewise addition and scalar multiplication $\bigoplus_{i\in I} \mathcal{H}_i$ becomes a vector space, and it becomes a Hilbert space by endowing it with the inner product*

$$\langle (\xi_i)_{i\in I}, (\eta_i)_{i\in I}\rangle = \sum_{i\in I} \langle \xi_i, \eta_i\rangle, \quad (\xi_i)_{i\in I}, (\eta_i)_{i\in I} \in \bigoplus_{i\in I} \mathcal{H}_i.$$

Proof. Let $\mathcal{H} = \bigoplus_{i\in I} \mathcal{H}_i$. For $(\xi_i)_{i\in I}, (\eta_i)_{i\in I} \in \mathcal{H}$ and $\lambda \in \mathbb{C}$, then since

$$\|\xi_i + \eta_i\|^2 \leq (\|\xi_i\| + \|\eta_i\|)^2 \leq 2\|\xi_i\|^2 + 2\|\eta_i\|^2$$

for all $i \in I$, we have that $(\xi_i + \eta_i)_{i\in I} \in \mathcal{H}$ and $(\lambda\xi_i)_{i\in I} \in \mathcal{H}$. Hence $\mathcal{H}$ is a vector space with the given operations.

Once we prove that the form $\langle\cdot,\cdot\rangle\colon \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ is well-defined, it is easy to check all the wanted properties of an inner product, namely that it is sesquilinear and positive semi-definite. Finally, if we assume that $\langle (\xi_i)_{i\in I}, (\xi_i)_{i\in I}\rangle = 0$ then $\sum_{i\in I} \|\xi_i\|^2 = 0$, so for all $i_0 \in I$ we have

$$0 \leq \|\xi_{i_0}\|^2 \leq \sum_{i\in I} \|\xi_i\|^2 = 0.$$

Thus $\xi_i = 0$ for all $i \in I$, so $(\xi_i)_{i\in I} = 0$.

To prove well-definedness, let $(\xi_i)_{i\in I}, (\eta_i)_{i\in I} \in \mathcal{H}$ and define

$$\xi_i' = \begin{cases} \xi_i & \text{if } \|\xi_i\| \geq \|\eta_i\| \\ 0 & \text{otherwise,} \end{cases} \qquad \eta_i' = \begin{cases} \eta_i & \text{if } \|\xi_i\| < \|\eta_i\| \\ 0 & \text{otherwise.} \end{cases}$$

It is clear for any $i \in I$ that $\|\xi_i'\| \leq \|\xi_i\|$ and $\|\eta_i'\| \leq \|\eta_i\|$, so we have $\sum_{i\in I} \|\xi_i'\|^2 \leq \sum_{i\in I} \|\xi_i\|^2$ and $\sum_{i\in I} \|\eta_i'\|^2 \leq \sum_{i\in I} \|\eta_i\|^2$. Hence $(\xi_i')_{i\in I}, (\eta_i')_{i\in I} \in \mathcal{H}$. Furthermore, we see that $\|\xi_i\| \leq \|\xi_i' + \eta_i'\|$ and $\|\eta_i\| \leq \|\xi_i' + \eta_i'\|$ for all $i \in I$ by mere construction, so we obtain

$$|\langle \xi_i, \eta_i\rangle| \leq \|\xi_i\|\|\eta_i\| \leq \|\xi_i' + \eta_i'\|^2, \quad i \in I.$$

Therefore $\sum_{i\in I} |\langle \xi_i, \eta_i\rangle|$ converges, since $(\xi_i' + \eta_i')_{i\in I}$ belongs to $\mathcal{H}$ because it is a vector space. Therefore the inner product is well-defined.

It remains to prove completeness of the metric induced by the inner product. Let $(\xi^n)_{n\geq 1}$ be a Cauchy sequence in $\mathcal{H}$ with $\xi^n = (\xi_i^n)_{i\in I}$ for $n \geq 1$. Then

$$\|\xi_{i_0}^n - \xi_{i_0}^m\|^2 \leq \sum_{i\in I} \|\xi_i^n - \xi_i^m\|^2$$

for all $n, m \geq 1$, so $(\xi_i^n)_{ngeq1}$ is a Cauchy sequence in $\mathcal{H}_i$ for all $i \in I$. Because all $\mathcal{H}_i$ are Hilbert spaces, we obtain the existence of an element $\xi_i \in \mathcal{H}_i$ such that $\xi_i^n \to \xi_i$ for all $i \in I$. Define $\xi = (\xi_i)_{i\in I} \in \prod_{i\in I} \mathcal{H}_i$. This will be our candidate for a limit, so it remains to prove that $\xi \in \mathcal{H}$ and that $\xi^n \to \xi$.

Start by fixing an $\varepsilon > 0$ and let $N \geq 1$ such that $\|\xi^n - \xi^m\|^2 < \frac{\varepsilon^2}{2}$ for $n, m \geq N$, possible since $(\xi^n)_{n\geq 1}$ is a Cauchy sequence. Then for all finite subsets $G \subseteq I$ and $n \geq N$, we have

$$\begin{aligned} \frac{\varepsilon^2}{2} &\geq \lim_{m\to\infty} \|\xi^n - \xi^m\|^2 = \lim_{m\to\infty} \sum_{i\in I} \|\xi_i^n - \xi_i^m\|^2 \\ &\geq \lim_{m\to\infty} \sum_{i\in G} \|\xi_i^n - \xi_i^m\|^2 = \sum_{i\in G} \lim_{m\to\infty} \|\xi_i^n - \xi_i^m\|^2 = \sum_{i\in G} \|\xi_i^n - \xi_i\|^2 \end{aligned}$$

Hence $\sum_{i\in I} \|\xi_i^n - \xi_i\|^2 \leq \frac{\varepsilon^2}{2}$ for $n \geq N$, so $\xi^N - \xi \in \mathcal{H}$. Hence $\xi = \xi^N - (\xi^N - \xi) \in \mathcal{H}$. Finally, it follows that $\|\xi^n - \xi\| < \varepsilon$ for $n \geq N$, so $\xi^n \to \xi$. Hence $\mathcal{H}$ is a Hilbert space. □

Along with the construction of the direct sum of Hilbert spaces come some natural maps with obvious but still neat properties. Let $\mathcal{H} = \bigoplus_{i\in I} \mathcal{H}_i$ and $j \in I$. Let $\iota_j \colon \mathcal{H}_j \to \mathcal{H}$ denote the natural injection of $\mathcal{H}_j$ into $\mathcal{H}$ for $j \in I$, i.e. $(\iota_j \xi)_j = \xi$ and $(\iota_j \xi)_i = 0$ for all $i \in I$ with $i \neq j$. Note that ι_j is isometric, so $\iota_j(\mathcal{H}_j)$ is a closed subspace of $\mathcal{H}$ for all $j \in I$. Furthermore, for $\xi \in \mathcal{H}_j$ and $(\eta_i)_{i\in I} \in \mathcal{H}$ then

$$\langle \iota_j \xi, (\eta_i)_{i\in I}\rangle = \langle \xi, \eta_j\rangle,$$

so $\iota_j^* \colon \mathcal{H} \to \mathcal{H}_j$ is the projection of $\mathcal{H}$ onto the j'th coordinate, and we denote $\pi_j = \iota_j^*$. It follows immediately that $\pi_j \iota_j \in B(\mathcal{H}_j)$ is the identity on $\mathcal{H}_j$ and that $E_j = \iota_j \pi_j \in B(\mathcal{H})$ is the orthogonal projection onto $\iota_j(\mathcal{H}_j)$. As all the E_i are orthogonal, note that any $\xi = (\xi_i)_{i\in I} \in \mathcal{H}$ satisfies

$$\xi = \sum_{i\in I} \iota_i(\xi_i) = \sum_{i\in I} E_i \xi,$$

so $1_{\mathcal{H}} = \sum_{i\in I} E_i$ where the series converges strongly.

We introduce some advantageous notation in the cases of some very particular direct sums. For a Hilbert space $\mathcal{H}$ and a non-empty set I, we denote the Hilbert space $\bigoplus_{i\in I} \mathcal{H}$ by $\mathcal{H}^I$. For $n \geq 1$ the Hilbert space $\bigoplus_{i=1}^n \mathcal{H}$ is denoted by $\mathcal{H}^n$.

The Gelfand-Neimark-Segal construction

For any Banach *-algebra $\mathcal{A}$ and $\varphi \in S(\mathcal{A})$, there exists a *GNS representation* of $\mathcal{A}$ corresponding to φ, consisting of a Hilbert space $\mathcal{H}_\varphi$, a *-homomorphism $\pi_\varphi\colon \mathcal{A} \to B(\mathcal{H}_\varphi)$ and a unit vector $\xi_\varphi \in \mathcal{H}_\varphi$ such that the subspace

$$\pi_\varphi(\mathcal{A})\xi_\varphi = \{\pi_\varphi(a)\xi_\varphi \mid a \in \mathcal{A}\}$$

is dense in $\mathcal{H}_\varphi$ and

$$\varphi(a) = \langle \pi_\varphi(a)\xi_\varphi, \xi_\varphi\rangle, \quad a \in \mathcal{A}.$$

If $\mathcal{A}$ is unital, then π_φ can be made unital. For a proof of this whole shenanigan, see [24, Theorem I.9.14]. $(\mathcal{H}_\varphi, \pi_\varphi, \xi_\varphi)$ is called the *GNS triple* associated with φ. It can be proved if φ is faithful that π_φ is faithful as well.

Consider now the Hilbert space $\mathcal{H} = \bigoplus_{\varphi\in S(\mathcal{A})} \mathcal{H}_\varphi$ and the map $\pi = \bigoplus_{\varphi\in S(\mathcal{A})} \pi_\varphi\colon \mathcal{A} \to B(\mathcal{H})$ given by

$$\pi(a)(\eta_\varphi)_{\varphi\in S(\mathcal{A})} = (\pi_\varphi(a)\eta_\varphi)_{\varphi\in S(\mathcal{A})}, \quad (\eta_\varphi)_{\varphi\in S(\mathcal{A})} \in \mathcal{H}.$$

π is a representation, and it is in fact faithful: if $\pi(a)(\eta_\varphi)_{\varphi\in S(\mathcal{A})} = 0$ for all $(\eta_\varphi)_{\varphi\in S(\mathcal{A})} \in \mathcal{H}$ and some $a \in \mathcal{A}$, then $\pi_\varphi(a)\xi_\varphi = 0$ for all $\varphi \in S(\mathcal{A})$. Therefore $\varphi(a) = 0$ for all $\varphi \in S(\mathcal{A})$, so $a = 0$ by Theorem 0.2.

Proposition 0.5. *Let $(e_\alpha)_{\alpha\in A}$ be an approximate identity for $\mathcal{A}$. Then for all $\eta \in \mathcal{H}$, with $\mathcal{H}$ as defined above, we have*

$$\|\pi(e_\alpha)\eta - \eta\| \to 0.$$

Proof. Let $\eta = (\eta_\varphi)_{\varphi\in S(\mathcal{A})} \in \mathcal{H}$ and $F \subseteq S(\mathcal{A})$ be a finite subset such that

$$\|\eta - (\eta_\varphi)_{\varphi\in F}\|^2 = \sum_{\varphi\notin F} \|\eta_\varphi\|^2 < \frac{\varepsilon^2}{25}.$$

For each $\varphi \in F$, take $x_\varphi \in \mathcal{A}$ such that $\|\pi_\varphi(x_\varphi)\xi_\varphi - \eta_\varphi\| \leq \frac{\varepsilon}{5}|F|^{-1}$, and take $\alpha_0 \in A$ such that $\|e_\alpha x_\varphi - x_\varphi\| < \frac{\varepsilon}{5}|F|^{-1}\|(\xi_\varphi)_{\varphi\in F}\|^{-1}$ for all $\varphi \in F$ and $\alpha \geq \alpha_0$. Then for all $\alpha \geq \alpha_0$,

$$\begin{aligned}
\|\pi(e_\alpha)\eta - \eta\| &\leq \|\pi(e_\alpha)\eta - \pi(e_\alpha)(\eta_\varphi)_{\varphi\in F}\| + \|\pi(e_\alpha)(\eta_\varphi)_{\varphi\in F} - \pi(e_\alpha)(\pi_\varphi(x_\varphi)\xi_\varphi)_{\varphi\in F}\| \\
&\quad + \|\pi(e_\alpha)(\pi_\varphi(x_\varphi)\xi_\varphi)_{\varphi\in F} - (\pi_\varphi(x_\varphi)\xi_\varphi)_{\varphi\in F}\| + \|(\pi_\varphi(x_\varphi)\xi_\varphi)_{\varphi\in F} - (\eta_\varphi)_{\varphi\in F}\| \\
&\quad + \|(\eta_\varphi)_{\varphi\in F} - \eta\| \\
&< \frac{\varepsilon}{5} + \sum_{\varphi\in F} \|\eta_\varphi - \pi_\varphi(x_\varphi)\xi_\varphi\| + \|(\xi_\varphi)_{\varphi\in F}\| \sum_{\varphi\in F} \|\pi_\varphi(e_\alpha x_\varphi - x_\varphi)\| \\
&\quad + \sum_{\varphi\in F} \|\pi_\varphi(x_\varphi)\xi_\varphi - \eta_\varphi\| + \frac{\varepsilon}{5} \\
&< \varepsilon,
\end{aligned}$$

since π and π_φ are contractive for all $\varphi \in F$ and (e_α) is an approximate identity. □

A consequence of the faithfulness of the representation π is this:

Proposition 0.6. *Let $\mathcal{A}$ be a C^*-algebra, and for $a, b \in \mathcal{A}_{\mathrm{sa}}$, write $a \leq b$ if $b - a \in \mathcal{A}_+$. Then $(\mathcal{A}_{\mathrm{sa}}, \leq)$ is a partially ordered set. In particular, finite sums of positive elements are positive.*

Proof. It is obvious that $\leq$ is reflexive. Letting $\pi = \bigoplus_{\varphi\in S(\mathcal{A})} \pi_\varphi\colon \mathcal{A} \to B(\mathcal{H})$ denote the above representation, then if $a \leq b$ and $b \leq a$, we have

$$\langle \pi_\varphi(b-a)\pi_\varphi(c)\xi_\varphi, \pi_\varphi(c)\xi_\varphi\rangle = \varphi(c^*(b-a)c) \geq 0$$

for all $\varphi \in S(\mathcal{A})$ so $\langle \pi_\varphi(b-a)\eta_\varphi, \eta_\varphi\rangle \geq 0$ for all $\eta_\varphi \in \mathcal{H}_\varphi$. Hence $\langle \pi(b-a)\eta, \eta\rangle \geq 0$ for all $\eta \in \mathcal{H}$. Likewise one can find that $-\langle \pi(b-a)\eta, \eta\rangle = \langle \pi(a-b)\eta, \eta\rangle \geq 0$ for all $\eta \in \mathcal{H}$, so $\langle \pi(b-a)\eta, \eta\rangle = 0$ for all $\eta \in \mathcal{H}$, implying $\pi(b-a) = 0$ and therefore $a = b$ by faithfulness of π. Finally, if $a \leq b$ and $b \leq c$, one likewise obtains that $\langle \pi(c-a)\eta, \eta\rangle \geq 0$ for all $\eta \in \mathcal{H}$, so $\pi(c-a)$ is positive. Since π is faithful, $\pi(\mathcal{A})$ is a C^*-algebra. Hence it follows that $\pi(c-a) = \pi(x)^*\pi(x) = \pi(x^*x)$ for some $x \in \mathcal{A}$, and therefore $c - a = x^*x \in \mathcal{A}_+$. □

Von Neumann algebras

The main concern in this project will be the wonderous world of von Neumann algebras, and we will start out by defining these pretty much from scratch. The main tools are the so-called weak and strong operator topology, although we will be expanding on these concepts in Chapter 2. If one needs a crash course on locally convex topological vector spaces, the author recommends [13].

Let $(T_\alpha)_{\alpha\in A}$ be a net in $B(\mathcal{H})$ and let $T \in \mathcal{H}$.

- The *weak operator topology* on $B(\mathcal{H})$ is the locally convex Hausdorff topology defined by the seminorms
$$T \mapsto |\langle T\xi, \eta\rangle|$$
for $\xi, \eta \in \mathcal{H}$. Hence $T_\alpha \to T$ in the weak operator topology (or *weakly*) if and only if
$$\langle T_\alpha \xi, \eta\rangle \to \langle T\xi, \eta\rangle$$
for all $\xi, \eta \in \mathcal{H}$.
- The *strong operator topology* on $B(\mathcal{H})$ is the locally convex Hausdorff topology defined by the seminorms
$$T \mapsto \|T\xi\|$$
where $\xi \in \mathcal{H}$. Hence $T_\alpha \to T$ in the strong operator topology (or *strongly*) if and only if
$$\|T_\alpha \xi - T\xi\| \to 0$$
for all $\xi \in \mathcal{H}$.

These topologies are special cases of the so-called point-norm and point-weak topology on $B(\mathfrak{X})$ for a Banach space $\mathfrak{X}$; these are defined in Appendix A. The reason that these more general topologies are relegated to an appendix is that we will only need them for a short while. Nonetheless, the results proved in Appendix A apply for the next couple of results (which we will need right away).

Note that norm convergence in $B(\mathcal{H})$ implies strong operator convergence which in turn implies weak operator convergence. The project revolves first and foremost around the following definition:

Definition 0.1. A *von Neumann algebra* is a *-subalgebra of $B(\mathcal{H})$ that contains the identity operator $1_\mathcal{H}$ and is closed in the strong operator topology.

Because norm convergence implies strong operator convergence, it follows that any von Neumann algebra is a unital C^*-subalgebra of $B(\mathcal{H})$. We will often treat a von Neumann algebra as an algebraic structure rather than a set of operators on a Hilbert space, and for this reason we might denote the unit of a von Neumann algebra $\mathscr{M}$ by $1_\mathscr{M}$. If we are considering a specific Hilbert space $\mathcal{H}$ on which $\mathscr{M}$ operates, we will oftentimes denote the unit of $\mathscr{M}$ by $1_\mathcal{H}$.

The following proposition is extremely useful for alternate characterisations of von Neumann algebras:

Proposition 0.7. *Let $\omega\colon B(\mathcal{H}) \to \mathbb{C}$ be a linear functional. Then the following are equivalent:*

(i) *ω is weakly continuous.*
(ii) *ω is strongly continuous.*
(iii) *There exist elements $\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_n$ of $\mathcal{H}$ such that $\omega(T) = \sum_{i=1}^n \langle T\xi_i, \eta_i\rangle$.*

Proof. See [31, Theorem 16.1] for a direct proof, or just Proposition A.5. □

This result implies that the strong operator and weak operator closures of a convex subset of $B(\mathcal{H})$ are the same (cf. [31, Theorem 16.2] or Corollary A.8), so the notion of strong operator closure in the definition of a von Neumann algebra can be freely replaced by the one of weak operator closure. If $\mathscr{M}$ is a self-adjoint subset of $B(\mathcal{H})$, then its commutant is a von Neumann algebra [31, Proposition 18.1].

The next result is similar to the fact that any bounded increasing sequence in $\mathbb{R}$ has a limit which is also its supremum, allowing for a notion of a supremum in $B(\mathcal{H})$, and it is no overstatement to say that we will use it *a lot*. For the record, if $(T_\alpha)_{\alpha\in A}$ in $B(\mathcal{H})$ is a net of self-adjoint operators, then we say that it is *bounded above* if $\sup_{\alpha\in A} \|T_\alpha\| < \infty$ or, equivalently, if there exists a self-adjoint operator $T \in B(\mathcal{H})$ such that $T_\alpha \le T$, and that it is *increasing* if $\alpha \le \beta$ implies $T_\alpha \le T_\beta$ for all $\alpha, \beta \in A$.

Theorem 0.8. *Let $\mathscr{M}$ be a strongly closed *-subalgebra of $B(\mathcal{H})$. If $(T_\alpha)_{\alpha\in A}$ is an increasing net of self-adjoint operators in $\mathscr{M}$ which is bounded above, then there exists $T \in \mathscr{M}$ such that T is the strong operator limit of $(T_\alpha)_{\alpha\in A}$. Moreover, T is the least upper bound of the net, i.e. if $S \in B(\mathcal{H})$ is self-adjoint and satisfies $T_\alpha \le S$ for all $\alpha \in A$, then $T \le S$.*

Proof. See [31, Theorem 17.1]. □

In the above case, T is called the *supremum* of $(T_\alpha)_{\alpha\in A}$ and is denoted by

$$\sup_{\alpha\in A} T_\alpha.$$

Infimums can be similarly defined. In particular, any family $(P_i)_{i\in I}$ of mutually orthogonal projections in $\mathscr{M}$ is *strong operator-summable*: the net $(\sum_{i\in F} P_i)_{F\subseteq I}$ (where each F is finite) converges strongly to a projection $P \in \mathscr{M}$ satisfying $\|P\xi\|^2 = \sum_{i\in I} \|P_i\xi\|^2$ [31, Corollary 17.4].

The next theorem will be "improved upon" in Chapter 2, but we will need its most basic form immediately: it characterizes von Neumann algebras by means of commutants.

Theorem 0.9 (The von Neumann bicommutant theorem, 1929)**.** *Let $\mathscr{M}$ be a *-subalgebra of $B(\mathcal{H})$ with $1_{\mathcal{H}} \in \mathscr{M}$. Then $\mathscr{M}$ is a von Neumann algebra if and only if $\mathscr{M} = \mathscr{M}''$.*

Proof. See [27, pp. 12-13]. □

Note that if $\mathscr{M}$ is a self-adjoint subset of $B(\mathcal{H})$ and $\mathscr{N}$ is a von Neumann algebra containing $\mathscr{M}$, then $\mathscr{M}'' \subseteq \mathscr{N}$. Hence $\mathscr{M}''$ is the smallest von Neumann algebra containing $\mathscr{M}$.

The direct sum of von Neumann algebras

Direct sums of Hilbert spaces of course yields a possibility of creating new von Neumann algebras in a very obvious but also very beautiful manner.

Let $(\mathcal{H}_i)_{i\in I}$ be a family of Hilbert spaces and let $(T_i)_{i\in I}$ be a family of operators with $T_i \in B(\mathcal{H}_i)$ that satisfies $\sup_{i\in I} \|T_i\| < \infty$. If $\mathcal{H} = \bigoplus_{i\in I} \mathcal{H}_i$, then the map $T\colon \mathcal{H} \to \mathcal{H}$ given by $(\xi_i)_{i\in I} \mapsto (T_i\xi_i)_{i\in I}$ defines a bounded linear operator, since

$$\|T((\xi_i)_{i\in I})\|^2 = \sum_{i\in I} \|T_i\xi_i\|^2 \le \left(\sup_{i\in I} \|T_i\|\right)^2 \sum_{i\in I} \|\xi_i\|^2 < \infty.$$

In this case T is also denoted by $(T_i)_{i\in I}$. Note that $\|T\| \le \sup_{i\in I} \|T_i\|$; in fact equality holds. If $i \in I$ and $\xi_i \in (\mathcal{H}_i)_1$, then let ξ_j be the zero vector in $\mathcal{H}_j$ for all $j \in I$, $j \neq i$. Then

$$\|T_i\xi_i\|^2 = \sum_{j\in I} \|T_j\xi_j\|^2 = \|T(\xi_j)_{j\in I}\|^2 \le \|T\|^2,$$

so $\|T_i\| \le \|T\|$ for all $i \in I$. Therefore

$$\|(T_i)_{i\in I}\| = \sup_{i\in I} \|T_i\|.$$

If $(\mathscr{M}_i)_{i\in I}$ is a family of *-algebras with $\mathscr{M}_i \subseteq B(\mathcal{H}_i)$ for all $i \in I$, then the subset of $B(\mathcal{H})$ given by

$$\bigoplus_{i\in I} \mathscr{M}_i := \left\{ T = (T_i)_{i\in I} \,|\, T_i \in \mathscr{M}_i \text{ for } i \in I \text{ and } \sup_{i\in I} \|T_i\| < \infty \right\}$$

is a *-subalgebra, where the linear operations, product and adjoint operation are coordinatewise.

Proposition 0.10. *If $(\mathscr{M}_i)_{i\in I}$ is a family of von Neumann algebras with $\mathscr{M}_i \subseteq B(\mathcal{H}_i)$ for all $i \in I$, then $\mathscr{M} = \bigoplus_{i\in I} \mathscr{M}_i$ is a von Neumann algebra on $\mathcal{H} = \bigoplus_{i\in I} \mathcal{H}_i$.*

Proof. We will use the maps ι_i, π_i and E_i defined after the proof of Proposition 0.4. Defining

$$\mathscr{B} = \left\{ B = (B_i)_{i\in I} \mid B_i \in \mathscr{M}_i' \text{ for } i \in I \text{ and } \sup_{i\in I} \|B_i\| < \infty \right\},$$

it is readily seen that $\mathscr{M}$ and $\mathscr{B}$ are unital *-algebras in $B(\mathcal{H})$ and that every operator in $\mathscr{M}$ commutes with any operator in $\mathscr{B}$, i.e. $\mathscr{M} \subseteq \mathscr{B}'$. Furthermore, $E_i \in \mathscr{B}$ for all $i \in I$. Supposing that $T \in B(\mathcal{H})$ commutes with any operator in $\mathscr{B}$, i.e. $T \in \mathscr{B}'$, then $TE_i = E_iT$ for all $i \in I$. If we define $T_i \colon \mathcal{H}_i \to \mathcal{H}_i$ by $T_i = \pi_i T \iota_i$, we see that $\|T_i\| \leq \|T\|$ for all $i \in I$; furthermore, for $j \in I$, then if $\xi \in \mathcal{H}_j$, we have

$$((T_i)_{i\in I}\iota_j(\xi))_j = \pi_j T \iota_j(\xi) = (T\iota_j(\xi))_j$$

and for $k \neq j$,

$$(T\iota_j(\xi))_k = (E_j T \iota_j(\xi))_k = (\iota_j T_j(\xi))_k = 0 = ((T_i)_{i\in I}\iota_j(\xi))_k,$$

implying $T\iota_j(\xi) = (T_i)_{i\in I}\iota_j(\xi)$. Since every element in $\mathcal{H}$ is a sum of elements $\iota_j(\xi)$, it follows that $T = (T_i)_{i\in I}$ by continuity. Hence if $B \in \mathscr{B}$ with $B = (B_i)_{i\in I}$, then $T_iB_i = B_iT_i$ for all $i \in I$. This implies that T_i commutes with all operators in $\mathscr{M}_i'$, so $T_i \in \mathscr{M}_i''$. Therefore

$$\mathscr{B}' \subseteq \bigoplus_{i\in I} \mathscr{M}_i''.$$

Since each $\mathscr{M}_i$ is a von Neumann algebra, we have $\mathscr{B}' \subseteq \mathscr{M}$ by the von Neumann bicommutant theorem and hence $\mathscr{M}$ is a von Neumann algebra. □

In the above case $\mathscr{M}$ is called *the direct sum of the von Neumann algebras* $(\mathscr{M}_i)_{i\in I}$. Let $j \in I$. Note that for $T = (T_i)_{i\in I}$ in $\mathscr{M}$, then $\pi_j (T_i)_{i\in I} \iota_j = T_j \in \mathscr{M}_j$. If $T \in \mathscr{M}_j$, then by defining

$$\tilde{T} = (T_i)_{i\in I}$$

where $T_j = T$ and $T_i = 0$ for all $i \neq j$ we see that $\tilde{T} \in \mathscr{M}$ and $T = \pi_j \tilde{T} \iota_j \in \pi_j \mathscr{M} \iota_j$, hence allowing us to identify $\mathscr{M}_j$ with $\pi_j \mathscr{M} \iota_j$. The proof above also yields the following important corollary:

Corollary 0.11. *If $(\mathscr{M}_i)_{i\in I}$ is a family of von Neumann algebras and $\mathscr{M} = \bigoplus_{i\in I} \mathscr{M}_i$, then*

$$\mathscr{M}' = \bigoplus_{i\in I} \mathscr{M}_i'.$$

CHAPTER 1

TENSOR PRODUCTS OF INVOLUTIVE ALGEBRAS

It may seem curious that the project contains a chapter on *tensor products*. Fact is that I have not encountered a definition of the algebraic tensor product I *really* liked in any of the material I have been assimilating for the project, so I thought I might give it my own spin. As one might have noticed from the table of contents, the chapter not only concerns algebraic tensor products, but related notions for C^*-algebras and von Neumann algebras. Of course these notions rely heavily on the vector space case, so there is really no good reason not to begin at the beginning.

1.1 The algebraic tensor product

We note first that the definition of tensor products will not differ in any way from the one encountered in basic homological algebra; the vector spaces can be replaced with modules over any associative ring with a multiplicative unit to define tensor products in a more general case, but as we will only be working with vector spaces, there is no need to generalize.

Definition 1.1. Let X and Y be vector spaces. An *algebraic tensor product* of X and Y is a vector space T together with a bilinear map $\tau\colon X \times Y \to T$ satisfying the following property: For any pair (V, σ) where V is a vector space and σ is a bilinear map $\sigma\colon X \times Y \to V$, there exists a unique linear map $\tilde{\sigma}\colon T \to V$ such that $\sigma = \tilde{\sigma} \circ \tau$, i.e. the following diagram commutes:

$$\begin{tikzcd} X \times Y \arrow[rr, "\tau"] \arrow[dr, "\sigma"'] & & T \arrow[dl, dotted, "!\tilde{\sigma}"] \\ & V & \end{tikzcd}$$

Keep in mind that there is another way to define it: we could have taken a *specific* vector space and proved that it indeed satisfied the needed properties of a tensor product. This is of course equivalent to proving that the tensor product exists – which we will later show that it actually does. The idea behind not beginning in this manner is that we do not really need to know what specific vector space it is to work with the tensor product. The following theorem possibly makes this even clearer.

Theorem 1.1 (Uniqueness of tensor products)**.** *Let X and Y be vector spaces, and let (T, τ) and (T', τ') be algebraic tensor products of X and Y. Then there is a unique isomorphism $\alpha\colon T \to T'$ such that $\tau' = \alpha \circ \tau$.*

Proof. The property of the tensor product (T, τ) used on the pair (T', τ') yields a unique linear map $\alpha\colon T \to T'$ such that $\tau' = \alpha \circ \tau$. Hence uniqueness is proven, and it only remains to show that it is an isomorphism. First of all, note that the property of (T', τ') used on (T, τ) likewise yields a unique linear map $\beta\colon T' \to T$ such that $\tau = \beta \circ \tau'$. Moreover, the property of (T, τ) used on itself yields a unique linear map $\gamma\colon T \to T$ such that $\tau = \gamma \circ \tau$. Because

$$(\beta \circ \alpha) \circ \tau = \beta \circ \tau' = \tau,$$

then both $\beta \circ \alpha$ and the identity on T also satisfy this equation, so they must be equal. Analoguously, one sees that $\alpha \circ \beta$ is the identity on T', proving that α is an isomorphism with inverse β. $\square$

The above theorem tells us that once we have constructed an algebraic tensor product of two vector spaces, we have determined the vector space structure of *any* algebraic tensor product completely. The property in Definition 1.1 hence fully characterizes them: it is their *universal property*. The uniqueness of tensor products also allows us to speak of *the* algebraic tensor product of two vector spaces X and

Y, and we will denote it by $X \odot Y$. Elements of $X \odot Y$ are called *tensors*, and for any $x \in X$ and $y \in Y$, we define

$$x \otimes y := \tau(x, y) \in X \odot Y;$$

such an element is called an *elementary tensor*. Some nice properties hold for these in particular:

Proposition 1.2 (Tensor calculus). *Let X and Y be vector spaces. Then*

(i) $(x_1 + x_2) \otimes y = x_1 \otimes y + x_2 \otimes y$ *for* $x_1, x_2 \in X$ *and* $y \in Y$,
(ii) $x \otimes (y_1 + y_2) = x \otimes y_1 + x \otimes y_2$ *for* $x \in X$ *and* $y_1, y_2 \in Y$ *and*
(iii) $(\lambda x) \otimes y = x \otimes (\lambda y) = \lambda(x \otimes y)$ *for* $x \in X$, $y \in Y$ *and* $\lambda \in \mathbb{C}$.

Proof. This just follows from bilinearity of τ. □

It turns out that any element of the algebraic tensor product is a finite sum of elementary tensors:

Proposition 1.3 (A picture of the tensor product). *Let X and Y be vector spaces and let $v \in X \odot Y$. Then there is a positive integer $n \geq 1$ as well as $x_1, \ldots, x_n \in X$ and $y_1, \ldots, y_n \in Y$ such that*

$$v = \sum_{i=1}^{n} x_i \otimes y_i.$$

Proof. Let V consist of all finite sums of elementary tensors, and define $\sigma\colon X \times Y \to V$ by

$$\sigma(x, y) = x \otimes y.$$

V is clearly a vector space in itself. Since σ is bilinear, the universal property of $X \odot Y$ yields a linear map $\tilde{\sigma}\colon X \odot Y \to V \subseteq X \odot Y$ such that $\sigma(x, y) = \tilde{\sigma}(\tau(x, y))$ for all $x \in X$ and $y \in Y$. Since $\sigma(x, y) = \mathrm{id}(\tau(x, y))$ as well, where id denotes the identity on $X \odot Y$, it follows by uniqueness that $\tilde{\sigma} = \mathrm{id}$. Hence it follows that the identity maps into V, implying $X \odot Y = V$. □

It is quite amazing how much we have already derived from the very simple defining property of tensor products. The above proposition also tells us that we might only need check properties of linear maps for elementary tensors.

We now prove a few theorems concerning maps from tensor products.

Proposition 1.4 (Tensor product maps). *Let X, Y, V and W be vector spaces. If $\varphi\colon X \to V$ and $\psi\colon Y \to W$ are linear maps, then there is a unique linear map*

$$\varphi \odot \psi\colon X \odot Y \to V \odot W$$

such that $\varphi \odot \psi(x \otimes y) = \varphi(x) \otimes \psi(y)$ for all $x \in X$ and $y \in Y$.

Proof. The map $X \times Y \to V \odot W$ given by $(x, y) \mapsto \varphi(x) \otimes \psi(y)$ is bilinear, so we just apply the universal property of $X \odot Y$. □

Proposition 1.5 (Product maps). *Let X and Y be vector spaces and let C be an algebra. If $\varphi\colon X \to C$ and $\psi\colon Y \to C$ are linear maps, then there is a unique linear map*

$$\varphi \times \psi\colon X \odot Y \to C$$

such that $\varphi \times \psi(x \otimes y) = \varphi(x)\psi(y)$ for all $x \in X$ and $y \in Y$.

Proof. The map $X \times Y \to C$ given by $(x, y) \mapsto \varphi(x)\psi(y)$ is bilinear, and we again apply the universal property of $X \odot Y$. □

Corollary 1.6 (Tensor product functionals). *Let X and Y be vector spaces. If $\varphi\colon X \to \mathbb{C}$ and $\psi\colon Y \to \mathbb{C}$ are linear functionals, then there is a unique linear functional*

$$\varphi \odot \psi\colon X \odot Y \to \mathbb{C}$$

such that $\varphi \odot \psi(x \otimes y) = \varphi(x)\psi(y)$ for all $x \in X$ and $y \in Y$.

Proof. This follows from Proposition 1.5 or from Proposition 1.4 using the fact that the map $\mathbb{C}\odot\mathbb{C} \to \mathbb{C}$ satisfying $a \otimes b \mapsto ab$ is an isomorphism (hence the notation). □

Corollary 1.7 (Conjugate linear tensor product functionals)**.** *Let X and Y be vector spaces. If $\varphi\colon X \to \mathbb{C}$ and $\psi\colon Y \to \mathbb{C}$ are conjugate linear functionals, i.e. additive maps that satisfy*

$$\varphi(\lambda x) = \overline{\lambda}\varphi(x), \quad \psi(\mu x) = \overline{\mu}\psi(x), \quad \lambda, \mu \in \mathbb{C},\ x \in X,\ y \in Y,$$

then there exists a conjugate linear tensor product functional

$$\varphi \odot \psi\colon X \odot Y \to \mathbb{C}$$

such that $\varphi \odot \psi(x \otimes y) = \varphi(x)\psi(y)$ for all $x \in X$ and $y \in Y$.

Proof. This follows from Corollary 1.6 applied to the linear functionals $\overline{\varphi}\colon X \to \mathbb{C}$ and $\overline{\psi}\colon Y \to \mathbb{C}$ given by $\overline{\varphi}(x) = \overline{\varphi(x)}$ and $\overline{\psi}(y) = \overline{\psi(y)}$ respectively, and then conjugating the resulting linear functional. □

All of this, however, does not diminish the fact that our deductions would have no purpose if tensor products did not exist. Luckily they do.

Theorem 1.8. *If X and Y are vector spaces, there exists an algebraic tensor product of X and Y.*

Proof. Considering $X \times Y$ as a discrete topological space, let $\widetilde{T} = C_c(X \times Y)$ denote the vector space of compactly supported functions over $X \times Y$. For $x \in X$ and $y \in Y$, let $\chi_{(x,y)}$ denote the characteristic function of the one-point set $\{(x,y)\} \subseteq X \times Y$. Since a compact subset of a discrete topological space is necessarily finite, it follows that the set

$$\{\chi_{(x,y)} \,|\, x \in X,\, y \in Y\}$$

constitutes a basis of $\widetilde{T}$, as any element therein is a unique finite linear combination of these elements. Let $\widetilde{T}_0$ denote the linear subspace of $\widetilde{T}$ spanned by all elements of the four following types:

(i) $\chi_{(x_1+x_2,y)} - \chi_{(x_1,y)} - \chi_{(x_2,y)}$ for $x_1, x_2 \in X$ and $y \in Y$;
(ii) $\chi_{(x,y_1+y_2)} - \chi_{(x,y_1)} - \chi_{(x,y_2)}$ for $x \in X$ and $y_1, y_2 \in Y$;
(iii) $\chi_{(\lambda x,y)} - \lambda\chi_{(x,y)}$ for $x \in X$, $y \in Y$ and $\lambda \in \mathbb{C}$;
(iv) $\chi_{(x,\lambda y)} - \lambda\chi_{(x,y)}$ for $x \in X$, $y \in Y$ and $\lambda \in \mathbb{C}$.

Now, define $T := \widetilde{T}/\widetilde{T}_0$ and let $\pi\colon \widetilde{T} \to T$ be the canonical quotient mapping, i.e.

$$\pi(f) = f + \widetilde{T}_0, \quad f \in \widetilde{T}.$$

Furthermore, define a map $\widetilde{\tau}\colon X \times Y \to \widetilde{T}$ by $\tau(x,y) = \chi_{(x,y)}$. From how we defined $\widetilde{T}_0$, it follows that $\tau := \pi \circ \widetilde{\tau}$ is bilinear. We claim that T together with τ is a tensor product of X and Y.

Now let V be a vector space and let $\sigma\colon X \times Y \to V$ be a bilinear map. Define a map $\hat{\sigma}\colon \widetilde{T} \to V$ through the identity $\hat{\sigma}(\chi_{(x,y)}) = \sigma(x,y)$ by extending linearly. Elements in $\widetilde{T}_0$ of the four aforementioned types are sent to the zero vector because of bilinearity of σ, so we obtain an induced linear map $\tilde{\sigma}\colon T \to V$ defined by

$$\tilde{\sigma}(f + \widetilde{T}_0) = \hat{\sigma}(f), \quad f \in \widetilde{T}.$$

Hence

$$\tilde{\sigma}(\tau(x,y)) = \tilde{\sigma}(\pi(\widetilde{\tau}(x,y))) = \tilde{\sigma}(\chi_{(x,y)} + \widetilde{T}_0) = \hat{\sigma}(\chi_{(x,y)}) = \sigma(x,y)$$

for all $x \in X$ and $y \in Y$, so $\sigma = \tilde{\sigma} \circ \tau$. If $\eta\colon T \to V$ is another linear map satisfying $\sigma = \eta \circ \tau$, note that $\eta(\tau(x,y)) = \tilde{\sigma}(\tau(x,y))$ and hence

$$\eta(\chi_{(x,y)} + \widetilde{T}_0) = \tilde{\sigma}(\chi_{(x,y)} + \widetilde{T}_0)$$

for all $x \in X$ and $y \in Y$. Since η and $\tilde{\sigma}$ agree on a spanning set of T, it follows that they are equal. □

We now turn our attention to C^*-algebras. As two C^*-algebras are vector spaces themselves, their algebraic tensor product exists, but the question is whether one can endow it with a C^*-algebra structure. To do this, one must of course give it a multiplication and an involution first, and that is what we will attempt to do, keeping in mind that we want these operations to look as natural (or obvious) as possible. The proofs required for this need some knowledge of when tensors are linearly independent, and Zorn's lemma will help us along the way: it is used in the proof of the next theorem.

Theorem 1.9. *Every linearly independent subset V of a vector space X can be extended to an algebraic basis for X.*

Proof. Let $\mathscr{S}$ be the collection of all linearly independent subsets of X containing V. $\mathscr{S}$ is non-empty since $V \in \mathscr{S}$, and if $\mathfrak{J}$ is a totally ordered subset of $\mathscr{S}$, let $J = \bigcup_{W \in \mathfrak{J}} W$. If $\lambda_1 x_1 + \ldots + \lambda_n x_n = 0$ for $x_1, \ldots, x_n \in J$, then $x_i \in W_i$ for some $W_i \in \mathfrak{J}$. Pick the largest of the W_i in $\mathfrak{J}$; since all x_i are in it and it is linearly independent, it follows that J is a linearly independent subset and an upper bound for all $W \in \mathfrak{J}$ inclusion-wise. Hence Zorn's lemma [13, Lemma 0.2] yields a maximal element S of $\mathscr{S}$. Let W be the linear span of S, and assume for contradiction that $W \neq X$. Then there exists $x \in X \setminus W$. If

$$\lambda x + \lambda_1 x_1 + \ldots + \lambda_n x_n = 0$$

for $x_1, \ldots, x_n \in S$, then $-\lambda x = \lambda_1 x_1 + \ldots + \lambda_n x_n$, so that $-\lambda x \in W$. Since $x \notin W$, we have $\lambda = 0$. Therefore $\lambda_1 = \ldots = \lambda_n = 0$, so $S \cup \{x\}$ is a linearly independent subset of X containing V, but this contradicts the maximality of S. Hence S is an algebraic basis for X. □

Proposition 1.10 (Linear independence)**.** *Let X and Y be vector spaces. If $x_1, \ldots, x_n \in X$ are arbitrary, $y_1, \ldots, y_n \in Y$ are linearly independent and $\sum_{i=1}^n x_i \otimes y_i = 0$, then $x_1 = \ldots = x_n = 0$.*

Proof. First of all, $\{y_1, \ldots, y_n\}$ can be extended to an algebraic basis $S = (s_\alpha)_{\alpha \in A}$ of Y by Theorem 1.9. Let α_j be the $\alpha \in \mathcal{A}$ such that $s_{\alpha_j} = y_j$ for all $j = 1, \ldots, n$. For each j, we may then define a linear functional φ_j on Y by

$$\varphi_j \left(\sum_\alpha \lambda_\alpha s_\alpha \right) = \sum_\alpha \lambda_\alpha t_\alpha^j,$$

where $\lambda_\alpha \in \mathbb{C}$, the sum $\sum_\alpha \lambda_\alpha s_\alpha$ is finite and the family $(t_\alpha^j)_{\alpha \in A}$ is given by $t_\alpha^j = 1$ for $\alpha = \alpha_j$ and $t_\alpha^j = 0$ for all $\alpha \neq \alpha_j$. It is clear that linear functionals on a vector space separate points in the above way, so it now suffices to prove that $\psi(x_j) = 0$ for all linear functionals ψ on X and all $j = 1, \ldots, n$. This follows from the construction of Corollary 1.6 as

$$0 = \psi \odot \varphi_j \left(\sum_{i=1}^n x_i \otimes y_i \right) = \sum_{i=1}^n \psi(x_i) \varphi_j(y_i) = \psi(x_j)$$

for any linear functional ψ on X. □

The above linear independence result has a lot of nice consequences: it tells us something about possible bases for algebraic tensor products and what happens to tensor product maps of injective maps. The next three results tell us what needs to be known in order to go further and are especially essential when uncovering tensor product notions for C^*-algebras.

Corollary 1.11 (Bases for tensor products)**.** *Let X and Y be vector spaces. If $(x_i)_{i \in I} \subseteq X$ and $(y_j)_{j \in J} \subseteq Y$ are bases, then $(x_i \otimes y_j)_{(i,j) \in I \times J} \subseteq X \odot Y$ is a basis of $X \odot Y$. In particular,*

$$\dim(X \odot Y) = \dim(X) \dim(Y).$$

Proof. Because $X \odot Y$ is spanned by elementary tensors, and any elementary tensor is a finite linear combination of elementary tensors of the form $x_i \otimes y_j$, $X \odot Y$ is spanned by the above set. Assume that

$$\sum_{(i,j) \in A} \lambda_{i,j} (x_i \otimes y_j) = 0$$

for some finite subset $A \subseteq I \times J$. Let $B \subseteq J$ denote the subset of all $j \in J$ such that $(i,j) \in A$ for some $i \in I$, and for $j \in B$, let I_j consist of all $i \in I$ such that $(i,j) \in A$. Then

$$0 = \sum_{(i,j) \in A} \lambda_{i,j} (x_i \otimes y_j) = \sum_{i \in B} \left(\sum_{j \in J_i} \lambda_{i,j} x_i \right) \otimes y_j.$$

It then follows from Proposition 1.10 that $\sum_{j\in J_i}\lambda_{i,j}x_j = 0$ for all $i \in B$. Hence for any $i \in B$ we obtain $\lambda_{i,j} = 0$ for all $j \in J_i$ using linear independence of the x_j. Hence $\lambda_{i,j} = 0$ for all $(i,j) \in A$. $\square$

The next corollary comes in handy later.

Corollary 1.12 (Unique representations). *Let X and Y be vector spaces. If $\{y_i\}_{i\in I} \subseteq Y$ is a basis and $z \in X \odot Y$, then there exists a unique set $(x_i)_{i\in I} \subseteq X$ such that $z = \sum_{i\in I} x_i \otimes y_i$, where only finitely many of the x_i are non-zero.*

Proof. Let $x'_j \in X$ and $y'_j \in Y$, $j = 1,\dots,n$, such that $z = \sum_{j=1}^n x'_j \otimes y'_j$. For each $j = 1,\dots,n$ there exists a family $(\lambda_{ji})_{i\in I}$ such that $y'_j = \sum_{i\in I}\lambda_{ji}y_i$, where only finitely many of the λ_{ji} are non-zero. Let I_0 denote the finite set of $i \in I$ such that $\lambda_{ji} \neq 0$ for some $j = 1,\dots,n$, so that $y'_j = \sum_{i\in I_0}\lambda_{ji}y_i$. Hence we can write z as a finite sum

$$z = \sum_{j=1}^n \sum_{i\in I_0} \lambda_{ji}x'_j \otimes y_i = \sum_{i\in I_0}\left(\sum_{j=1}^n \lambda_{ji}x'_j\right)\otimes y_i = \sum_{i\in I}\left(\sum_{j=1}^n \lambda_{ji}x'_j\right)\otimes y_i,$$

so by putting $x_i = \sum_{j=1}^n \lambda_{ji}x'_j$ we have found a family of the wanted form. This family is unique: if $\sum_{i\in I} x_i \otimes y_i = \sum_{i\in I} x'_i \otimes y_i$, then by letting I_1 consist of the i such that either $x_i \neq 0$ or $x'_i \neq 0$, I_1 is finite and Proposition 1.10 yields $x_i = x'_i$ for all $i \in I_1$. For $i \in I \setminus I_1$, $x_i = x'_i = 0$, so the families $(x_i)_{i\in I}$ and $(x'_i)_{i\in I}$ are equal. $\square$

Proposition 1.13. *Let X, Y, V and W be vector spaces. If $\varphi\colon X \to V$ and $\psi\colon Y \to W$ are injective linear maps, then the tensor product map $\varphi \odot \psi\colon X \odot Y \to V \odot W$ is injective.*

Proof. Let $v \in X \odot Y$ such that $\varphi \odot \psi(v) = 0$ and write $v = \sum_{i=1}^n x_i \otimes y_i$ for $x_i \in V$ and $y_i \in W$, where $i = 1,\dots,n$. Choose a basis $(x'_j)_{i=1}^m$ for the linear span of the x_i, so that

$$v = \sum_{j=1}^n x'_j \otimes \left(\sum_{i=1}^n \lambda_{ij}y_i\right)$$

for numbers $\lambda_{ij} \in \mathbb{C}$. As

$$\sum_{j=1}^n \varphi(x'_j) \otimes \psi\left(\sum_{i=1}^n \lambda_{ij}y_i\right) = 0$$

by assumption and φ is injective, it follows from Proposition 1.10 that $\psi(\sum_{i=1}^n \lambda_{ij}y_i) = 0$ and hence $\sum_{i=1}^n \lambda_{ij}y_i$ for all $j = 1,\dots,m$, so $v = 0$. $\square$

We are now finally ready to jump right onto the C^*-algebra train. As one might guess, there are natural ways to define the multiplication and involution of a tensor product of *-algebras – not just C^*-algebras – and the proofs concern whether these natural operations are well-defined.

Proposition 1.14. *Let $\mathcal{A}$ and $\mathcal{B}$ be *-algebras. The tensor product $\mathcal{A} \odot \mathcal{B}$ has a multiplication defined by*

$$\left(\sum_i a_i \otimes b_i\right)\left(\sum_j c_j \otimes d_j\right) = \sum_{i,j}(a_ic_j) \otimes (b_id_j).$$

Proof. Once we prove that the above multiplication is well-defined, one can straightforwardly check that it indeed satisfies the axioms required for it to be a true multiplication. We will construct the above multiplication using the universal property of the tensor product.

Consider first the vector space $L(\mathcal{A} \odot \mathcal{B})$ of all linear maps $\mathcal{A} \odot \mathcal{B} \to \mathcal{A} \odot \mathcal{B}$. Let $M_a\colon \mathcal{A} \to \mathcal{A}$ be left multiplication by $a \in \mathcal{A}$, i.e. the map $x \mapsto ax$, and let $M_b\colon \mathcal{B} \to \mathcal{B}$ denote left multiplication by $b \in \mathcal{B}$. Now Proposition 1.4 yields a unique linear map $M_a \odot M_b \in L(\mathcal{A} \odot \mathcal{B})$ such that

$$M_a \odot M_b(a' \otimes b') = (aa') \otimes (bb')$$

for all $a' \in \mathcal{A}$, $b' \in \mathcal{B}$. Define $\varphi\colon \mathcal{A}\times\mathcal{B} \to L(\mathcal{A}\odot\mathcal{B})$ by $\varphi(a,b) = M_a \odot M_b$. We claim that φ is actually bilinear. For instance, for $a_1, a_2 \in \mathcal{A}$ and $b \in \mathcal{B}$, note that the sum $M_{a_1} \odot M_b + M_{a_2} \odot M_b$ is also a linear map $\mathcal{A}\odot\mathcal{B} \to \mathcal{A}\odot\mathcal{B}$. For all $a' \in \mathcal{A}$ and $b' \in \mathcal{B}$ we then have

$$(M_{a_1} \odot M_b + M_{a_2} \odot M_b)(a' \otimes b') = ((a_1 + a_2)a') \otimes (bb') = M_{a_1+a_2} \odot M_b(a' \otimes b'),$$

but since $M_{a_1+a_2} \odot M_b$ was unique with this property, we must have

$$\varphi(a_1 + a_2, b) = M_{(a_1+a_2)} \odot M_b = M_{a_1} \odot M_b + M_{a_2} \odot M_b = \varphi(a_1, b) + \varphi(a_2, b).$$

The rest of the desired properties follow similarly.

Universality now yields a linear map $M\colon \mathcal{A}\odot\mathcal{B} \to L(\mathcal{A}\odot\mathcal{B})$ such that $M(a\otimes b) = M_a \odot M_b$. The map $(\mathcal{A}\odot\mathcal{B})^2 \to \mathcal{A}\odot\mathcal{B}$ given by $(x,y) \mapsto M(x)y$ then defines the above multiplication; indeed,

$$M\left(\sum_i a_i \otimes b_i\right)\left(\sum_j c_j \otimes d_j\right) = \sum_{i,j} M_{a_i} \odot M_{b_i}(c_j \otimes d_j) = \sum_{i,j}(a_i c_j) \otimes (b_i d_j).$$

This proves that $\mathcal{A}\odot\mathcal{B}$ can be endowed with an algebra structure. □

The case of the involution is a little trickier.

Proposition 1.15. *Let $\mathcal{A}$ and $\mathcal{B}$ be *-algebras. There exists a unique involution $^*\colon \mathcal{A}\odot\mathcal{B} \to \mathcal{A}\odot\mathcal{B}$ such that*

$$(a\otimes b)^* = a^* \otimes b^*, \quad a \in \mathcal{A},\ b \in \mathcal{B}.$$

Proof. Uniqueness follows from the requirement that it satisfies the above equation, and indeed the equation ensures that there is only one possible way to define it, namely

$$\left(\sum_{i=1}^n a_i \otimes b_i\right)^* = \sum_{i=1}^n a_i^* \otimes b_i^*.$$

The only concern we might have is whether this map is well-defined, and it boils down to proving that $\sum_{i=1}^n a_i \otimes b_i = 0$ implies $\sum_{i=1}^n a_i^* \otimes b_i^* = 0$. Choose a basis $(e_i)_{i=1}^m$ of the linear span of the set $\{b_1, \dots, b_n\}$ and write $b_i = \sum_{k=1}^m \lambda_{ij} e_j$ with $\lambda_{ij} \in \mathbb{C}$ for $1 \le i \le n$. Then

$$0 = \sum_{i=1}^n a_i \otimes \left(\sum_{j=1}^m \lambda_{ij} e_j\right) = \sum_{i=1}^n \left(\sum_{j=1}^m \lambda_{ij} a_i\right) \otimes e_j.$$

Hence, Proposition 1.10 yields $\sum_{j=1}^m \lambda_{ij} a_i = 0$ and thus $\sum_{j=1}^m \overline{\lambda_{ij}} a_i^* = 0$ for all $1 \le i \le n$. Therefore

$$\sum_{i=1}^n a_i^* \otimes b_i^* = \sum_{i=1}^n a_i^* \otimes \left(\sum_{j=1}^m \overline{\lambda_{ij}} e_j^*\right) = \sum_{i=1}^n \left(\sum_{j=1}^m \overline{\lambda_{ij}} a_i^*\right) \otimes e_j^* = 0,$$

so the map is well-defined. The axioms are easily checked, proving that it is indeed an involution. □

Before something nearly completely different, we introduce two propositions about tensor product maps over *-algebras.

Proposition 1.16 (Tensor product *-homomorphisms)**.** *Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ be *-algebras and let $\varphi\colon \mathcal{A} \to \mathcal{C}$ and $\psi\colon \mathcal{B} \to \mathcal{D}$ be *-homomorphisms. Then the tensor product map $\varphi \odot \psi\colon \mathcal{A}\odot\mathcal{B} \to \mathcal{C}\odot\mathcal{D}$ is also a *-homomorphism.*

Proof. This follows from straightforward calculations. □

Proposition 1.17 (Product *-homomorphisms)**.** *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be *-algebras and let $\pi_{\mathcal{A}}\colon \mathcal{A} \to \mathcal{C}$ and $\pi_{\mathcal{B}}\colon \mathcal{B} \to \mathcal{C}$ be *-homomorphisms with commuting ranges. Then the product map $\varphi \times \psi\colon \mathcal{A}\odot\mathcal{B} \to \mathcal{C}$ is a *-homomorphism.*

Proof. This is also straightforward. The requirement of commuting ranges is used for proving that $\varphi \times \psi$ preserves adjoints. □

We now need to make an important digression in order to use our tensor products of *-algebras and hence C^*-algebras to the fullest. Recall that any C^*-algebra has a faithful representation on a Hilbert space (see viii). Since we can construct the algebraic tensor product of two Hilbert spaces, one could then ask if there for any two C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ with representations on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ exists a representation of $\mathcal{A} \odot \mathcal{B}$ on a Hilbert space related to $\mathcal{H} \odot \mathcal{K}$. However, this is a rather big question to ask at the moment: we do not even know whether $\mathcal{A} \odot \mathcal{B}$ has a well-defined norm, let alone a C^*-algebra structure, and how would we construct an inner product space structure on $\mathcal{H} \odot \mathcal{K}$? We cannot have our cake and eat it too, so we will digress for a moment and look into the Hilbert space situation.

Recall what we mean by a *Hilbert space completion*: if V is an inner product space, then V has a unique Hilbert space completion $(\mathcal{H}, \rho)$, where $\mathcal{H}$ is a Hilbert space and ρ is a linear and inner product-preserving map $V \to \mathcal{H}$ with dense range [5, Theorem 23]. It is commonplace to denote $\rho(x) \in \mathcal{H}$ by x for all $x \in V$, implying that V lies inside $\mathcal{H}$ (although strictly speaking, it is not always so).

If $x \in \mathcal{H} \odot \mathcal{K}$ for two Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, then we can write $x = \sum_{i=1}^n \xi_i \otimes \eta_i$. Choosing an orthonormal basis in $\mathcal{H}$ for the Hilbert space spanned by the vectors $\xi_1, \dots, \xi_n$, moving around terms yields that x is of the form $x = \sum_{j=1}^m e_j \otimes \eta_j'$ for an orthonormal set of vectors $(e_j)_{j=1}^m$ and some set of vectors $(\eta_j')_{j=1}^m$ in $\mathcal{K}$. We use this in some of the upcoming proofs.

Proposition 1.18 (Tensor product of Hilbert spaces). *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. Then $\mathcal{H} \odot \mathcal{K}$ is an inner product space with respect to the inner product*

$$\left\langle \sum_{i\in I} \xi_i \otimes \eta_i, \sum_{j\in J} \xi_j' \otimes \eta_j' \right\rangle = \sum_{i,j} \langle \xi_i, \xi_j' \rangle \langle \eta_i, \eta_j' \rangle. \tag{1.1}$$

The Hilbert space completion of $\mathcal{H} \odot \mathcal{K}$ is denoted $\mathcal{H} \otimes \mathcal{K}$.

Proof. The question is whether it is even possible to define an inner product as in (1.1). We will show that there exists a sesquilinear form $\langle \cdot, \cdot \rangle \colon \mathcal{H} \odot \mathcal{K} \times \mathcal{H} \odot \mathcal{K} \to \mathbb{C}$ that satisfies (1.1). Let $\mathcal{C} = (\mathcal{H} \odot \mathcal{K})^{*,c}$ be the vector space of conjugate linear functionals on $\mathcal{H} \odot \mathcal{K}$. For $(\xi, \eta) \in \mathcal{H} \times \mathcal{K}$, the maps $x \mapsto \langle \xi, x \rangle$ and $y \mapsto \langle \eta, y \rangle$ are conjugate linear functionals, so by Corollary 1.7, there exists $f_{(\xi,\eta)} \in \mathcal{C}$ such that

$$f_{(\xi,\eta)}(x \otimes y) = \langle \xi, x \rangle \langle \eta, y \rangle, \quad x \in \mathcal{H},\ y \in \mathcal{K}.$$

We now claim that the map $\sigma \colon \mathcal{H} \times \mathcal{K} \to \mathcal{C}$ given by $\varphi(\xi, \eta) = f_{(\xi,\eta)}$ is bilinear. For instance, for $\xi_1, \xi_2, x \in \mathcal{H}$ and $\eta, y \in \mathcal{K}$, note that

$$f_{(\xi_1+\xi_2,\eta)}(x \otimes y) = \langle \xi_1 + \xi_2, x \rangle \langle \eta, y \rangle = \langle \xi_1, x \rangle \langle \eta, y \rangle + \langle \xi_2, x \rangle \langle \eta, y \rangle = f_{(\xi_1,\eta)}(x \otimes y) + f_{(\xi_2,\eta)}(x \otimes y),$$

so the conjugate linear functionals $f_{(\xi_1+\xi_2,\eta)}$ and $f_{(\xi_1,\eta)} + f_{(\xi_2,\eta)}$ agree on all elementary tensors. Hence they must agree on all of $\mathcal{H} \odot \mathcal{K}$, yielding equality. The rest of the properties of a bilinear map follows similarly.

Using the universal property of $\mathcal{H} \odot \mathcal{K}$ there exists a unique linear map $\tilde{\sigma} \colon \mathcal{H} \odot \mathcal{K} \to \mathcal{C}$ such that $\tilde{\sigma}(\xi \otimes \eta) = f_{(\xi,\eta)}$ for all $\xi \in \mathcal{H}$ and $\eta \in \mathcal{K}$. Define $\langle \cdot, \cdot \rangle \colon \mathcal{H} \odot \mathcal{K} \times \mathcal{H} \odot \mathcal{K} \to \mathbb{C}$ by

$$\langle v, w \rangle = \tilde{\sigma}(v)(w), \quad v, w \in \mathcal{H} \odot \mathcal{K}.$$

Then it is easy to see that $\langle \cdot, \cdot \rangle$ is a sesquilinear form satisfying (1.1) and that $\langle v, w \rangle = \overline{\langle w, v \rangle}$ for all $v, w \in \mathcal{H} \odot \mathcal{K}$. Furthermore, if $(e_i)_{i\in I}$ is a finite set of orthonormal vectors in $\mathcal{H}$ and $(\eta_i)_{i\in I}$ is a family of vectors in mK, we see that

$$\left\langle \sum_{i\in I} e_i \otimes \eta_i, \sum_{i\in I} e_i \otimes \eta_i \right\rangle = \sum_{i,j\in I} \langle e_i, e_j \rangle \langle \eta_i, \eta_j \rangle = \sum_{i\in I} \|\eta_i\|^2.$$

This proves that $\langle v, v \rangle \geq 0$ for *all* $v \in \mathcal{H} \odot \mathcal{K}$ by the remark made before the statement of the proposition. Moreover, if the above sum equals 0, then $\eta_i = 0$ for all $i \in I$ and hence $\sum_{i\in I} e_i \otimes \eta_i = 0$, yielding that $\langle \cdot, \cdot \rangle$ is an inner product. □

Thus we obtain our Hilbert space! Note that for $\xi \in \mathcal{H}$ and $\eta \in \mathcal{K}$, the equality

$$\|\xi \otimes \eta\| = \|\xi\|\|\eta\|$$

holds: we express this by saying that the norm is a *cross-norm*.

It is worth noting that for an arbitrary Hilbert space $\mathcal{H}$, the inner product space $\mathcal{H} \odot \mathbb{C}^n$ is already complete with respect to the norm. Since any element $v \in \mathcal{H} \odot \mathbb{C}^n$ has a unique representation of the form

$$v = \sum_{i=1}^{n} \xi_i \otimes e_i, \quad \xi_i \in \mathcal{H}$$

with $(e_i)_{i=1}^n$ denoting the canonical basis of $\mathbb{C}^n$, then the map $\mathcal{H} \odot \mathbb{C}^n \to \mathcal{H}^n$ by

$$\sum_{i=1}^{n} \xi_i \otimes e_i \mapsto (\xi_1, \dots, \xi_n)$$

is an inner product-preserving isomorphism, so $\mathcal{H} \odot \mathcal{C}^n$ is a complete metric space. Hence we obtain

$$\mathcal{H} \otimes \mathbb{C}^n = \mathcal{H} \odot \mathbb{C}^n.$$

We now want to look into whether we can create operators over tensor products from given operators. If $S \in B(\mathcal{H})$ and $T \in B(\mathcal{K})$, then we can consider the tensor product map $S \odot T \colon \mathcal{H} \odot \mathcal{K} \to \mathcal{H} \odot \mathcal{K}$ by Proposition 1.4. It turns out that it can be extended to $\mathcal{H} \otimes \mathcal{K}$ and has some nice additional properties.

Proposition 1.19 (Tensor product operators). *If $S \in B(\mathcal{H})$ and $T \in B(\mathcal{K})$, then there is a unique linear operator $S \otimes T \in B(\mathcal{H} \otimes \mathcal{K})$ such that*

$$S \otimes T(x \otimes y) = Sx \otimes Ty, \quad x \in \mathcal{H},\ y \in \mathcal{K}.$$

Moreover, $\|S \otimes T\| = \|S\|\|T\|$.

Proof. We first consider $S = 1_{\mathcal{H}}$. If $v \in \mathcal{H} \odot \mathcal{K}$, write $v = \sum_{i=1}^n e_i \otimes \eta_i$ for an orthonormal set $(e_i)_{i=1}^n$ in $\mathcal{H}$ and a set of vectors $(\eta_i)_{i=1}^n$ in $\mathcal{K}$ and note that $\|v\|^2 = \sum_{i=1}^n \|\eta_i\|^2$. Then

$$\|1_{\mathcal{H}} \odot T(v)\|^2 = \left\|\sum_{i=1}^{n} e_i \otimes T\eta_i\right\|^2 = \sum_{i,j=1}^{n} \langle e_i, e_j\rangle\langle T\eta_i, T\eta_j\rangle = \sum_{i=1}^{n} \|T\eta_i\|^2 \le \|T\|^2 \sum_{i=1}^{n} \|\eta_i\|^2 = \|T\|^2\|v\|^2.$$

Then by Proposition A.1 there exists a unique bounded operator $1_{\mathcal{H}} \otimes T \in B(\mathcal{H} \otimes \mathcal{K})$ such that $1_{\mathcal{H}} \otimes T|_{\mathcal{H}\odot\mathcal{K}} = 1_{\mathcal{H}} \odot T$ with $\|1_{\mathcal{H}} \otimes T\| \le \|T\|$. In the same manner one obtains an unique extension $S \otimes 1_{\mathcal{K}} \in B(\mathcal{H} \otimes \mathcal{K})$ of $S \odot 1_{\mathcal{K}}$ with $\|S \otimes 1_{\mathcal{K}}\| \le \|S\|$. We now *define*

$$S \otimes T := (S \otimes 1_{\mathcal{K}})(1_{\mathcal{H}} \otimes T).$$

It then follows that $\|S \otimes T\| \le \|S\|\|T\|$ and that $S \otimes T|_{\mathcal{H}\odot\mathcal{K}} = S \odot T$. Hence it is also unique with the elementary tensor property. To prove $\|S \otimes T\| \ge \|S\|\|T\|$, take sequences $(\xi_n)_{n\ge1}$ and $(\eta_n)_{n\ge1}$ in $(\mathcal{H})_1$ and $(\mathcal{K})_1$ respectively such that $\|S\| = \lim_{n\to\infty} \|S\xi_n\|$ and $\|T\| = \lim_{n\to\infty} \|T\eta_n\|$. As $\|\xi_n \otimes \eta_n\| = \|\xi_n\|\|\eta_n\| \le 1$ for all $n \ge 1$ and

$$\|S \otimes T(\xi_n \otimes \eta_n)\| = \|S\xi_n\|\|T\eta_n\| \to \|S\|\|T\|,$$

we obtain $\|S \otimes T\| \ge \|S\|\|T\|$, completing the proof. □

It is easily seen that these tensor product operators actually behave well:

Proposition 1.20 (Tensor product operator calculus). *It holds that*

(i) $(S_1 + S_2) \otimes T = S_1 \otimes T + S_2 \otimes T$ *for* $S_1, S_2 \in B(\mathcal{H})$ *and* $T \in B(\mathcal{K})$,
(ii) $S \otimes (T_1 + T_2) = S \otimes T_1 + S \otimes T_2$ *for* $S \in B(\mathcal{H})$ *and* $T_1, T_2 \in B(\mathcal{K})$,
(iii) $(\lambda S) \otimes T = S \otimes (\lambda T) = \lambda(S \otimes T)$ *for* $S \in B(\mathcal{H})$, $T \in B(\mathcal{K})$ *and* $\lambda \in \mathbb{C}$,
(iv) $(S \otimes T)^* = S^* \otimes T^*$ *for* $S \in B(\mathcal{H})$ *and* $T \in B(\mathcal{K})$, *and*
(v) $(S_1 \otimes S_2)(T_1 \otimes T_2) = (S_1T_1) \otimes (S_2T_2)$ *for* $S_1, S_2 \in B(\mathcal{H})$ *and* $T_1, T_2 \in B(\mathcal{K})$.

(vi) $1_{\mathcal{H}} \otimes 1_{\mathcal{K}} = 1_{\mathcal{H}\otimes\mathcal{K}}$.

Proof. Most of the above follow from the uniqueness criterion in Proposition 1.19; for (iv), note that for $\xi_1, \xi_2 \in \mathcal{H}$ and $\eta_1, \eta_2 \in \mathcal{K}$ we have

$$\begin{aligned}\langle (S\otimes T)^*(\xi_1 \otimes \eta_1), \xi_2 \otimes \eta_2\rangle &= \langle \xi_1 \otimes \eta_1, S\xi_2 \otimes T\eta_2\rangle \\ &= \langle \xi_1, S\xi_2\rangle\langle \eta_1, T\eta_2\rangle \\ &= \langle S^*\xi_1, \xi_2\rangle\langle T^*\eta_1, \eta_2\rangle \\ &= \langle S^*\xi_1 \otimes T^*\eta_1, \xi_2 \otimes \eta_2\rangle \\ &= \langle (S^*\otimes T^*)(\xi_1 \otimes \eta_1), \xi_2 \otimes \eta_2\rangle.\end{aligned}$$

By linearity and continuity, it follows that $(S\otimes T)^*(\xi\otimes\eta) = (S^*\otimes T^*)(\xi\otimes\eta)$ for all $\xi \in \mathcal{H}$ and $\eta \in \mathcal{K}$, so that (iv) follows from uniqueness. □

We remark that we have *-isomorphisms

$$\begin{aligned}B(\mathcal{H}) &\cong B(\mathcal{H}) \otimes \mathbb{C}1_{\mathcal{K}} := \{S \otimes 1_{\mathcal{K}} \mid S \in B(\mathcal{H})\} \subseteq B(\mathcal{H}\otimes\mathcal{K}) \\ B(\mathcal{K}) &\cong \mathbb{C}1_{\mathcal{H}} \otimes B(\mathcal{K}) := \{1_{\mathcal{H}} \otimes T \mid T \in B(\mathcal{K})\} \subseteq B(\mathcal{H}\otimes\mathcal{K})\end{aligned}$$

by considering the maps $S \mapsto S \otimes 1_{\mathcal{K}}$ and $T \mapsto 1_{\mathcal{H}} \otimes T$. Also, $B(\mathcal{H}) \otimes \mathbb{C}1_{\mathcal{K}}$ and $\mathbb{C}1_{\mathcal{H}} \otimes B(\mathcal{K})$ are commuting *-subalgebras in $B(\mathcal{H}\otimes\mathcal{K})$, giving us the following result.

Corollary 1.21. *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. There is a natural injective *-homomorphism*

$$\pi\colon B(\mathcal{H}) \odot B(\mathcal{K}) \to B(\mathcal{H}\otimes\mathcal{K})$$

satisfying $\pi(S\otimes T) = S\otimes T$ for all $S \in B(\mathcal{H})$ and $T \in B(\mathcal{K})$.

Proof. Use Proposition 1.17 on the aforementioned *-isomorphisms to obtain the *-homomorphism. To show that it is injective, we must show that if $\sum_{i=1}^n S_i \otimes T_i \in B(\mathcal{H}\otimes\mathcal{K})$ is zero, then the corresponding sum of tensors $\sum_{i=1}^n S_i \otimes T_i \in B(\mathcal{H}) \odot B(\mathcal{K})$ is zero as well. By using the same method as in Proposition 1.13, we may assume that the operators $S_1, \dots, S_n$ are linearly independent. By letting $\xi_1, \xi_2 \in \mathcal{H}$ and $\eta_1, \eta_2 \in \mathcal{K}$ and noting that

$$\begin{aligned}0 &= \left\langle \left(\sum_{i=1}^n S_i \otimes T_i\right)\xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2\right\rangle \\ &= \sum_{i=1}^n \langle (S_i \otimes T_i)\xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2\rangle \\ &= \sum_{i=1}^n \langle S_i\xi_1, \xi_2\rangle\langle T_i\eta_1, \eta_2\rangle \\ &= \left\langle \sum_{i=1}^n \langle T_i\eta_1, \eta_2\rangle S_i\xi_1, \xi_2\right\rangle,\end{aligned}$$

then since the above holds for all $\xi_2 \in \mathcal{H}$, we must have $\sum_{i=1}^n \langle T_i\eta_1, \eta_2\rangle S_i\xi_1 = 0$ for all $\xi_1 \in \mathcal{H}$. Hence

$$\sum_{i=1}^n \langle T_i\eta_1, \eta_2\rangle S_i = 0,$$

so by linear independence of the S_i, we have $\langle T_i\eta_1, \eta_2\rangle = 0$ for all $\eta_1, \eta_2 \in \mathcal{K}$, so $T_i = 0$ for all $i = 1, \dots, n$. Hence π is injective. □

This then yields the following important corollary.

Corollary 1.22. *Given two representations $\pi_{\mathcal{A}}\colon \mathcal{A} \to B(\mathcal{H})$ and $\pi_{\mathcal{B}}\colon \mathcal{B} \to B(\mathcal{K})$ of *-algebras $\mathcal{A}$ and $\mathcal{B}$, there is an induced representation*

$$\pi_{\mathcal{A}} \odot \pi_{\mathcal{B}}\colon \mathcal{A} \odot \mathcal{B} \to B(\mathcal{H}\otimes\mathcal{K})$$

such that $\pi_{\mathcal{A}} \odot \pi_{\mathcal{B}}(a\otimes b) = \pi_{\mathcal{A}}(a) \otimes \pi_{\mathcal{B}}(b)$ for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$. If $\pi_{\mathcal{A}}$ and $\pi_{\mathcal{B}}$ are faithful, then $\pi_{\mathcal{A}} \odot \pi_{\mathcal{B}}$ is faithful as well.

Proof. Combine Proposition 1.16, Corollary 1.21 and Proposition 1.13. □

In the above case, we have that the image $\pi_{\mathcal{A}} \odot \pi_{\mathcal{B}}(\mathcal{A} \odot \mathcal{B})$ is a *-subalgebra of $B(\mathcal{H} \otimes \mathcal{K})$, consisting of finite sums of elementary tensors $\sum_{i=1}^n \pi_{\mathcal{A}}(a_i) \otimes \pi_{\mathcal{B}}(b_i)$. If $\mathcal{A}$ and $\mathcal{B}$ are *-subalgebras of $B(\mathcal{H})$ and $B(\mathcal{K})$ respectively, we are then able to consider $\mathcal{A} \odot \mathcal{B}$ as a subset of $B(\mathcal{H} \otimes \mathcal{K})$ by means of the above corollary used on the inclusion maps.

And hence ends our pursuit of elementary results for *-algebra tensor products. Nothing particularly surprising of course, but that is precisely what we need: nothing too fancy so far. But fancy it will be.

1.2 Matrix algebras

Any algebraist will likely hit upon a place where he or she will need to construct new algebraic structures from already given ones. It is a song no mathematician ever stops singing, and the preceding section was just another verse in that song: new *-algebra structures were created from given pairs of *-algebras. This section will manage to create a ton of new *C^*-algebras* – not just *-algebras – from a given one. The small setback is that it turns out later that we have already seen them before – they are in fact algebraic tensor products – but since the original new class of *-algebras have norms, these algebraic tensor product acquire norms as well, so time is not exactly wasted. Let us get started right away.

Let $\mathcal{A}$ be a C^*-algebra and let $n \geq 1$ be an integer. We construct the *matrix algebra* $M_n(\mathcal{A})$, a new C^*-algebra derived from $\mathcal{A}$, as follows. First, let $M_n(\mathcal{A})$ be the set of all matrices $(x_{ij})_{i,j=1}^n$ where each entry x_{ij} belongs to $\mathcal{A}$. Addition and scalar multiplication in $M_n(\mathcal{A})$ are then given by the usual pointwise operations, and the product is given by the standard way of multiplying matrices, i.e.

$$\begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix} \begin{pmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{pmatrix} = \begin{pmatrix} z_{11} & z_{12} & \cdots & z_{1n} \\ z_{21} & z_{22} & \cdots & z_{2n} \\ \vdots & \vdots & & \vdots \\ z_{n1} & z_{n2} & \cdots & z_{nn} \end{pmatrix}$$

where

$$z_{ij} = \sum_{k=1}^n x_{ik} y_{kj}, \quad i, j = 1, 2, \ldots, n.$$

It is straightforward to check that $M_n(\mathcal{A})$ becomes an algebra with these operations. An involution is then defined by

$$\begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix}^* = \begin{pmatrix} x_{11}^* & x_{21}^* & \cdots & x_{n1}^* \\ x_{12}^* & x_{22}^* & \cdots & x_{n2}^* \\ \vdots & \vdots & & \vdots \\ x_{1n}^* & x_{2n}^* & \cdots & x_{nn}^* \end{pmatrix},$$

and one can check that the involution axioms indeed hold, so that $M_n(\mathcal{A})$ becomes a *-algebra. We now only need a C^*-algebra norm on $M_n(\mathcal{A})$ to yield a proper new C^*-algebra, and the question is: from where do we get such a norm (a good one would be preferable)?

The answer is that we can use the fact that any C^*-algebra has a faithful representation on a Hilbert space (see page viii). Let $\pi\colon \mathcal{A} \to B(\mathcal{H})$ be one such on a Hilbert space $\mathcal{H}$. We now define a map $\hat{\pi}\colon M_n(\mathcal{A}) \to B(\mathcal{H}^n)$ by

$$\hat{\pi}\begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{pmatrix} = \begin{pmatrix} \pi(x_{11})\xi_1 + \pi(x_{12})\xi_2 + \cdots + \pi(x_{1n})\xi_n \\ \pi(x_{21})\xi_1 + \pi(x_{22})\xi_2 + \cdots + \pi(x_{2n})\xi_n \\ \vdots \\ \pi(x_{n1})\xi_1 + \pi(x_{n2})\xi_2 + \cdots + \pi(x_{nn})\xi_n \end{pmatrix}.$$

This actually gives us a norm, provided we prove the following proposition first.

Proposition 1.23. *$\hat{\pi}$ is a faithful representation of $M_n(\mathcal{A})$ on $\mathcal{H}^n$.*

Proof. It is clear that $\hat{\pi}$ is linear. Furthermore, it is well-defined since for $x = (x_{ij})_{i,j=1}^n \in M_n(\mathcal{A})$ and $\xi = (\xi_1, \dots, \xi_n) \in \mathcal{H}^n$, we have

$$\|\hat{\pi}(x)\xi\|^2 = \sum_{i=1}^n \left\| \sum_{j=1}^n \pi(x_{ij})\xi_j \right\|^2 \leq \sum_{i=1}^n \left(\sum_{j=1}^n \|\pi(x_{ij})\xi_j\| \right)^2 \leq m^2 \sum_{i,j=1}^n \|\xi_j\|^2 = nm^2\|\xi\|^2,$$

where $m = \max_{i,j} \|x_{ij}\|$. Multiplicativity follows from letting $x = (x_{ij})_{i,j=1}^n$, $y = (y_{ij})_{i,j=1}^n \in M_n(\mathcal{A})$, defining the matrix $z = xy = (z_{ij})_{i,j=1}^n$ and noting that

$$\begin{aligned}
\hat{\pi}(x)\hat{\pi}(y)\begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{pmatrix} &= \hat{\pi}(x)\begin{pmatrix} \sum_{j=1}^n \pi(y_{1j})\xi_j \\ \sum_{j=1}^n \pi(y_{2j})\xi_j \\ \vdots \\ \sum_{j=1}^n \pi(y_{nj})\xi_j \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^n \pi(x_{1k})[\sum_{j=1}^n \pi(y_{kj})\xi_j] \\ \sum_{k=1}^n \pi(x_{2k})[\sum_{j=1}^n \pi(y_{kj})\xi_j] \\ \vdots \\ \sum_{k=1}^n \pi(x_{nk})[\sum_{j=1}^n \pi(y_{kj})\xi_j] \end{pmatrix} \\
&= \begin{pmatrix} \sum_{j=1}^n \sum_{k=1}^n \pi(x_{1k}y_{kj})\xi_j \\ \sum_{j=1}^n \sum_{k=1}^n \pi(x_{2k}y_{kj})\xi_j \\ \vdots \\ \sum_{j=1}^n \sum_{k=1}^n \pi(x_{nk}y_{kj})\xi_j \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^n \pi(\sum_{k=1}^n x_{1k}y_{kj})\xi_j \\ \sum_{j=1}^n \pi(\sum_{k=1}^n x_{2k}y_{kj})\xi_j \\ \vdots \\ \sum_{j=1}^n \pi(\sum_{k=1}^n x_{nk}y_{kj})\xi_j \end{pmatrix} \\
&= \begin{pmatrix} \sum_{j=1}^n \pi(z_{1j})\xi_j \\ \sum_{j=1}^n \pi(z_{2j})\xi_j \\ \vdots \\ \sum_{j=1}^n \pi(z_{nj})\xi_j \end{pmatrix} = \hat{\pi}(z)\begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{pmatrix}.
\end{aligned}$$

Note also that for $\xi = (\xi_1, \dots, \xi_n)$ and $(\eta_1, \dots, \eta_n)$ in $\mathcal{H}^n$, we have

$$\langle \hat{\pi}(x^*)\xi, \eta \rangle = \sum_{j=1}^n \left\langle \sum_{k=1}^n \pi(x_{kj}^*)\xi_k, \eta_j \right\rangle = \sum_{j=1}^n \sum_{k=1}^n \langle \pi(x_{kj})^*\xi_k, \eta_j \rangle = \sum_{k=1}^n \left\langle \xi_k, \sum_{j=1}^n \pi(x_{kj})\eta_j \right\rangle = \langle \xi, \hat{\pi}(x)\eta \rangle,$$

so $\hat{\pi}$ preserves adjoints (note here that we explicitly use how the adjoints of $M_n(\mathcal{A})$ are defined). Finally, $\hat{\pi}$ is injective. Assume that $\hat{\pi}(x) = 0$ for some $x = (x_{ij}) \in M_n(\mathcal{A})$, and let $\xi_0 \in \mathcal{H}$. Because $\hat{\pi}(x)(\xi_0, 0, \dots, 0) = 0$ we then obtain $\pi(x_{11})\xi_0 = \cdots = \pi(x_{n1})\xi_0$, and since ξ_0 was arbitrary, we must have $\pi(x_{11}) = \cdots = \pi(x_{n1}) = 0$ whence $x_{11} = \cdots = x_{n1} = 0$ since π is faithful. That the other columns of x consist only of zero vectors is proved in the exact same way. $\square$

It follows that we can define an algebra norm on $M_n(\mathcal{A})$ by $\|x\| := \|\hat{\pi}(x)\|$ for all $x \in M_n(\mathcal{A})$. That $\|x\| = 0$ implies $x = 0$ follows from $\hat{\pi}$ being injective, and additionally

$$\|x^*x\| = \|\hat{\pi}(x)^*\hat{\pi}(x)\| = \|\hat{\pi}(x)\|^2 = \|x\|^2,$$

so that the C^*-axiom is also satisfied. However, we are not entirely done; it still remains to show that $M_n(\mathcal{A})$ is complete under this norm.

Lemma 1.24. *Let $\mathcal{H}$ be a Hilbert space and $T \in B(\mathcal{H})$. Then*

$$\|T\| = \sup\{|\langle T\xi, \eta \rangle| \mid \xi, \eta \in (\mathcal{H})_1\} = \sup\{|\langle T\xi, \eta \rangle| \mid \xi, \eta \in \mathcal{H},\ \|\xi\| = \|\eta\| = 1\}.$$

Proof. Let m_1 and m_2 denote the supremums in the order above; note that $m_2 \leq m_1$. If $T = 0$, the equations clearly hold, so assume that $T \neq 0$. Note first that $|\langle T\xi, \eta \rangle| \leq \|T\|\|\xi\|\|\eta\| = \|T\|$ for all $\xi, \eta \in (\mathcal{H})_1$, so $m_1 \leq \|T\|$. For $0 < \varepsilon < \|T\|$ choose $\xi \in \mathcal{H}$ with $\|\xi\| = 1$ such that $\|T\xi\| + \varepsilon \geq \|T\|$. Then $T\xi \neq 0$, so by letting $\eta = \|T\xi\|^{-1}T\xi$, we obtain $|\langle T\xi, \eta \rangle| + \varepsilon \geq \|T\|$. Hence $\|T\| \leq m_2$, completing the proof. $\square$

Lemma 1.25. *For all $x = (x_{ij})_{i,j=1}^n \in M_n(\mathcal{A})$, it holds that*

$$\max_{i,j=1,\dots,n} \|x_{ij}\| \leq \|x\| \leq \sum_{i,j=1}^n \|x_{ij}\|.$$

Proof. Let α be the above maximum. Then $\alpha = \|x_{ij}\| = \|\pi(x_{ij})\|$ for some i, j, so Lemma 1.24 yields vectors $\xi, \eta \in (\mathcal{H})_1$ such that $\alpha - \varepsilon \leq |\langle \pi(x_{ij})\xi, \eta\rangle|$. If we let $v = (0, \dots, 0, \xi, 0, \dots, 0) \in \mathcal{H}^n$ with ξ at the j'th place and $w \in \mathcal{H}^n$ likewise with η at the i'th place, we first and foremost obtain

$$\hat{\pi}(x)v = \begin{pmatrix} \pi(x_{1j})\xi \\ \pi(x_{2j})\xi \\ \vdots \\ \pi(x_{nj})\xi \end{pmatrix},$$

and hence $\alpha - \varepsilon \leq |\langle \pi(x_{ij})\xi, \eta\rangle| = |\langle \hat{\pi}(x)v, w\rangle| \leq \|\hat{\pi}(x)\|$. Since $\varepsilon > 0$ was arbitrary, we obtain $\|\hat{\pi}(x)\| \geq \alpha$, proving the first inequality.

For the second inequality, we remark that for $\xi = (\xi_1, \dots, \xi_n), \eta = (\eta_1, \dots, \eta_n) \in (\mathcal{H}^n)_1$ we have $\|\xi_i\| \leq \|\xi\|$ as well as $\|\eta_i\| \leq \|\eta\|$ for all i, and hence

$$|\langle \pi(x)\xi, \eta\rangle| = \left|\sum_{i,j=1}^{n} \langle \pi(x_{ij})\xi_j, \eta_i\rangle\right| \leq \sum_{i,j=1}^{n} |\langle \pi(x_{ij})\xi_j, \eta_i\rangle| \leq \sum_{i,j=1}^{n} \|\pi(x_{ij})\| \leq \sum_{i,j=1}^{n} \|x_{ij}\|$$

since π is a *-homomorphism, so Lemma 1.24 completes the proof. □

Proposition 1.26. *$M_n(\mathcal{A})$ is complete under $\|\cdot\|$ and hence becomes a C^*-algebra under this norm.*

Proof. Let $(x^\lambda)_{\lambda \geq 1}$ be a Cauchy sequence of matrices in $M_n(\mathcal{A})$. For any $i, j = 1, \dots, n$, Lemma 1.25 yields

$$\|x^\lambda_{ij} - x^\mu_{ij}\| \leq \max_{i,j=1,\dots,n} \|x^\lambda_{ij} - x^\mu_{ij}\| \leq \|x^\lambda - x^\mu\|$$

for all $\lambda, \mu \geq 1$, so (x^λ_{ij}) is a Cauchy sequence and hence converges to some $x_{ij} \in \mathcal{A}$ for $\lambda \to \infty$ since $\mathcal{A}$ is a Banach space. Let x be the matrix $(x_{ij})_{i,j=1}^n \in M_n(\mathcal{A})$; then Lemma 1.25 tells us that

$$\|x^\lambda - x\| \leq \sum_{i,j=1}^{n} \|x^\lambda_{ij} - x_{ij}\| \to 0$$

for $\lambda \to \infty$, so (x^λ) converges and therefore $(M_n(\mathcal{A}), \|\cdot\|)$ is a Banach space, hence a C^*-algebra. □

Before commenting on what we have found, we prove one small lemma first.

Lemma 1.27. *Let $(\mathcal{B}, \|\cdot\|_1)$ be a C^*-algebra and let $\|\cdot\|_2$ be another C^*-algebra norm on $\mathcal{B}$. Then $\|\cdot\|_1 = \|\cdot\|_2$.*

Proof. Define $\mathcal{B}' = (\mathcal{B}, \|\cdot\|_1)$. Then the map $\pi \colon \mathcal{B} \to \mathcal{B}'$ given by $\pi(x) = x$ is an injective *-homomorphism and hence an isometry, yielding the result. □

Hence the C^*-algebra norm is independent of which representation on some Hilbert space we choose – since the norms derived from any two representations make $M_n(\mathcal{A})$ into a C^*-algebra, they must be equal. This is great news, but there is more: By identifying the elements of $M_n(\mathbb{C})$ with the bounded linear operators on $\mathbb{C}^n$, we obtain a C^*-algebra structure on $M_n(\mathbb{C})$. The canonical basis for $M_n(\mathbb{C})$ is the set of *matrix units* $(e_{ij})_{i,j=1}^n$ where e_{ij} is the matrix with 1 at position (i, j) and 0 everywhere else. Corollary 1.12 now gives us for any C^*-algebra $\mathcal{A}$ that any element $v \in \mathcal{A} \odot M_n(\mathbb{C})$ can be written uniquely in the form

$$v = \sum_{i,j=1}^{n} a_{ij} \otimes e_{ij}, \quad a_{ij} \in \mathcal{A}, \quad i, j = 1, \dots, n.$$

Defining a map $\varphi \colon \mathcal{A} \odot M_n(\mathbb{C}) \to M_n(\mathcal{A})$ by

$$\varphi\left(\sum_{i,j=1}^{n} a_{ij} \otimes e_{ij}\right) = (a_{ij})_{i,j=1}^n,$$

it becomes evident that φ is a *-isomorphism of *-algebras. Indeed, it follows from the unique representation of tensors that the map is a bijection, it is clearly linear, and by using the matrix unit equality $e_{ij}e_{kl} = \delta_{jk}e_{il}$ (δ_{jk} denoting the *Kronecker delta*, i.e. $\delta_{jk} = 1$ if $j = k$ and $\delta_{jk} = 0$ if $j \neq k$) and how we defined the products and involutions in the separate *-algebras, one can easily show that it is also multiplicative and *-preserving.

Corollary 1.28. *For any C^*-algebra $\mathcal{A}$, there exists a unique C^*-algebra norm on the algebraic tensor product $\mathcal{A} \odot M_n(\mathbb{C})$. $\mathcal{A} \odot M_n(\mathbb{C})$ equipped with the norm is denoted by $\mathcal{A} \otimes M_n(\mathbb{C})$.*

Proof. The *-isomorphism $\mathcal{A} \odot M_n(\mathbb{C}) \cong M_n(\mathcal{A})$ induces a C^*-algebra norm on $\mathcal{A} \odot M_n(\mathbb{C})$, and Lemma 1.27 yields uniqueness of this norm. □

Having completed the construction of the matrix algebra, it must be remembered that when one works with matrices containing complex numbers, one does not usually work with only quadratic ones. In the same manner, it is possible for us to define some useful "shrinkings" of general matrix C^*-algebras.

Definition 1.2. Let $n \geq 1$. Then we define three closed subspaces of $M_n(\mathcal{A})$ by

(i) $M_{n,1}(\mathcal{A}) = \{(a_{ij})_{i,j=1}^n \,|\, a_{ij} = 0 \text{ for all } (i,j) \text{ with } i \geq 1,\ j \geq 2\}$;
(ii) $M_{1,n}(\mathcal{A}) = \{(a_{ij})_{i,j=1}^n \,|\, a_{ij} = 0 \text{ for all } (i,j) \text{ with } i \geq 2,\ j \geq 1\}$;
(iii) $M_{1,1}(\mathcal{A}) = \{(a_{ij})_{i,j=1}^n \,|\, a_{ij} = 0 \text{ for all } (i,j) \neq (1,1)\}$.

Hence $x \in M_{n,1}(\mathcal{A})$ and $y \in M_{1,n}(\mathcal{A})$ are of the form

$$x = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ a_2 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ a_n & 0 & \cdots & 0 \end{pmatrix}, \quad y = \begin{pmatrix} b_1 & b_2 & \cdots & b_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

for $a_1, b_1, \ldots, a_n, b_n \in \mathcal{A}$. We will specify elements of $M_{1,n}(\mathcal{A})$ and $M_{n,1}(\mathcal{A})$ by simply writing the n elements of $\mathcal{A}$ that make up the column or row in the obvious ordering, i.e. $x \in M_{n,1}(\mathcal{A})$ is written

$$x = (a_1, \ldots, a_n).$$

It is easy to see that

$$M_{1,n}(\mathcal{A})M_n(\mathcal{A}) \subseteq M_{1,n}(\mathcal{A}), \quad M_n(\mathcal{A})M_{n,1}(\mathcal{A}) \subseteq M_{n,1}(\mathcal{A}), \quad M_{1,n}(\mathcal{A})M_{n,1}(\mathcal{A}) \subseteq M_{1,1}(\mathcal{A}),$$

as well as $M_{n,1}(\mathcal{A})^* = M_{1,n}(\mathcal{A})$ and $M_{1,n}(\mathcal{A})^* = M_{n,1}(\mathcal{A})$. Finally, note that $M_{1,1}(\mathcal{A})$ is a C^*-subalgebra of $M_n(\mathcal{A})$ that is *-isomorphic to $\mathcal{A}$; this allows us to write $M_{1,1}(\mathcal{A}) = \mathcal{A}$.

So far, we have found a way of inducing a norm on the *-algebra tensor product $\mathcal{A} \odot M_n(\mathbb{C})$. We are not entirely ready for the general case, as we would perhaps like to know a bit more about special cases of matrix algebras. As the construction made use of a connection to Hilbert spaces, one might ask what would happen if the C^*-algebra was a subset of $B(\mathcal{H})$ for some $\mathcal{H}$ – or better yet, a von Neumann algebra? The next section clears that up, along with a whole lot of other things...

1.3 Tensor products of von Neumann algebras

As von Neumann algebras require some topological concerns, then if one would take tensor products of two von Neumann algebras, one would not come a very long way. Of course Corollary 1.22 will embed their algebraic tensor product in the bounded linear operators in a Hilbert space, and taking the double commutant of the resulting *-algebra will get us a long way. To prove things about this double commutant, which we will give a name in a short while, one will not come very far if the only thing known about it is that it is a von Neumann algebra. The first couple of pages of this section may therefore seem irrelevant, but they give us the information we need to actually prove some, and I cannot stress this enough, *very* nice properties of this new von Neumann algebra.

We start out slow but don't worry, everything will soon be really complicated.

Proposition 1.29. *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces and let $U\colon \mathcal{H} \to \mathcal{K}$ be an isometric isomorphism. If $\mathscr{S}$ is any subset of $B(\mathcal{H})$, then*

$$(U\mathscr{S}U^{-1})' = U\mathscr{S}'U^{-1}.$$

In particular, if $\mathscr{M} \subseteq B(\mathcal{H})$ is a von Neumann algebra, then $\mathscr{N} = U\mathscr{M}U^{-1} \subseteq B(\mathcal{K})$ is a von Neumann algebra, and the map $\mathscr{M} \to \mathscr{N}$ given by $T \mapsto UTU^{-1}$ is a *-isomorphism between these von Neumann algebras.

Proof. For all $T \in B(\mathcal{K})$ we have

$$\begin{aligned} T \in (U\mathscr{S}U^{-1})' &\Leftrightarrow TUSU^{-1} = USU^{-1}T \text{ for all } S \in \mathscr{S} \\ &\Leftrightarrow U^{-1}TUS = SU^{-1}TU \text{ for all } S \in \mathscr{S} \\ &\Leftrightarrow U^{-1}TU \in \mathscr{S}' \\ &\Leftrightarrow T \in U\mathscr{S}'U^{-1}. \end{aligned}$$

The second part easily follows. □

Definition 1.3. If $U\colon \mathcal{H} \to \mathcal{K}$ is an isometric isomorphism, then a *-isomorphism of von Neumann algebras induced by U as in Proposition 1.29 is called a *spatial isomorphism*.

In the following, let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. Let $(e_i)_{i\in I}$ and $(f_j)_{j\in J}$ be orthonormal bases for $\mathcal{H}$ and $\mathcal{K}$ respectively, and define a map $U_j\colon \mathcal{H} \to \mathcal{H}\otimes\mathcal{K}$ by $U_j(\xi) = \xi \otimes f_j$ for $j \in J$. U_j is then a linear isometry and if $j \neq k$, U_j and U_k have orthogonal ranges. Defining a map $U\colon \mathcal{H}^J \to \mathcal{H}\otimes\mathcal{K}$ by

$$U((\xi_j)_{j\in J}) = \sum_{j\in J} U_j(\xi_j) = \sum_{j\in J} \xi_j \otimes f_j,$$

we see that U is a well-defined linear isometry. This particular map is very important for our understanding of the concepts coming up, so keep the following in mind:

We will keep U and all U_j defined in the same way above for Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ (the latter with an orthonormal basis indexed by a set J) for the remainder of this section.

Proposition 1.30. *Let $(e_i)_{i\in I}$ and $(f_j)_{j\in J}$ be orthonormal bases for $\mathcal{H}$ and $\mathcal{K}$ respectively. Then the set $\{e_i \otimes f_j \mid i \in I,\ j \in J\}$ is an orthonormal basis for $\mathcal{H}\otimes\mathcal{K}$.*

Proof. It is clear that the set is orthonormal. Let $v \in \mathcal{H}\odot\mathcal{K}$. Then $w = \sum_{n=1}^m \xi_n \otimes \eta_n$ for vectors $\xi_1,\dots,\xi_m \in \mathcal{H}$ and $\eta_1,\dots,\eta_m \in \mathcal{K}$. We then have $\xi_n = \sum_{i\in I}\lambda_i^n e_i$ and $\eta_n = \sum_{j\in J}\mu_j^n f_j$ for all $n = 1,\dots,m$ where $(\lambda_i^n)_{i\in I}$ and $(\mu_j^n)_{j\in J}$ are square-summable sequences. For finite subsets $F \subseteq I$ and $G \subseteq J$

$$\left\| \xi_n \otimes \eta_n - \left(\sum_{i\in F}\lambda_i^n e_i\right) \otimes \left(\sum_{j\in G}\mu_j^n f_j\right) \right\| \le \|\xi_n\| \left\| \eta_n - \sum_{j\in G}\mu_j^n f_j \right\| + \left\| \xi_n - \sum_{i\in F}\lambda_i^n e_i \right\| \left(\sum_{j\in J}|\mu_j^n|^2\right)^{1/2},$$

which can be made arbitrarily small by choosing F and G appropriately. Hence $\xi_n\otimes\eta_n$ and therefore w is contained in the closure of the linear span of the $\{e_i\otimes f_j \mid i \in I,\ j\in J\}$. Thus $\mathcal{H}\odot\mathcal{K}$ is contained in this span, so $\mathcal{H}\otimes\mathcal{K}$ is contained in the span as well, so $\{e_i\otimes f_j \mid i\in I,\ j\in J\}$ must be an orthonormal basis of $\mathcal{H}\otimes\mathcal{K}$. □

Note that the image of U contains this basis; we will now show that U is in fact surjective. Take $w \in \mathcal{H}\otimes\mathcal{K}$ and take a family $(\lambda_{ij})_{(i,j)\in I\times J}$ of complex numbers such that

$$\sum_{(i,j)\in I\times J} |\lambda_{ij}|^2 < \infty, \quad w = \sum_{(i,j)\in I\times J} \lambda_{ij}(e_j\otimes f_j).$$

For all finite subsets $F\subseteq I$ and $G \subseteq J$, let $w_{F,G} = \sum_{(i,j)\in F\times G}\lambda_{ij}(e_j\otimes f_j)$ and define

$$\xi_{F,G} = \sum_{j\in G}\sum_{i\in F}\iota_j(\lambda_{ij}e_j),$$

where ι_j is the inclusion of the j'th replica of $\mathcal{H}$ into $\mathcal{H}^J$. Then $U(\xi_{F,G}) = w_{F,G}$. Letting $\mathcal{F}_I$ and $\mathcal{F}_J$ denote the set of finite subsets of I and J respectively, we make $\mathcal{F}_I\times\mathcal{F}_J$ into a directed set by coordinatewise inclusion. Let $\varepsilon > 0$ and take a finite subset $A \subseteq I\times J$ such that

$$\sum_{(i,j)\notin A} |\lambda_{ij}|^2 < \frac{\varepsilon^2}{4}.$$

We can now take a finite subset $F_0 \subseteq I$ and $G_0 \subseteq J$ such that $A \subseteq F_0 \times G_0$. For all $(F, G) \geq (F_0, G_0)$, we now have

$$\|\xi_{F,G} - \xi_{F_0,G_0}\|^2 = \left\| \sum_{j \in G \setminus G_0} \iota_j \left(\sum_{i \in F \setminus F_0} \lambda_{ij} e_j \right) \right\|^2 = \sum_{(i,j) \in (F \times G) \setminus (F_0 \times G_0)} |\lambda_{ij}|^2 \leq \sum_{(i,j) \in A} |\lambda_{ij}|^2 < \frac{\varepsilon^2}{4}.$$

Hence for all (F_1, G_1) and (F_2, G_2) larger than (F_0, G_0), we have $\|\xi_{F_1,G_1} - \xi_{F_2,G_2}\| < \varepsilon$, so $(\xi_{F,G})$ is a Cauchy net and therefore converges to some $\xi \in \mathcal{H}^J$ by [11, Proposition 1.7]; since $\|w_{F,G} - U(\xi)\|$ is equal to $\|\xi_{F,G} - \xi\|$, it follows that $U(\xi) = w$. Therefore U is onto.

In short, we have now proved the following statement:

Proposition 1.31. *If $(f_j)_{j \in J}$ is an orthonormal basis for $\mathcal{K}$, then any $w \in \mathcal{H} \otimes \mathcal{K}$ can be uniquely represented in the form*

$$w = \sum_{j \in J} \xi_j \otimes f_j,$$

where $(\xi_j)_{j \in J}$ is a family of elements in $\mathcal{H}$ satisfying $\sum_{j \in J} \|\xi_j\|^2 < \infty$.

If U_i is defined as above, we then have

$$U_i^* \left(\sum_{j \in J} h_j \otimes f_j \right) = h_i$$

for $i \in J$, so

$$U = \sum_{j \in J} U_j \pi_j, \quad U^{-1} = \sum_{j \in J} \iota_j U_j^*,$$

where ι_j is as before, π_j is the projection from $\mathcal{H}^J$ to the j'th copy and the series are strongly convergent. Note that $U_i^* U_i = 1_{\mathcal{H}}$ for all $i \in J$ and that $U_j^* U_i = 0$ for all $i, j \in J$ with $i \neq j$. For $T \in B(\mathcal{H} \otimes \mathcal{K})$, then by defining bounded operators $T_{ij} = U_j^* T U_i \colon \mathcal{H} \to \mathcal{H}$ for $i, j \in J$ we have

$$U^{-1} T U = \sum_{j \in J} \sum_{i \in J} \iota_j T_{ij} \pi_i.$$

Formulated differently, T defines a matrix of operators in $B(\mathcal{H})$.

Suppose now that an operator $T \in B(\mathcal{H} \otimes \mathcal{K})$ is of the form $T_1 \otimes 1_{\mathcal{K}}$ and let $i, j \in J$. Considering $U_j^* T U_i \colon \mathcal{H} \to \mathcal{H}$ for $i, j \in J$, we find that $U_j^* T U_i = T_1$ for $i = j$ and $U_j^* T U_i = 0$ if $i \neq j$. If we now consider $U_i U_j^* \colon \mathcal{H} \otimes \mathcal{K} \to \mathcal{H} \otimes \mathcal{K}$, $U_i U_j^*$ is a bounded linear operator and for $\xi = \sum_{j \in J} \xi_j \otimes f_j$, we have

$$U_i U_j^* T \xi = U_i T_1 \xi_j = T_1 \xi_j \otimes f_i, \quad T U_i U_j^* \xi = T U_i \xi_j = T_1 \xi_j \otimes f_i.$$

Hence all operators in $B(\mathcal{H} \otimes \mathcal{K})$ of the form $T_1 \otimes 1_{\mathcal{K}}$ commute with all operators of the form $U_i U_j^*$. Conversely, suppose that $T \in B(\mathcal{H} \otimes \mathcal{K})$ commutes with all $U_i U_j^*$ for all $i, j \in J$. Then for all $i, j \in J$ with $i \neq j$, we have $U_i U_j^* T U_j = T U_i U_j^* U_j = T U_i$, so $U_i^* T U_i = U_i^* U_i U_j^* T U_j = U_j^* T U_j$. Moreover, $U_j^* T U_i = U_j^* T U_i (U_i^* U_i) = (U_j^* U_i) U_i^* T U_i = 0$ since $U_j^* U_i = 0$. Define $T_1 = U_j^* T U_j$ for some $j \in J$. Then for any $\xi = (\xi_j)_{j \in J} \in \mathcal{H}^J$, we have

$$U^{-1} T U \xi = \sum_{j \in J} \iota_j T_1 \pi_j \xi = (T_1 \xi_j)_{j \in J} = U^{-1} \left(\sum_{j \in J} T_1 \xi_j \otimes f_j \right) = U^{-1} (T_1 \otimes 1_{\mathcal{K}}) U \xi.$$

Hence $T = T_1 \otimes 1_{\mathcal{K}}$, so we conclude

$$\{U_i U_j^* \mid i, j \in J\}' = B(\mathcal{H}) \otimes \mathbb{C} 1_{\mathcal{K}}.$$

It is now time to define the tensor product in the case of von Neumann algebras.

Definition 1.4. If $\mathscr{M} \subseteq B(\mathcal{H})$ and $\mathscr{N} \subseteq B(\mathcal{K})$ are *-algebras, then the *von Neumann algebra tensor product* $\mathscr{M} \mathbin{\overline{\otimes}} \mathscr{N}$ *of* $\mathscr{M}$ *and* $\mathscr{N}$ is the von Neumann algebra generated by all operators of the form $S \otimes T \in B(\mathcal{H} \otimes \mathcal{K})$ for $S \in \mathscr{M}$ and $T \in \mathscr{N}$. In other words,

$$\mathscr{M} \mathbin{\overline{\otimes}} \mathscr{N} = \{S \otimes T \mid S \in \mathscr{M},\ T \in \mathscr{N}\}''.$$

Assuming additionally that $\mathscr{M}$ and $\mathscr{N}$ are unital, then if we denote the set of finite linear combinations of operators $S \otimes T \in B(\mathcal{H} \otimes \mathcal{K})$ for $S \in \mathscr{M}$ and $T \in \mathscr{N}$ by $\mathscr{M} \odot \mathscr{N}$, then $\mathscr{M} \mathbin{\overline{\otimes}} \mathscr{N}$ is in fact the strong operator closure of $\mathscr{M} \odot \mathscr{N}$ by von Neumann's bicommutant theorem (Theorem 0.9).

Proposition 1.32. *Define* $\pi\colon B(\mathcal{H}) \to B(\mathcal{H} \otimes \mathcal{K})$ *by* $\pi(T) = T \otimes 1_{\mathcal{K}}$. *If* $\mathscr{S} \subseteq B(\mathcal{H})$ *is any subset, then* $\pi(\mathscr{S}'') = \pi(\mathscr{S})''$, *so* π *maps von Neumann algebras to von Neumann algebras.*

Proof. For all $i, j \in J$ then $U_i U_j^*(S \otimes 1_{\mathcal{K}}) = (S \otimes 1_{\mathcal{K}}) U_i U_j^*$ for all $S \in \mathscr{S}$, so $U_i U_j^* \in \pi(\mathscr{S})'$. Hence $\pi(\mathscr{S})'' \subseteq \pi(B(\mathcal{H}))$ by what we found above. Since π is an isomorphism of $B(\mathcal{H})$ onto $\pi(B(\mathcal{H}))$, then for $S \in \pi(\mathscr{S})''$ there exists $S_1 \in B(\mathcal{H})$ such that $\pi(S_1) = S$. For $T \in \mathscr{S}'$ and $\pi(R) \in \pi(\mathscr{S})$, we have $\pi(T)\pi(R) = \pi(R)\pi(T)$, so $\pi(T) \in \pi(\mathscr{S})'$. Therefore

$$\pi(S_1 T) = S\pi(T) = \pi(T)S = \pi(TS_1),$$

so $S_1 T = TS_1$, and hence $S_1 \in \mathscr{S}''$. On the other hand, if $T \in \mathscr{S}''$, then for all $S \in \pi(\mathscr{S})'$, we have $\pi(T)S = S\pi(T)$, so $\pi(\mathscr{S}'') \subseteq \pi(\mathscr{S})''$. □

The above proposition then says that for any von Neumann algebra $\mathscr{M} \subseteq B(\mathcal{H})$,

$$\mathscr{M} \mathbin{\overline{\otimes}} \mathbb{C}1_{\mathcal{K}} = \pi(\mathscr{M})'' = \pi(\mathscr{M}) = \mathscr{M} \otimes \mathbb{C}1_{\mathcal{K}}.$$

Hence $\mathscr{M} \otimes \mathbb{C}1_{\mathcal{K}}$ is a von Neumann algebra acting on $\mathcal{H} \otimes \mathcal{K}$; π is in the above case called an *amplification* of $\mathscr{M}$. For a *-subalgebra $\mathscr{M} \subseteq B(\mathcal{H})$ and a nonempty set J, define the following subsets of $B(\mathcal{H}^J)$ as follows:

- ❖ $\mathscr{M}^J = \bigoplus_{j \in J} \mathscr{M}$.
- ❖ $\Delta_J(\mathscr{M}) = \{T = (T_j)_{j \in J} \in \mathscr{M}^J \mid \text{there exists an } S \in \mathscr{M} \text{ such that } T_j = S \text{ for all } j \in J\}$.
- ❖ $M_J(\mathscr{M}) = \{T \in B(\mathcal{H}^J) \mid \pi_i T \iota_j \in \mathscr{M} \text{ for all } i, j \in J\}$.

We clearly have the inclusion

$$\Delta_J(\mathscr{M}) \subseteq \mathscr{M}^J \subseteq M_J(\mathscr{M}).$$

In the case where J is finite, we might want to exchange J in the symbolisms above for the cardinality of J. However, this creates a glaring problem: if $\mathscr{M}$ is a C^*-algebra, is $M_n(\mathscr{M})$ in the above sense different from $M_n(\mathscr{M})$ in the matrix algebra sense in the previous section? We will get this problem out of the way immediately, in a manner that allows for great flexibility later on. Let $n \geq 1$, let $\mathcal{A}$ be the matrix algebra $M_n(\mathscr{M})$ and let $\mathcal{B} \subseteq B(\mathcal{H}^n)$ be $M_n(\mathscr{M})$ as defined as above. The inclusion $\mathscr{M} \to B(\mathcal{H})$ allows for a representation $\pi\colon \mathcal{A} \to B(\mathcal{H}^n)$ given by

$$\pi \begin{pmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ T_{21} & T_{22} & \cdots & T_{2n} \\ \vdots & \vdots & & \vdots \\ T_{n1} & T_{n2} & \cdots & T_{nn} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{pmatrix} = \begin{pmatrix} T_{11}\xi_1 + T_{12}\xi_2 + \cdots + T_{1n}\xi_n \\ T_{21}\xi_1 + T_{22}\xi_2 + \cdots + T_{2n}\xi_n \\ \vdots \\ T_{n1}\xi_1 + T_{n2}\xi_2 + \cdots + T_{nn}\xi_n \end{pmatrix},$$

proven to be a faithful representation in Proposition 1.23. If $T = (T_{ij})_{i,j=1}^n \in \mathcal{A}$ then we have

$$\pi_i \pi(T) \iota_j(\xi) = \pi_i \begin{pmatrix} T_{1j}\xi \\ T_{2j}\xi \\ \vdots \\ T_{nj}\xi \end{pmatrix} = T_{ij}\xi$$

for all $i, j = 1, \ldots, n$, so that $\pi_i \pi(T) \iota_j = T_{ij} \in \mathscr{M}$, hence proving that $\pi(\mathcal{A}) \subseteq \mathcal{B}$. On the other hand, if $T \in \mathcal{B}$ then by putting $T_{ij} = \pi_i T \iota_j$ for $i, j = 1, \ldots, n$, we have for all $\xi = (\xi_1, \ldots, \xi_n) \in \mathcal{H}^n$ that

$$\hat{\pi}\begin{pmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ T_{21} & T_{22} & \cdots & T_{2n} \\ \vdots & \vdots & & \vdots \\ T_{n1} & T_{n2} & \cdots & T_{nn} \end{pmatrix}\xi = \begin{pmatrix} \pi_1(T\iota_1\xi_1 + \cdots + T\iota_n\xi_n) \\ \pi_2(T\iota_1\xi_1 + \cdots + T\iota_n\xi_n) \\ \vdots \\ \pi_n(T\iota_1\xi_1 + \cdots + T\iota_n\xi_n) \end{pmatrix} = \begin{pmatrix} \pi_1(T\xi) \\ \pi_2(T\xi) \\ \vdots \\ \pi_n(T\xi) \end{pmatrix} = T\xi,$$

yielding $\pi((T_{ij})_{i,j=1}^n) = T$ whence π is surjective. Since π induces the norm on $\mathcal{A}$, π is isometric as well, so $\mathcal{A}$ and $\mathcal{B}$ are isometrically *-isomorphic as normed *-algebras. This enables us to identify the matrix algebra $M_n(\mathscr{M})$ of $\mathscr{M} \subseteq B(\mathcal{H})$ with a *-subalgebra of $B(\mathcal{H}^n)$, and moreover, as $\pi_i T\iota_j \in B(\mathcal{H})$ for all $T \in B(\mathcal{H}^n)$, it follows that

$$M_n(B(\mathcal{H})) \cong B(\mathcal{H}^n).$$

Hence our problem is out of the way by means of a natural *-isomorphism. We will return to this matter at the end of this section.

We now continue from where we left off. For the sake of notation, then for any $T \in \mathscr{M}$ we define the operator $\Delta(T) \in \Delta_J(\mathscr{M})$ by $\Delta(T) = (T_j)_{j\in J}$ where $T_j = T$ for all $j \in J$.

Proposition 1.33. *Let $\mathscr{M}$ be a *-subalgebra of $B(\mathcal{H})$ and $\mathcal{K}$ a Hilbert space with orthonormal basis indexed by a set J. With U defined as above, the following statements hold:*

(i) $\Delta_J(\mathscr{M}) = U^{-1}(\mathscr{M} \otimes \mathbb{C}1_{\mathcal{K}})U$.

(ii) *If $\mathscr{M}$ is a von Neumann algebra, then $M_J(\mathscr{M})' = \Delta_J(\mathscr{M}')$ and $M_J(\mathscr{M}') = \Delta_J(\mathscr{M})'$, implying that $M_J(\mathscr{M})$ and $\Delta_J(\mathscr{M})$ are von Neumann algebras.*

(iii) *For any subset $\mathscr{S} \subseteq B(\mathcal{H}^J)$, we have $(U\mathscr{S}U^{-1})' = U\mathscr{S}'U^{-1}$. If $\mathscr{M}$ is a von Neumann algebra, then $UM_J(\mathscr{M})U^{-1}$ is a von Neumann algebra acting on $\mathcal{H} \otimes \mathcal{K}$.*

Proof. (i) For $T \in \mathscr{M}$, we have

$$U^{-1}(T \otimes 1_{\mathcal{K}})U(\xi_j)_{j\in J} = (T\xi_j)_{j\in J} = \Delta(T)(\xi_j)_{j\in J}.$$

(ii) For all $T \in \mathscr{M}'$ and $S \in M_J(\mathscr{M})$, note that for any $\xi = (\xi_j)_{j\in J}$ we have

$$\pi_i(S\Delta(T)\xi) = \sum_{j\in J} \pi_i S\iota_j(T\xi_j) = \sum_{j\in J} T\pi_i S\iota_j\xi_j = T\pi_i S\xi$$

for all $i \in J$, so

$$S\Delta(T)\xi = \sum_{i\in J} \iota_i(\pi_i(S\Delta(T)\xi)) = \sum_{i\in J} \iota_i(T\pi_i S\xi) = (T\pi_i S\xi)_{i\in J} = \Delta(T)S\xi.$$

Hence $\Delta_J(\mathscr{M}') \subseteq M_J(\mathscr{M})'$. Assuming instead that $T \in \mathscr{M}$ and $S \in M_J(\mathscr{M}')$, we see that the above equations are still true, so we also have $\Delta_J(\mathscr{M})' \subseteq M_J(\mathscr{M}')$. Note that these inclusions do not require $\mathscr{M}$ to be a von Neumann algebra, but at most just a *-subalgebra. This also implies that for *all* *-subalgebras $\mathscr{M}$ of $B(\mathcal{H})$, we have

$$\Delta_J(\mathscr{M}'') \subseteq M_J(\mathscr{M}')' \subseteq \Delta_J(\mathscr{M})''. \tag{1.2}$$

Assume that $S \in M_J(\mathscr{M})'$. Then $S \in (\mathscr{M}^J)' = (\mathscr{M}')^J$ by Corollary 0.11, so $S = (S_j)_{j\in J}$ where $S_j \in \mathscr{M}'$ for all $j \in J$. Define $E_{ij} = \iota_i\pi_j$ for $i,j \in J$. Then $\pi_i E_{ij}\iota_j = 0$ if $i,j \in J$ and $i \neq j$ and $\pi_i E_{ii}\iota_i = 1_{\mathcal{H}}$ for $i \in I$, so $E_{ij} \in M_J(\mathscr{M})$. Hence S commutes with all E_{ij}. For $i,j \in J$ with $i \neq j$, we only need to prove that $S_i = S_j$ in order to prove the first statement, but as

$$S_i\xi = \pi_i S\iota_i\xi = \pi_i S\iota_i(\pi_j\iota_j)\xi = \pi_i SE_{ij}\iota_j\xi = \pi_i E_{ij}S\iota_j\xi = \pi_i(\iota_i\pi_j)S\iota_j\xi = \pi_j S\iota_j\xi = S_j\xi$$

for $\xi \in \mathcal{H}$ we hence obtain $S_i = S_j$ and thus $S \in \Delta_J(\mathscr{M}')$. Finally, for $S \in \Delta_J(\mathscr{M})'$, then for all $T \in \mathscr{M}$, $i,j \in J$ and $\xi \in \mathcal{H}$ we have $\pi_i\Delta(T) = T\pi_i$ and hence

$$\pi_i S\iota_j T\xi = \pi_i S\Delta(T)\iota_j\xi = \pi_i\Delta(T)S\iota_j\xi = T\pi_i S\iota_j\xi,$$

so $\pi_i S\iota_j \in \mathscr{M}'$ for all $i,j \in J$, and hence $S \in M_J(\mathscr{M}')$. As we now obtain

$$\mathscr{M}_J(\mathscr{M})'' = \Delta_J(\mathscr{M}')' = M_J(\mathscr{M}), \quad \Delta_J(\mathscr{M})'' = M_J(\mathscr{M}')' = \Delta_J(\mathscr{M}),$$

we see that $M_J(\mathscr{M})$ and $\Delta_J(\mathscr{M})$ are von Neumann algebras.

(iii) The first part follows from Proposition 1.29, and therefore

$$(UM_J(\mathscr{M})U^{-1})'' = UM_J(\mathscr{M})U^{-1}$$

by (ii), since $\mathscr{M}$ is a von Neumann algebra. Therefore $UM_J(\mathscr{M})U^{-1}$ is a von Neumann algebra acting on $\mathcal{H} \otimes \mathcal{K}$. $\square$

Lemma 1.34. *Let $\mathscr{M}$ be a von Neumann algebra of $B(\mathcal{H})$ and $\mathcal{K}$ a Hilbert space with orthonormal basis indexed by a set J. With U defined as above, then $U^{-1}(\mathscr{M} \mathbin{\overline{\otimes}} B(\mathcal{K}))U = M_J(\mathscr{M})$. If J is finite, then $\mathscr{M} \mathbin{\overline{\otimes}} B(\mathcal{K}) = \mathscr{M} \odot B(\mathcal{K})$.*

Proof. For all $S \in \mathscr{M}$, we have $U^{-1}(S \otimes 1_{\mathcal{K}})U \in \Delta_J(\mathscr{M})$ by Proposition 1.33. By defining $u_{ij} \colon \mathcal{K} \to \mathcal{K}$ by $u_{ij}\xi = \langle \xi, f_j \rangle f_i$, then for all $T \in B(\mathcal{K})$ we have

$$T = \sum_{i \in J} \sum_{j \in J} \langle Tf_i, f_j \rangle u_{ij}$$

where the series in the above expression are strong operator convergent. As $U^{-1}(1_{\mathcal{H}} \otimes u_{ij})U = \iota_i \pi_j$ and $\iota_i \pi_j \in ((\mathscr{M}')^J)' = \mathscr{M}^J$ for all $i, j \in J$ by Corollary 0.11, then because $\mathscr{M}^J$ is strongly closed and the map $S \mapsto U^{-1}SU$ for $S \in B(\mathcal{H} \otimes \mathcal{K})$ is strongly-to-strongly continuous, it follows that $U^{-1}(1_{\mathcal{H}} \otimes T)U \in \mathscr{M}^J$ for all $T \in B(\mathcal{K})$. Hence $U^{-1}(S \otimes T)U \in \mathscr{M}^J$ for all $S \in \mathscr{M}$ and $T \in B(\mathcal{K})$, so

$$U^{-1}(\mathscr{M} \mathbin{\overline{\otimes}} B(\mathcal{K}))U \subseteq \mathscr{M}^J \subseteq M_J(\mathscr{M})$$

by the preceding paragraph, as $\mathscr{M} \mathbin{\overline{\otimes}} B(\mathcal{K})$ is the smallest von Neumann algebra containing all operators $S \otimes T$ and $U\mathscr{M}^J U^{-1}$ is a von Neumann algebra by Proposition 1.33.

If $T = (T_j)_{j \in J} \in M_J(\mathscr{M})$ and let $T_{ij} = \pi_i T \iota_j \in \mathscr{M}$ for all $i, j \in I$. For a finite subset $F \subseteq J$, let $E_F = \sum_{i \in F} \iota_i \pi_i$. As

$$\begin{aligned} U^{-1}\left(\sum_{i,j \in F} (T_{ij} \otimes u_{ij}) \right) U(\xi_k)_{k \in J} &= \sum_{i,j \in F} U^{-1}\left(\sum_{k \in J} T_{ij}\xi_k \otimes u_{ij}(f_k) \right) \\ &= \sum_{i,j \in F} U^{-1}\left(T_{ij}\xi_j \otimes f_i \right) \\ &= \sum_{i,j \in F} \iota_i(T_{ij}\xi_j) \\ &= E_F T E_F (\xi_k)_{k \in J}. \end{aligned}$$

Hence $UE_F T E_F U^{-1} \in \mathscr{M} \mathbin{\overline{\otimes}} B(\mathcal{K})$. Because E_F converges strongly to the identity operator of $\mathcal{H}^J$, then $E_F T E_F \to T$ strongly, and since $S \mapsto USU^{-1}$ is strongly-to-strongly continuous, we find that $UE_F T E_F U^{-1} \to UTU^{-1}$ strongly, and hence $T \in U^{-1}(\mathscr{M} \mathbin{\overline{\otimes}} B(\mathcal{K}))U$, as $\mathscr{M} \mathbin{\overline{\otimes}} B(\mathcal{K})$ is strongly closed. Hence $U^{-1}(\mathscr{M} \mathbin{\overline{\otimes}} B(\mathcal{K}))U = M_J(\mathscr{M})$.

In the case where J is finite, then E_J is the identity operator, so

$$T = U^{-1}\left(\sum_{i,j \in F} (T_{ij} \otimes u_{ij}) \right) U \in U^{-1}(\mathscr{M} \odot B(\mathcal{K}))U$$

with T, T_{ij} and u_{ij} as above. Hence $\mathscr{M} \mathbin{\overline{\otimes}} B(\mathcal{K}) = UM_J(\mathscr{M})U^{-1} \subseteq \mathscr{M} \odot B(\mathcal{K})$, yielding the second statement. $\square$

Proposition 1.35. *For any Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ and any von Neumann algebra $\mathscr{M} \subseteq B(\mathcal{H})$, we have*

(i) $(\mathscr{M} \otimes \mathbb{C}1_{\mathcal{K}})' = \mathscr{M}' \mathbin{\overline{\otimes}} B(\mathcal{K})$.
(ii) $(\mathscr{M} \mathbin{\overline{\otimes}} B(\mathcal{K}))' = \mathscr{M}' \otimes \mathbb{C}1_{\mathcal{K}}$.

(iii) $B(\mathcal{H}) \mathbin{\overline{\otimes}} B(\mathcal{K}) = B(\mathcal{H} \otimes \mathcal{K})$.

Proof. Assuming that $\mathcal{K}$ has an orthonormal basis indexed by J, then if U is defined as above, we have

$$\mathscr{M}' \otimes \mathbb{C}1_{\mathcal{K}} = U\Delta_J(\mathscr{M}')U^{-1} = UM_J(\mathscr{M})'U^{-1} = (UM_J(\mathscr{M})U^{-1})' = (\mathscr{M} \mathbin{\overline{\otimes}} B(\mathcal{K}))'$$

from Proposition 1.33 and Lemma 1.34. Hence we have (ii), from which (i) follows immediately. For (iii), note that Lemma 1.34 yields

$$U^{-1}(B(\mathcal{H}) \mathbin{\overline{\otimes}} B(\mathcal{K}))U = M_J(B(\mathcal{H})) = B(\mathcal{H}^J),$$

so $B(\mathcal{H}) \mathbin{\overline{\otimes}} B(\mathcal{K}) = B(\mathcal{H} \otimes \mathcal{K})$. □

Up until now, we have been working with two Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ and focusing exclusively on the algebraic properties of the tensor product of von Neumann algebras in $B(\mathcal{H})$ and "trivial" subalgebras of $B(\mathcal{K})$. This is not because the results turn false once we shift our focal points and work with non-trivial von Neumann algebras in $B(\mathcal{K})$; the entire section can essentially be "proved again" for these, but we will not do this, for the simple reason that it would be very tedious and completely useless (a stronger variant of useless). What would the use be, indeed: we have nowhere assumed that our von Neumann algebras and Hilbert spaces have had magical properties. Suffice to say that the reader can probably accept the claim that similar properties hold in the other case.

Our last result will concern what happens with strong operator closures of *-subalgebras $\mathscr{M}$ of $B(\mathcal{H})$ when passing to matrix algebras. The statement is quite elementary, so we will skip right ahead to the proof.

Proposition 1.36. *Let $\mathscr{M}$ be a *-subalgebra of $B(\mathcal{H})$ with strong (or weak) operator closure $\mathscr{N}$ such that $\mathscr{N}$ is a von Neumann algebra. Then $M_n(\mathscr{N}) \subseteq B(\mathcal{H}^n)$ is the strong (or weak) operator closure of $M_n(\mathscr{M})$.*

Proof. Since the strong operator closures and weak operator closures of convex sets are equal, it suffices to show the result for the strong operator case. Let $T \in M_2(\mathscr{N})$. Then

$$T = \begin{pmatrix} T_{11} & \cdots & T_{1n} \\ \vdots & \ddots & \vdots \\ T_{n1} & \cdots & T_{nn} \end{pmatrix},$$

where $T_{ij} \in \mathscr{N}$ for all $i, j = 1, \ldots, n$. For all $i, j = 1, \ldots, n$, there exist nets $(T_{ij}^{\alpha})_{\alpha \in A_{ij}}$ of $\mathscr{M}$ such that $T_{ij}^{\alpha} \to T$ strongly. Make

$$\mathbb{A} = \prod_{i,j=1}^{n} A_{ij}$$

into a directed set by defining $(\alpha_{ij})_{i,j=1}^n \leq (\alpha'_{ij})_{i,j=1}^n$ if $\alpha_{ij} \leq \alpha'_{ij}$ for all $i, j = 1, \ldots, n$. We thus obtain a net of matrices

$$\left\{ \begin{pmatrix} T_{11}^{\alpha_{11}} & \cdots & T_{1n}^{\alpha_{1n}} \\ \vdots & \ddots & \vdots \\ T_{n1}^{\alpha_{n1}} & \cdots & T_{nn}^{\alpha_{nn}} \end{pmatrix} \right\}_{(\alpha_{ij})_{i,j=1}^n \in \mathbb{A}}$$

in $M_n(\mathscr{M})$. Let $(\xi_1, \ldots, \xi_n) \in \mathcal{H}^n$ and $\varepsilon > 0$, and pick $(\alpha'_{ij})_{i,j=1}^n \in \mathbb{A}$ such that

$$\|T_{ij}^{\alpha_{ij}}\xi_j - T_{ij}\xi_j\| < \frac{\varepsilon}{n^2}$$

for $(\alpha_{ij})_{i,j=1}^n \geq (\alpha'_{ij})_{i,j=1}^n$. Then

$$\begin{aligned} \left\| \begin{pmatrix} T_{11}^{\alpha_{11}} & \cdots & T_{1n}^{\alpha_{1n}} \\ \vdots & \ddots & \vdots \\ T_{n1}^{\alpha_{n1}} & \cdots & T_{nn}^{\alpha_{nn}} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix} - T \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix} \right\| &= \left\| \begin{pmatrix} (T_{11}^{\alpha_{11}} - T_{11})\xi_1 + \ldots + (T_{1n}^{\alpha_{1n}} - T_{1n})\xi_n \\ \vdots \\ (T_{n1}^{\alpha_{n1}} - T_{n1})\xi_1 + \ldots + (T_{nn}^{\alpha_{nn}} - T_{nn})\xi_n \end{pmatrix} \right\| \\ &\leq \sqrt{\sum_{i,j=1}^{n} \|(T_{ij}^{\alpha_{ij}} - T_{ij})\xi_j\|^2} \\ &\leq \sum_{i,j=1}^{n} \|(T_{ij}^{\alpha_{ij}} - T_{ij})\xi_j\| < \varepsilon. \end{aligned}$$

Hence the net converges to T strongly, proving that $M_2(\mathscr{N})$ is contained in the strong operator closure of $M_2(\mathscr{M})$. Conversely, if T of the above form is contained in the strong operator closure of $M_n(\mathscr{M})$, then there exists a net of matrices

$$T_\alpha = \left\{ \begin{pmatrix} T_{11}^\alpha & \cdots & T_{1n}^\alpha \\ \vdots & \ddots & \vdots \\ T_{n1}^\alpha & \cdots & T_{nn}^\alpha \end{pmatrix} \right\}_{\alpha\in A}$$

with all entries in $\mathscr{M}$, such that T_α converges to T strongly. For $\xi \in \mathcal{H}$ and any $i, j = 1, \ldots, n$, note that

$$\|(T_{ij}^\alpha - T_{ij})\xi\|^2 \le \sum_{i=1}^n \|(T_{ij}^\alpha - T_{ij})\xi\|^2 = \|(T_\alpha - T)\iota_i(\xi)\|^2 \to 0,$$

where ι_i denotes the inclusion of $\mathcal{H}$ into the i'th copy in $\mathcal{H}^n$. Hence $T_{ij}^\alpha \to T_{ij}$ strongly for all i, j, so $T \in M_n(\mathscr{N})$. □

1.4 Tensor products of C^*-algebras and algebraic states

Having quite thoroughly dealt with matrix algebras and tensor products of von Neumann algebras in the previous two sections, we now proceed to the general case of C^*-algebras. In this case, one could just define norms on algebraic tensor products making the completions into C^*-algebras, but we will go slower and take a more subtle and cumulative approach, revealing how these norms actually arise in the process.

We will first look into norms of Banach space tensor products and define one that works wonders.

Definition 1.5. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces. A norm p on $\mathfrak{X} \odot \mathfrak{Y}$ is a *cross-norm* if

$$p(x \otimes y) = \|x\|\|y\|, \quad x \in \mathfrak{X},\ y \in \mathfrak{Y}.$$

The completion of $\mathfrak{X} \odot \mathfrak{Y}$ with respect to the norm p is denoted by $\mathfrak{X} \otimes_p \mathfrak{Y}$.

If $\mathfrak{X}$ and $\mathfrak{Y}$ are Banach spaces, then for any $w = \sum_i x_i \otimes y_i \in \mathfrak{X} \odot \mathfrak{Y}$, note that the map $\mathfrak{X}^* \times \mathfrak{Y}^* \to \mathbb{C}$ given by

$$(\varphi, \psi) \mapsto \sum_{i=1}^n \varphi(x_i)\psi(y_i)$$

is bilinear and hence induces a linear functional $\alpha\colon \mathfrak{X}^* \odot \mathfrak{Y}^* \to \mathbb{C}$ by universality. If $v \in \mathfrak{X} \odot \mathfrak{Y}$ has the form $v = \sum_j x_j' \otimes y_j'$ and $v = w$, then v similarly induces a linear functional $\beta\colon \mathfrak{X}^* \odot \mathfrak{Y}^* \to \mathbb{C}$. For any $\varphi \in \mathfrak{X}^*$ and $\psi \in \mathfrak{Y}^*$, Corollary 1.6 then tells us that

$$\alpha(\varphi \otimes \psi) = \sum_i \varphi(x_i)\psi(y_i) = \sum_i (\varphi \otimes \psi)(x_i \otimes y_i) = \sum_j (\varphi \otimes \psi)(x_j' \otimes y_j') = \sum_j \varphi(x_j')\psi(y_j') = \beta(\varphi \otimes \psi),$$

so that any $w \in \mathfrak{X} \odot \mathfrak{Y}$ gives rise to a *unique* linear functional $\alpha_w\colon \mathfrak{X}^* \odot \mathfrak{Y}^* \to \mathbb{C}$. Moreover,

$$|\alpha_w(\varphi \otimes \psi)| \le \|\varphi\|\|\psi\| \sum_{i=1}^n \|x_i\|\|y_i\|.$$

Definition 1.6. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces. The *projective tensor norm* on $\mathfrak{X} \odot \mathfrak{Y}$ is given by

$$\gamma(w) = \inf \left\{ \sum_{i=1}^n \|x_i\|\|y_i\| \;\middle|\; w = \sum_{i=1}^n x_i \otimes y_i \right\}, \quad w \in \mathfrak{X} \odot \mathfrak{Y}.$$

The completion of $\mathfrak{X} \odot \mathfrak{Y}$ with respect to γ is denoted by $\mathfrak{X} \odot_\gamma \mathfrak{Y}$.

Hence for all $w \in \mathfrak{X} \odot \mathfrak{Y}$, we have $|\alpha_w(\varphi \otimes \psi)| \le \|\varphi\|\|\psi\|\gamma(w)$ for all $\varphi \in \mathfrak{X}$ and $\psi \in \mathfrak{Y}$. If p is a cross-norm on $\mathfrak{X} \odot \mathfrak{Y}$ and $w = \sum_i x_i \otimes y_i \in \mathfrak{X} \odot \mathfrak{Y}$, we have

$$p(w) = p\left(\sum_i x_i \otimes y_i\right) \le \sum_i \|x_i\|\|y_i\|,$$

so by taking infimums over all possible representations of w in $\mathfrak{X} \odot \mathfrak{Y}$, we see that $p(w) \leq \gamma(w)$. Once we prove that the projective tensor norm is a cross-norm, we will then know that it is the largest cross-norm on $\mathfrak{X} \odot \mathfrak{Y}$.

Proposition 1.37. *The projective tensor norm is actually a cross-norm. If $\mathfrak{X}$ and $\mathfrak{Y}$ are Banach *-algebras, then γ is a *-algebra norm, i.e. γ also satisfies*

$$\gamma(vw) \leq \gamma(v)\gamma(w), \quad \gamma(v^*) = \gamma(v), \quad v, w \in \mathfrak{X} \odot \mathfrak{Y}.$$

Proof. For $w \in \mathfrak{X} \odot \mathfrak{Y}$, then by writing $w = \sum_i x_i \otimes y_i$ we have

$$\rho(\lambda w) \leq \sum_i \|\lambda x_i\| \|y_i\| = |\lambda| \sum_i \|x_i\| \|y_i\|$$

for $\lambda \in \mathbb{C}$; since the decomposition of w was arbitrary, we have $\rho(\lambda w) \leq |\lambda\|\rho(w)$. Equality is clear if $\lambda = 0$. If $\lambda \neq 0$, it follows from the above result that $\rho(w) = \rho(\lambda^{-1}(\lambda w)) \leq |\lambda|^{-1}\rho(\lambda w)$, so we finally have $\rho(\lambda w) = |\lambda|\rho(w)$.

For $v, w \in \mathfrak{X} \odot \mathfrak{Y}$, write $v = \sum_{i=i}^n x_i \otimes y_i$ and $w = \sum_{i=n+1}^m x_i \otimes y_i$ for $x_i \in \mathfrak{X}$ and $y_i \in \mathfrak{Y}$, $i = 1, \ldots, m$. Thus

$$\gamma(v + w) \leq \sum_{i=1}^m \|x_i\| \|y_i\| \leq \sum_{i=1}^n \|x_i\| \|y_i\| + \sum_{i=n+1}^m \|x_i\| \|y_i\|,$$

and therefore $\gamma(v + w) \leq \gamma(v) + \gamma(w)$, again by noting that the decomposition was arbitrary. If $w = \sum_i x_i \otimes y_i \in \mathfrak{X} \odot \mathfrak{Y}$ satisfies $\gamma(w) = 0$, then for all $\varphi \in \mathfrak{X}^*$ and $\psi \in \mathfrak{Y}^*$ we have

$$\sum_i \varphi(x_i)\psi(y_i) = \alpha_w(\varphi \otimes \psi) = 0.$$

For any $i = 1, \ldots, n$, the Hahn-Banach theorem [13, Theorem 5.8] yields $\psi_i \in \mathfrak{Y}^*$ such that $\psi_i(y_i) \neq 0$ and $\psi_i(y_j) = 0$ for all $j \neq i$, so that we must have $\varphi(x_i) = 0$ for all $\varphi \in \mathfrak{X}^*$. Therefore $x_i = 0$ for all $i = 1, \ldots, n$, so $w = 0$, proving that γ is a norm. It is in fact a cross-norm: for $x \in \mathfrak{X}$ and $y \in \mathfrak{Y}$, we clearly have $\gamma(w) \leq \|x\|\|y\|$ if $w = x \otimes y$. The Hahn-Banach theorem yields $\varphi \in \mathfrak{X}^*$ and $\psi \in \mathfrak{Y}^*$ such that $\|\varphi\| = \|\psi\| = 1$, $\varphi(x) = \|x\|$ and $\psi(y) = \|y\|$, telling us that

$$\|x\|\|y\| = \varphi(x)\psi(y) = |\alpha_w(\varphi \otimes \psi)| \leq \|\varphi\|\|\psi\||\gamma(w)| = |\gamma(w)|.$$

Thus γ is a cross-norm.

If $\mathfrak{X}$ and $\mathfrak{Y}$ are involutive Banach algebras, then it is clear that $\gamma(w^*) \leq \gamma(w)$ for all $w \in \mathfrak{X} \odot \mathfrak{Y}$, proving that $\gamma(w) = \gamma(w^*)$. Furthermore, note that

$$\gamma(vw) \leq \sum_{i,j} \|x_i x'_j\| \|y_i y'_j\| \leq \sum_{i,j} \|x_i\| \|x'_j\| \|y_i\| \|y'_j\| \leq \left(\sum_i \|x_i\| \|y_i\| \right) \left(\sum_j \|x'_j\| \|y'_j\| \right)$$

if $v = \sum_i x_i \otimes y_i$ and $w = \sum_j x'_j \otimes y'_j$ so that $\gamma(vw) \leq \gamma(v)\gamma(w)$. This completes the proof. □

Some of the properties of the projective tensor product norm in the Banach *-algebra case have a name of their own, so let us get them straight.

Definition 1.7. Let $\mathcal{A}$ and $\mathcal{B}$ be Banach *-algebras. A (semi-)norm p on $\mathcal{A} \odot \mathcal{B}$ is called *submultiplicative* if it satisfies

$$p(xy) \leq p(x)p(y), \quad x, y \in \mathcal{A} \odot \mathcal{B}.$$

A *C^*-norm* (resp. *C^*-seminorm*) p on $\mathcal{A} \odot \mathcal{B}$ is a submultiplicative norm (resp. semi-norm) if

$$p(x^*) = p(x), \quad p(x^*x) = p(x)^2, \quad x \in \mathcal{A} \odot \mathcal{B}.$$

The following result will take us a long way since approximate identities has some nice properties with respect to *certain* linear functionals.

Proposition 1.38. *Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras. If p is a cross-norm on $\mathcal{A} \odot \mathcal{B}$ and $(e_\alpha)_{\alpha\in A}$ and $(f_\beta)_{\beta\in B}$ are bounded approximate identities for $\mathcal{A}$ and $\mathcal{B}$ respectively, then $(e_\alpha \otimes f_\beta)_{\alpha,\beta}$ is a bounded approximate identity for $\mathcal{A} \otimes_p \mathcal{B}$.*

Proof. For $x = \sum_{i=1}^n a_i \otimes b_i \in \mathcal{A} \odot \mathcal{B}$, we have

$$\begin{aligned}
p(x(e_\alpha \otimes f_\beta) - x) &= p\left(\sum_{i=1}^n (a_i e_\alpha \otimes b_i f_\beta - a_i \otimes b_i)\right) \\
&= p\left(\sum_{i=1}^n ((a_i e_\alpha - a_i) \otimes b_i f_\beta + a_i \otimes (b_i - b_i f_\beta))\right) \\
&\le \sum_{i=1}^n (\|a_i e_\alpha - a_i\|\|b_i f_\beta\| + \|a_i\|\|b_i - b_i f_\beta\|) \\
&\le \sum_{i=1}^n (\|a_i e_\alpha - a_i\|\|b_i\| + \|a_i\|\|b_i - b_i f_\beta\|) \to 0.
\end{aligned}$$

Hence $p(x(e_\alpha \otimes f_\beta) - x) \to 0$ for all $x \in \mathcal{A} \odot \mathcal{B}$. Since $p(e_\alpha \otimes f_\beta) = \|e_\alpha\|\|f_\beta\| \le 1$, it is not hard to show by an $\frac{\varepsilon}{3}$ argument that the same holds for arbitrary $x \in \mathcal{A} \otimes_p \mathcal{B}$. ☐

What happens with the completions of $\mathcal{A} \odot \mathcal{B}$ with respect to p if p is a certain type of norm can hence be described as follows:

- ❖ If p is a submultiplicative norm on $\mathcal{A} \odot \mathcal{B}$ that satisfies $p(x^*) = p(x)$, then $\mathcal{A} \otimes_p \mathcal{B}$ is a Banach *-algebra.
- ❖ If p is a cross-norm on $\mathcal{A} \odot \mathcal{B}$, then $\mathcal{A} \otimes_p \mathcal{B}$ has a bounded approximate identity. (This is just Proposition 1.38.)
- ❖ If p is a C^*-norm on $\mathcal{A} \odot \mathcal{B}$, then $\mathcal{A} \otimes_p \mathcal{B}$ is a C^*-algebra.

We recall that if p is a norm on $\mathcal{A} \odot \mathcal{B}$ such that $\mathcal{A} \otimes_p \mathcal{B}$ becomes a Banach *-algebra, then a continuous positive linear functional $\varphi \in \mathcal{A} \otimes_p \mathcal{B}$ is said to be a state if $\|\varphi\| = 1$ and the set of states is denoted by $S(\mathcal{A} \otimes_p \mathcal{B})$ (as it should be). As promised, we will now start uncovering from where C^*-norms on algebraic tensor products arise. Perhaps surprisingly, we will see that the place of birth is in fact the space of states. Of course, algebraic tensor products are not brought into the world with norms, so we will define a new type of state that nonetheless resembles our original definition quite a bit.

Definition 1.8. Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras. A linear functional $\varphi\colon \mathcal{A} \odot \mathcal{B} \to \mathbb{C}$ is *algebraically positive* if $\varphi(x^*x) \ge 0$ for all $x \in \mathcal{A} \odot \mathcal{B}$. If φ is algebraically positive and

$$\|\varphi\|_{\text{alg}} := \sup\{|\varphi(a \otimes b)| \mid a \in (\mathcal{A})_1,\ b \in (\mathcal{B})_1\} = 1,$$

φ is called an *algebraic state*. The set of algebraic states on $\mathcal{A} \odot \mathcal{B}$ is denoted by $S(\mathcal{A} \odot \mathcal{B})$.

The next result resembles one we already know for unital C^*-algebras.

Proposition 1.39. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras. Then an algebraically positive linear functional $\varphi\colon \mathcal{A} \odot \mathcal{B} \to \mathbb{C}$ is an algebraic state if and only if $\varphi(1_\mathcal{A} \otimes 1_\mathcal{B}) = 1$.*

Proof. For $a \in (\mathcal{A})_1$ and $b \in (\mathcal{B})_1$, then $|\varphi(a \otimes b)|^2 \le \varphi(a^*a \otimes b^*b)\varphi(1_\mathcal{A} \otimes 1_\mathcal{B})$ by the Cauchy-Schwarz inequality. As $a^*a \le 1_\mathcal{A}$ and $b^*b \le 1_\mathcal{B}$, then $1_\mathcal{A} - a^*a = x^*x$ and $1_\mathcal{B} - b^*b = y^*y$ for some $x \in \mathcal{A}$ and $y \in \mathcal{B}$, implying

$$\begin{aligned}
0 &\le \varphi((a \otimes y)^*(a \otimes y)) + \varphi((x \otimes 1_\mathcal{B})^*(x \otimes 1_\mathcal{B})) \\
&= \varphi(a^*a \otimes (1_\mathcal{B} - b^*b) + (1_\mathcal{A} - a^*a) \otimes 1_\mathcal{B}) \\
&= \varphi(a^*a \otimes 1_\mathcal{B} - a^*a \otimes b^*b + 1_\mathcal{A} \otimes 1_\mathcal{B} - a^*a \otimes 1_\mathcal{B}) \\
&= \varphi(1_\mathcal{A} \otimes 1_\mathcal{B}) - \varphi(a^*a \otimes b^*b),
\end{aligned}$$

so $|\varphi(a \otimes b)| \le \varphi(1_\mathcal{A} \otimes 1_\mathcal{B})$ for all $a \in (\mathcal{A})_1$ and $b \in (\mathcal{B})_1$. This implies $\|\varphi\|_{\text{alg}} = \varphi(1_\mathcal{A} \otimes 1_\mathcal{B})$, so the result follows. ☐

And so it begins... or does it? As one may recall, all states on C^*-algebras are contractive by definition (which is in itself a very helpful defining property). The definition of algebraic states depends only on the C^*-algebras of which tensor products are taken, but as we can most likely define a lot of different norms on C^*-norms, there is no way of ensuring that all algebraic states are contractive with respect to these. We need to get this hurdle out of the way, and the natural way is the following.

Definition 1.9. Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras. The set of algebraic states on the tensor product $\mathcal{A}\odot\mathcal{B}$ that are contractive with respect to a cross-norm p is denoted by $S_p(\mathcal{A}\odot\mathcal{B})$, i.e. $\varphi\in S_p(\mathcal{A}\odot\mathcal{B})$ if

$$|\varphi(x)|\le p(x),\quad x\in\mathcal{A}\odot\mathcal{B}.$$

Now it begins.

Proposition 1.40. *If $\mathcal{A}\otimes_p\mathcal{B}$ becomes a Banach *-algebra for some cross-norm p on $\mathcal{A}\odot\mathcal{B}$, then*

$$S_p(\mathcal{A}\odot\mathcal{B})=\{\varphi|_{\mathcal{A}\odot\mathcal{B}}\,|\,\varphi\in S(\mathcal{A}\otimes_p\mathcal{B})\}\,.$$

Proof. Assume first that $\varphi\in S(\mathcal{A}\otimes_p\mathcal{B})$ and let χ be its restriction to $\mathcal{A}\odot\mathcal{B}$. Then χ is algebraically positive, contractive with respect to p and

$$|\chi(a\otimes b)|\le p(a\otimes b)=\|a\|\|b\|,\quad a\in\mathcal{A},\ b\in\mathcal{B}.$$

If $(e_\alpha)_{\alpha\in A}$ and $(f_\beta)_{\beta\in B}$ are bounded approximate identities for $\mathcal{A}$ and $\mathcal{B}$ respectively, then $(e_\alpha\otimes f_\beta)_{\alpha,\beta}$ is a bounded approximate identity for $\mathcal{A}\otimes_p\mathcal{B}$ by Proposition 1.38. Proposition 0.3 now yields

$$\chi(e_\alpha\otimes f_\beta)=\varphi(e_\alpha\otimes f_\beta)\to\|\varphi\|=1.$$

Hence $\|\chi\|_{\mathrm{alg}}=1$, so $\chi\in S_p(\mathcal{A}\odot\mathcal{B})$.

Conversely, if $\chi\in S_p(\mathcal{A}\odot\mathcal{B})$, then χ extends to a contractive linear functional $\tilde{\chi}$ on $\mathcal{A}\otimes_p\mathcal{B}$ by Proposition A.1. Moreover, if $x\in\mathcal{A}\otimes_p\mathcal{B}$ and $p(x_n-x)\to0$ for some sequence $(x_n)_{n\ge1}$ in $\mathcal{A}\odot\mathcal{B}$, then $p(x_n^*x_n-x^*x)\to0$, implying that $\tilde{\chi}$ is positive since we then have

$$\tilde{\xi}(x^*x)=\lim_{n\to\infty}\xi(x_n^*x_n)\ge0.$$

As $\|\chi\|_{\mathrm{alg}}=1$ we see that $\|\tilde{\chi}\|\ge1$, but since $\tilde{\chi}(e_\alpha\otimes f_\beta)\to\|\tilde{\chi}\|$ as before, we also have $\|\tilde{\chi}\|\le1$ and hence $\tilde{\chi}\in S(\mathcal{A}\otimes_p\mathcal{B})$. Moreover, if χ is the restriction of $\varphi\in S(\mathcal{A}\otimes_p\mathcal{B})$ to $\mathcal{A}\odot\mathcal{B}$ then $\tilde{\chi}=\varphi$ by the uniqueness part of Proposition A.1, yielding a bijective correspondence between the two spaces. ☐

The bijective correspondence of the last proof yields a way of identifying $S_p(\mathcal{A}\odot\mathcal{B})$ and $S(\mathcal{A}\otimes_p\mathcal{B})$. The weak* topology on $S(\mathcal{A}\otimes_p\mathcal{B})$ as a subspace of $(\mathcal{A}\otimes_p\mathcal{B})^*$ hence induces a topology on $S_p(\mathcal{A}\odot\mathcal{B})$ by restriction to $\mathcal{A}\odot\mathcal{B}$ which we shall call the *weak* topology on $S_p(\mathcal{A}\odot\mathcal{B})$*.

Now note that for $\varphi\in S(\mathcal{A}\odot\mathcal{B})$ and $x=\sum_{i=1}^n a_i\otimes b_i\in\mathcal{A}\odot\mathcal{B}$, we have

$$|\varphi(x)|\le\sum_{i=1}^n|\varphi(a_i\otimes b_i)|\le\sum_{i=1}^n\|a_i\|\|b_i\|$$

and hence $\varphi\in S_\gamma(\mathcal{A}\odot\mathcal{B})$ where γ is the projective tensor norm on $\mathcal{A}\odot\mathcal{B}$. Noting that γ indeed satisfies the conditions of Proposition 1.40, we have

$$S(\mathcal{A}\odot\mathcal{B})=\{\varphi|_{\mathcal{A}\odot\mathcal{B}}\,|\,\varphi\in S(\mathcal{A}\otimes_\gamma\mathcal{B})\}.$$

Let $\varphi\in S(\mathcal{A}\odot\mathcal{B})$. As $\mathcal{A}\otimes_\gamma\mathcal{B}$ is a Banach *-algebra, we obtain a GNS triple $(\mathcal{H}_\varphi,\pi_\varphi,\xi_\varphi)$ associated with φ (or rather, the extension of φ to a state on $\mathcal{A}\otimes_\gamma\mathcal{B}$) such that for all $x\in\mathcal{A}\odot\mathcal{B}$, we have

$$\varphi(x)=\langle\pi_\varphi(x)\xi_\varphi,\xi_\varphi\rangle.$$

Therefore

$$|\varphi(x)|\le\|\pi_\varphi(x)\|\le\gamma(x),\quad x\in\mathcal{A}\odot\mathcal{B}. \tag{1.3}$$

To summarize, all this means that any $\varphi\in S(\mathcal{A}\odot\mathcal{B})$ admits a GNS triple $(\mathcal{H}_\varphi,\pi_\varphi,\xi_\varphi)$. As $\mathcal{A}\odot\mathcal{B}$ is dense in $\mathcal{A}\otimes_\gamma\mathcal{B}$, it follows that $\pi_\varphi(\mathcal{A}\odot\mathcal{B})\xi_\varphi$ is dense in $\mathcal{H}_\varphi$.

Once we start working with C^*-norms, we obtain the following very surprising result, for which the proof is nonetheless very easy.

Proposition 1.41. *Let p be a C^*-norm on $\mathcal{A}\odot\mathcal{B}$. Then*

$$p(x) = \sup\{\varphi(x^*x)^{1/2} \,|\, \varphi \in S_p(\mathcal{A}\odot\mathcal{B})\}, \quad x \in \mathcal{A}\odot\mathcal{B}.$$

Proof. For any $\varphi \in S_p(\mathcal{A}\odot\mathcal{B})$, Proposition 1.40 and the GNS construction on $\mathcal{A}\otimes_p\mathcal{B}$ together yield a GNS triple $(\mathcal{H}_\varphi, \pi_\varphi, \xi_\varphi)$ where ξ_φ is a unit vector satisfying $\varphi(x) = \langle \pi_\varphi(x)\xi_\varphi, \xi_\varphi\rangle$ for all $x \in \mathcal{A}\odot\mathcal{B}$. In particular, $\varphi(x^*x) \le \|\pi_\varphi(x)\|^2 \le p(x)^2$ for all $x \in \mathcal{A}\odot\mathcal{B}$. For the converse, note that there exists a faithful representation $\pi\colon \mathcal{A}\otimes_p\mathcal{B} \to B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. For any unit vector $\xi \in \mathcal{H}$, the functional $x \mapsto \langle \pi(x)\xi, \xi\rangle$ is a state on $\mathcal{A}\otimes_p\mathcal{B}$. Therefore Proposition 1.40 yields

$$\sup\{\varphi(x^*x)^{1/2} \,|\, \varphi \in S_p(\mathcal{A}\odot\mathcal{B})\} \ge \sup\{\|\pi(x)\xi\| \,|\, \xi \in (\mathcal{H})_1\} = \|\pi(x)\| = p(x)$$

for all $x \in \mathcal{A}\odot\mathcal{B}$ and hence the proof is complete. □

The aim is now to find a general method for constructing a C^*-norm on the tensor product. If $\varphi \in S(\mathcal{A}\odot\mathcal{B})$, then by defining $p_\varphi(x) = \|\pi_\varphi(x)\|$ for $x \in \mathcal{A}\odot\mathcal{B}$, where π_φ is the GNS representation associated to φ as found above, we clearly obtain a C^*-seminorm. If $\Gamma \subseteq S(\mathcal{A}\odot\mathcal{B})$, we define

$$p_\Gamma(x) = \sup_{\varphi\in\Gamma} \|\pi_\varphi(x)\|, \quad x \in \mathcal{A}\odot\mathcal{B}.$$

As $\|\pi_\varphi(x)\| \le \gamma(x)$ for all $\varphi \in \Gamma$ and $x \in \mathcal{A}\odot\mathcal{B}$, p_Γ is a well-defined C^*-seminorm.

If p_Γ is in fact a norm, we say that Γ is *separating* and denote the completion of $\mathcal{A}\odot\mathcal{B}$ by $\mathcal{A}\otimes_\Gamma\mathcal{B}$. The set of algebraic states on $\mathcal{A}\odot\mathcal{B}$ that are contractive with respect to p_Γ is denoted by $S_\Gamma(\mathcal{A}\odot\mathcal{B})$. If $\varphi \in \Gamma$, then

$$|\varphi(x)| \le \|\pi_\varphi(x)\| \le p_\Gamma(x), \quad x \in \mathcal{A}\odot\mathcal{B},$$

so $\Gamma \subseteq S_\Gamma(\mathcal{A}\odot\mathcal{B})$. In fact, any C^*-norm p on $\mathcal{A}\odot\mathcal{B}$ can be obtained in this way; since $\mathcal{A}\otimes_p\mathcal{B}$ is a C^*-algebra, then we know that

$$\pi = \bigoplus_{\varphi\in S(\mathcal{A}\otimes_p\mathcal{B})} \pi_\varphi$$

is a faithful representation of $\mathcal{A}\otimes_p\mathcal{B}$ (see page viii). Faithfulness of π then yields

$$p(x) = \|\pi(x)\| = \sup_{\varphi\in S(\mathcal{A}\otimes_p\mathcal{B})} \|\pi_\varphi(x)\|, \quad x \in \mathcal{A}\odot\mathcal{B},$$

so by letting $\Gamma = S_p(\mathcal{A}\odot\mathcal{B})$, we obtain $p = p_\Gamma$.

If $\mathcal{A}$ and $\mathcal{B}$ are unital and Γ is separating we have the following result, the proof of which needs a result from Appendix A.

Lemma 1.42. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras and let $\Gamma \subseteq S(\mathcal{A}\odot\mathcal{B})$ be convex and separating. Let*

$$\Gamma' = \{\varphi \in S(\mathcal{A}\otimes_\Gamma\mathcal{B}) \,|\, \varphi|_{\mathcal{A}\odot\mathcal{B}} \in \Gamma\}.$$

Assume furthermore that it holds for all $\varphi \in \Gamma$ and $y \in \mathcal{A}\odot\mathcal{B}$ that there exists $\psi \in \Gamma$ such that

$$\varphi(y^*xy) = \varphi(y^*y)\psi(x), \quad x \in \mathcal{A}\odot\mathcal{B}.$$

Then Γ' is weak-dense in $S(\mathcal{A}\otimes_\Gamma\mathcal{B})$ implying that Γ is weak*-dense in $S_\Gamma(\mathcal{A}\odot\mathcal{B})$, and*

$$p_\Gamma(x) = \sup\{\varphi(x^*x)^{1/2} \,|\, \varphi \in \Gamma\}.$$

Proof. If $\varphi \in \Gamma'$ and $y \in \mathcal{A}\odot\mathcal{B}$, then there exists $\psi \in \Gamma$ such that $\varphi(y^*xy) = \varphi(y^*y)\psi(x)$ for all $x \in \mathcal{A}\odot\mathcal{B}$. ψ is a restriction of a state $\psi' \in \Gamma'$ on $\mathcal{A}\otimes_\Gamma\mathcal{B}$ by Proposition 1.40, in turn yielding $\varphi(y^*xy) = \varphi(y^*y)\psi'(x)$ for $x \in \mathcal{A}\otimes_\Gamma\mathcal{B}$ by continuity.

Let $x \in (\mathcal{A}\otimes_\Gamma\mathcal{B})_{\mathrm{sa}}$ and assume that $\varphi(x) \ge 0$ for all $\varphi \in \Gamma'$. Then on the grounds of what we just found, then for any $\varphi \in \Gamma'$ and $y \in \mathcal{A}\odot\mathcal{B}$, there is a state $\psi \in \Gamma'$ such that

$$\langle \pi_\varphi(x)\pi_\varphi(y)\xi_\varphi, \pi_\varphi(y)\xi_\varphi\rangle = \varphi(y^*xy) = \varphi(y^*y)\psi(x).$$

As $\varphi(y^*y)\psi(x) \geq 0$, we see that

$$\langle \pi_\varphi(x)\pi_\varphi(y)\xi_\varphi, \pi_\varphi(y)\xi_\varphi \rangle \geq 0$$

for all $\varphi \in \Gamma'$ and $y \in \mathcal{A} \odot \mathcal{B}$. As $\mathcal{A} \odot \mathcal{B}$ is dense in $\mathcal{A} \otimes_\Gamma \mathcal{B}$, it follows that $\pi_\varphi(x)$ is positive for all $\varphi \in \Gamma'$. Hence if $\pi = \bigoplus_{\varphi \in S(\mathcal{A} \otimes_\Gamma \mathcal{B})} \pi_\varphi$, we clearly have $\pi(x) \geq 0$, so since π is faithful, we see that $x \geq 0$. But now Lemma A.4 tells us that Γ' is weak*-dense in $S(\mathcal{A} \otimes_\Gamma \mathcal{B})$ from which the second density statement follows. Finally, Proposition 1.41 yields

$$p_\Gamma(x) = \sup\{\varphi(x^*x)^{1/2} \mid \varphi \in S_\Gamma(\mathcal{A} \odot \mathcal{B})\} = \sup\{\varphi(x^*x)^{1/2} \mid \varphi \in \Gamma\},$$

because of weak* density. □

So far we have worked quite a lot with C^*-norms but have not defined any. The Rolling Stones and Beatles of C^*-norms on algebraic tensor products of C^*-algebras are the following.

Definition 1.10 (Maximal norm)**.** Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras. The *maximal norm* on $\mathcal{A} \odot \mathcal{B}$ is defined by

$$\|x\|_{\max} = \sup\{\|\pi(x)\| \mid \pi \text{ is a representation of } \mathcal{A} \odot \mathcal{B}\}.$$

We let $\mathcal{A} \otimes_{\max} \mathcal{B}$ denote the completion of $\mathcal{A} \odot \mathcal{B}$ with respect to $\|\cdot\|_{\max}$.

Definition 1.11 (Minimal norm)**.** Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras. Then the *minimal norm* on $\mathcal{A} \odot \mathcal{B}$ is given by

$$\left\| \sum_i a_i \otimes b_i \right\|_{\min} = \left\| \sum_i \pi(a_i) \otimes \rho(b_i) \right\|_{B(\mathcal{H} \otimes \mathcal{K})}$$

where $\pi\colon \mathcal{A} \to B(\mathcal{H})$ and $\rho\colon \mathcal{B} \to B(\mathcal{K})$ are *faithful* representations. We let $\mathcal{A} \otimes_{\min} \mathcal{B}$ denote the completion of $\mathcal{A} \odot \mathcal{B}$ with respect to $\|\cdot\|_{\min}$.

It takes a bit of work to prove that the maximal and minimal norm are actually C^*-norms and that the minimal norm is independent of the choice of representations, and we will not embark on this journey here; for comments on this, see [4]. However, we will state the most important facts about the two norms (and others) in the following theorem, to be used fervently later on.

Theorem 1.43. *Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras. Then the following statements hold:*

(i) *$\|\cdot\|_{\min}$ is independent of the choice of representations of $\mathcal{A}$ and $\mathcal{B}$.*
(ii) *$\|\cdot\|_{\max}$ and $\|\cdot\|_{\min}$ are C^*-norms on $\mathcal{A} \odot \mathcal{B}$.*
(iii) *If p is a C^*-norm on $\mathcal{A} \odot \mathcal{B}$, then*

$$\|\cdot\|_{\min} \leq p \leq \|\cdot\|_{\max}.$$

(The first inequality is known as Takesaki's theorem.)
(iv) *If p is a C^*-norm on $\mathcal{A} \odot \mathcal{B}$, then it is automatically a cross-norm.*
(v) *If $\mathcal{C}$ is a C^*-algebra and $\mathcal{A} \subseteq \mathcal{C}$, then there is a natural isometric inclusion $\mathcal{A} \otimes_{\min} \mathcal{B} \to \mathcal{C} \otimes_{\min} \mathcal{B}$.*
(vi)

Proof. See [4, Sections 3.3 and 3.4] and [4, Proposition 3.6.1]. □

For any two C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ and $\varphi \in S(\mathcal{A} \odot \mathcal{B})$, then equation (1.3) immediately yields

$$|\varphi(x)| \leq \|x\|_{\max}, \quad x \in \mathcal{A} \odot \mathcal{B},$$

so we can conclude $S(\mathcal{A} \odot \mathcal{B}) = S_{\max}(\mathcal{A} \odot \mathcal{B})$.

The aim of the last part of this section is to prove that the minimal norm can be expressed differently, by defining a new C^*-norm which turns out to be equal to the minimal norm.

Lemma 1.44. *Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras. If $\varphi \in S(\mathcal{A})$ and $\psi \in S(\mathcal{B})$, then $\varphi \odot \psi \in S(\mathcal{A} \odot \mathcal{B})$.*

Proof. Let $(\mathcal{H}_\varphi, \pi_\varphi, \xi_\varphi)$ and $(\mathcal{H}_\psi, \pi_\psi, \xi_\psi)$ be GNS representations of $\mathcal{A}$ and $\mathcal{B}$ corresponding to φ and ψ respectively. For all $a \in \mathcal{A}$ and $b \in \mathcal{B}$, we obtain a bounded linear operator $\pi_\varphi(a) \otimes \pi_\psi(b) \in B(\mathcal{H}_\varphi \otimes \mathcal{H}_\psi)$ by Corollary 1.22 satisfying

$$\pi_\varphi(a) \otimes \pi_\psi(b)(\xi_\varphi \otimes \xi_\psi) = \pi_\varphi(a)\xi_\varphi \otimes \pi_\psi(b)\xi_\psi.$$

For $x = \sum_{i=1}^n a_i \otimes b_i \in \mathcal{A} \odot \mathcal{B}$ with $a_i \in \mathcal{A}$ and $b_i \in \mathcal{B}$ for $i = 1, \dots, n$, we have

$$\begin{aligned}
(\varphi \odot \psi)(x) &= \sum_{i=1}^n \varphi(a_i)\psi(b_i) \\
&= \sum_{i=1}^n \langle \pi_\varphi(a_i)\xi_\varphi, \xi_\varphi \rangle_{\mathcal{H}_\varphi} \langle \pi_\psi(b_i)\xi_\psi, \xi_\psi \rangle_{\mathcal{H}_\psi} \\
&= \sum_{i=1}^n \langle \pi_\varphi(a_i)\xi_\varphi \otimes \pi_\psi(b_i)\xi_\psi, \xi_\varphi \otimes \xi_\psi \rangle_{\mathcal{H}_\varphi \otimes \mathcal{H}_\psi} \\
&= \sum_{i=1}^n \langle \pi_\varphi(a_i) \otimes \pi_\psi(b_i)(\xi_\varphi \otimes \xi_\psi), \xi_\varphi \otimes \xi_\psi \rangle_{\mathcal{H}_\varphi \otimes \mathcal{H}_\psi} \\
&= \sum_{i=1}^n \langle \pi_\varphi \odot \pi_\psi(a_i \otimes b_i)(\xi_\varphi \otimes \xi_\psi), \xi_\varphi \otimes \xi_\psi \rangle_{\mathcal{H}_\varphi \otimes \mathcal{H}_\psi} \\
&= \langle \pi_\varphi \odot \pi_\psi(x), \xi_\varphi \otimes \xi_\psi \rangle_{\mathcal{H}_\varphi \otimes \mathcal{H}_\psi}.
\end{aligned}$$

Hence if $x = y^*y$ for some $y \in \mathcal{A} \odot \mathcal{B}$, we have

$$(\varphi \odot \psi)(x) = \|(\pi_\varphi \odot \pi_\psi)(y)\|^2_{\mathcal{H}_\varphi \otimes \mathcal{H}_\psi} \geq 0,$$

yielding positivity. Moreover, it is clear that $\|\varphi \odot \psi\|_{\text{alg}} \leq 1$. Taking sequences $(a_n)_{n\geq 1}$ in $\mathcal{A}$ and $(b_n)_{n\geq 1}$ in $\mathcal{B}$ such that $|\varphi(a_n)| \to \|\varphi\| = 1$ and $|\psi(b_n)| \to \|\psi\| = 1$, we then have $|\varphi \odot \psi(a_n \otimes b_n)| \to 1$, so $\|\varphi \odot \psi\|_{\text{alg}} \geq 1$, proving that $\varphi \odot \psi \in S(\mathcal{A} \odot \mathcal{B})$. □

The above result allows for the following definition:

Definition 1.12. For C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, we define a norm

$$\|x\|_\kappa = \sup\{\|\pi_{\varphi\odot\psi}(x)\| \mid \varphi \in S(\mathcal{A}),\ \psi \in S(\mathcal{B})\}, \quad x \in \mathcal{A} \odot \mathcal{B}.$$

The completion of $\mathcal{A} \odot \mathcal{B}$ with respect to $\|\cdot\|_\kappa$ is denoted by $\mathcal{A} \otimes_\kappa \mathcal{B}$.

We may of course inquire whether the above definition really yields a norm, and the most essential things to know about $\|\cdot\|_\kappa$ are the following, which will be stated without proof. Note first that we already know that $\|\cdot\|_\kappa$ is a C^*-seminorm since it is equal to p_Γ for

$$\Gamma = S(\mathcal{A}) \otimes S(\mathcal{B}) = \{\varphi \odot \psi \mid \varphi \in S(\mathcal{A}),\ \psi \in S(\mathcal{B})\}.$$

The proof itself requires knowledge about the enveloping von Neumann algebra which we will learn about in Chapter 2.

Proposition 1.45. *$\|\cdot\|_\kappa$ is a crossnorm and all linear functionals in $\mathcal{A}^* \odot \mathcal{B}^*$, i.e. the linear span of all linear functionals on $\mathcal{A} \odot \mathcal{B}$ obtained by Corollary 1.6, are bounded with respect to $\|\cdot\|_\kappa$.*

Proof. See [28, Proposition 1.23]. □

Therefore $\|\cdot\|_\kappa$ is a C^*-norm. We denote the set of states on $\mathcal{A} \odot \mathcal{B}$ that are contractive with respect to $\|\cdot\|_\kappa$ by $S_\kappa(\mathcal{A} \odot \mathcal{B})$.

Definition 1.13. For C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, we define $\mathbb{M}(\mathcal{A}, \mathcal{B}) = \mathcal{A}^* \odot \mathcal{B}^* \cap S(\mathcal{A} \odot \mathcal{B})$. When the C^*-algebras are clear from the context, we will just write $\mathbb{M} = \mathbb{M}(\mathcal{A}, \mathcal{B})$.

It immediately follows from Lemma 1.44 and Proposition 1.45 that

$$S(\mathcal{A}) \otimes S(\mathcal{B}) \subseteq \mathbb{M} \subseteq S_\kappa(\mathcal{A} \odot \mathcal{B}).$$

Hence

$$\|x\|_\kappa \leq p_{\mathbb{M}}(x) = \sup_{\varphi \in \mathbb{M}} \|\pi_\varphi(x)\| \leq \sup_{\varphi \in S(\mathcal{A} \otimes_\kappa \mathcal{B})} \|\pi_\varphi(x)\| = \|x\|_\kappa, \quad x \in \mathcal{A} \odot \mathcal{B},$$

so $\|\cdot\|_\kappa = p_{\mathbb{M}}$.Assume now that $\mathcal{A}$ and $\mathcal{B}$ are unital. Then $S(\mathcal{A} \odot \mathcal{B})$ is convex by Lemma 1.39. For any $\varphi \in \mathbb{M}$ and $y \in \mathcal{A} \odot \mathcal{B}$ such that $\varphi(y^*y) \neq 0$, write $\varphi = \sum_{i=1}^n \omega_i \odot \chi_i$ for $\omega_1, \ldots, \omega_n \in \mathcal{A}^*$ and $\chi_1, \ldots, \chi_n \in \mathcal{B}^*$ and $y = \sum_{j=1}^m x_i \otimes y_i$. For $i = 1, \ldots, n$, $j, k = 1, \ldots, m$, define

$$\omega'_{ijk}(a) = \varphi(y^*y)^{-1/2}\omega_i(x_j^* a x_k), \quad \chi'_{ijk}(b) = \varphi(y^*y)^{-1/2}\chi_i(y_j^* b y_k), \quad a \in \mathcal{A},\ b \in \mathcal{B}.$$

Clearly these functionals are linear and continuous. Then

$$\sum_{i=1}^n \sum_{j,k=1}^m \omega'_{ijk}(a)\chi'_{ijk}(b) = \sum_{i=1}^n \frac{\sum_{j,k=1}^n \omega_i(x_j^* a x_k)\chi_i(y_j^* b y_k)}{\varphi(y^*y)} = \frac{\varphi(y^*(a \otimes b)y)}{\varphi(y^*y)}.$$

Defining $\psi = \sum_{i=1}^n \sum_{j,k=1}^m \omega'_{ijk} \otimes \chi'_{ijk}$, then $\psi \in \mathcal{A}^* \odot \mathcal{B}^*$ and

$$\psi(x)\varphi(y^*y) = \varphi(y^*xy), \quad x \in \mathcal{A} \odot \mathcal{B}.$$

As ψ is then algebraically positive and $\psi(1_{\mathcal{A}} \otimes 1_{\mathcal{B}}) = 1$, it follows that $\psi \in S(\mathcal{A} \odot \mathcal{B})$, so $\mathbb{M}$ satisfies the conditions of Lemma 1.42, yielding the following proposition:

Proposition 1.46. *If $\mathcal{A}$ and $\mathcal{B}$ are unital C^*-algebras, then*

$$\|x\|_\kappa = \sup\{\varphi(x^*x)^{1/2} \mid \varphi \in \mathbb{M}\}, \quad x \in \mathcal{A} \odot \mathcal{B}.$$

We dive headlong into our next result.

Proposition 1.47. *$\|\cdot\|_\kappa$ and $\|\cdot\|_{\min}$ are equal C^*-norms.*

Proof. Let $\pi\colon \mathcal{A} \to B(\mathcal{H})$ and $\rho\colon \mathcal{A} \to B(\mathcal{K})$ be faithful representations. Let

$$\mathfrak{X}_1 = \{\omega_\xi \circ \pi \mid \xi \in \mathcal{H},\ \|\xi\| = 1\}, \quad \mathfrak{X}_2 = \{\omega_\eta \circ \rho \mid \eta \in \mathcal{K},\ \|\eta\| = 1\},$$

where $\omega_\xi\colon B(\mathcal{H}) \to \mathbb{C}$ and $\omega_\eta\colon B(\mathcal{K}) \to \mathbb{C}$ are given by

$$\omega_\xi(S) = \langle S\xi, \xi\rangle, \quad \omega_\eta(T) = \langle T\eta, \eta\rangle, \quad S \in B(\mathcal{H}),\ T \in B(\mathcal{K}).$$

[9, Proposition 3.4.2] then yields that the convex hull of $\mathfrak{X}_1$ (resp. $\mathfrak{X}_2$) is weak*-dense in $S(\mathcal{A})$ (resp. $S(\mathcal{B})$). For any $\xi \in \mathcal{H}$ and $\eta \in \mathcal{K}$ of norm 1 we have for all $x \in \mathcal{A} \odot \mathcal{B}$ of the form $x = \sum_{i=1}^n a_i \otimes b_i$ that

$$\begin{aligned} |(\omega_\xi \circ \pi) \otimes (\omega_\eta \circ \rho)(x)| &= \left|\sum_{i=1}^n \langle(\pi(a_i) \otimes \rho(b_i))\xi \otimes \eta, \xi \otimes \eta\rangle\right| \\ &\leq |\langle(\pi \otimes \rho)(x)\xi \otimes \eta, \xi \otimes \eta\rangle| \\ &\leq \|(\pi \otimes \rho)(x)\| = \|x\|_{\min}. \end{aligned}$$

This implies that $\varphi \odot \psi \in S_{\min}(\mathcal{A} \odot \mathcal{B})$ for all $\varphi \in S(\mathcal{A})$ and $\psi \in S(\mathcal{B})$. Thus

$$\|x\|_\kappa = \sup\{\|\pi_{\varphi\odot\psi}(x)\| \mid \varphi \in S(\mathcal{A}),\ \psi \in S(\mathcal{B})\} \leq \sup\{\|\pi_\omega(x)\| \mid \omega \in S_{\min}(\mathcal{A} \odot \mathcal{B})\} \leq \|x\|_{\min}.$$

But since $\|\cdot\|_{\min}$ is the least possible C^*-norm by Takesaki's theorem (see Theorem 1.43), we must have $\|\cdot\|_\kappa = \|\cdot\|_{\min}$. ☐

The preceding two propositions now tell us the following:

Corollary 1.48. *For unital C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, we have*

$$\|x\|_{\min} = \sup\{\varphi(x^*x)^{1/2} \mid \varphi \in \mathbb{M}\}, \quad x \in \mathcal{A} \odot \mathcal{B}.$$

Definition 1.14. A C^*-algebra $\mathcal{A}$ is said to be $\otimes$-*nuclear* if there is a unique C^*-norm on $\mathcal{A} \odot \mathcal{B}$ for any C^*-algebra $\mathcal{B}$.

The last result is (unfortunately) stated without proof.

Theorem 1.49. *Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras.*

(i) *If $\mathcal{A}$ is non-unital and $\mathcal{B}$ is unital, then there is a unique C^*-norm on $\mathcal{A} \odot \mathcal{B}$ if and only if there is a unique C^*-norm on $\tilde{\mathcal{A}} \odot \mathcal{B}$.*

(ii) *If $\mathcal{A}$ and $\mathcal{B}$ are both non-unital, then there is a unique C^*-norm on $\mathcal{A} \odot \mathcal{B}$ if and only if there is a unique C^*-norm on $\tilde{\mathcal{A}} \odot \tilde{\mathcal{B}}$.*

(iii) *If $\mathcal{A}$ is unital and $\mathcal{B}$ is non-unital, then there is a unique C^*-norm on $\mathcal{A} \odot \mathcal{B}$ if and only if there is a unique C^*-norm on $\mathcal{A} \odot \tilde{\mathcal{B}}$.*

Proof. See [28, Theorem 1.36]. □

That this section leaves out a lot of important proofs irritates me a great deal, but the fact is that we will not be using any of the results until Chapter 5 and even then the use of them will be downright minimal. Hopefully the reader will have understood the point of being introduced to algebraic states, for they really do explain a great deal about norms on algebraic tensor products, and the proof of Corollary 1.48 is not exactly trivial. However that does not matter now; we have far more important things to attend to.

CHAPTER 2

THE ULTRAWEAK OPERATOR TOPOLOGY

In a beginner's course on von Neumann algebras, one starts out by being introduced to the weak and strong operator topologies on $B(\mathcal{H})$ and a neophyte in the subject is perhaps fooled into thinking that these topologies suffice for proving all worthwhile results about von Neumann algebras. Experience heals imbecility in this case; it might indeed be interesting to investigate whether topologies finer than the weak or strong operator topology and coarser than the norm topology exist, not only as a test for the curious, but perhaps for an altogether new approach to understanding the properties of certain subsets of bounded linear operators on Hilbert space.

In this chapter we will construct two other locally convex Hausdorff topologies on $B(\mathcal{H})$ for a Hilbert space $\mathcal{H}$, proving not only immensely useful, but also necessary for understanding just how flexible a von Neumann algebra actually is. The last statement will be reflected in a wide array of concepts for Hilbert spaces, von Neumann algebras and linear functionals on von Neumann algebras, not all of which are directly related, but which nevertheless combine into the idea expressed in Section 2.11, namely the enveloping von Neumann algebra of a C^*-algebra. The chapter also includes three intermezzos presenting concepts that are not directly related at all to the ultraweak operator topology, but are put here for three reasons: *(1)* for the greater good of the project structure-wise, *(2)* because they would not fit in anywhere else and *(3)* because the ideas are too relevant to be relegated to an appendix. In any case, the three intermezzos are in themselves very much related to one another and are absolutely essential for the concepts introduced and proofs given in Chapter 4 and 5.

2.1 Towards finer topologies

The details in the definition below are easily checked.

Definition 2.1. The *ultraweak topology* on $B(\mathcal{H})$ is the locally convex Hausdorff topology determined by the separating family of seminorms

$$T \mapsto \left| \sum_{n=1}^{\infty} \langle T\xi_n, \eta_n \rangle \right|, \quad T \in B(\mathcal{H})$$

for sequences $(\xi_n)_{n\geq 1}$ and $(\eta_n)_{n\geq 1}$ of $\mathcal{H}$ with $\sum_{n=1}^{\infty} \|\xi_n\|^2 < \infty$ and $\sum_{n=1}^{\infty} \|\eta_n\|^2 < \infty$.

The *ultrastrong topology* on $B(\mathcal{H})$ is the locally convex topology determined by the separating family of seminorms

$$T \mapsto \left[\sum_{n=1}^{\infty} \|T\xi_n\|^2 \right]^{1/2}$$

for $T \in B(\mathcal{H})$, where $(\xi_n)_{n\geq 1}$ is a sequence of $\mathcal{H}$ with $\sum_{n=1}^{\infty} \|\xi_n\|^2 < \infty$.

It is clear from the outset that $T_\alpha \to T$ ultrastrongly implies $T_\alpha \to T$ strongly and that $T_\alpha \to T$ ultraweakly implies $T_\alpha \to T$ weakly. Furthermore, if $T_\alpha \to T$ ultrastrongly in $B(\mathcal{H})$, then $T_\alpha \to T$ ultraweakly as well. Indeed, for square-summable sequences $(\xi_n)_{n\geq 1}$ and $(\eta_n)_{n\geq 1}$ in $\mathcal{H}$ and $T \in B(\mathcal{H})$, then

$$\left| \sum_{n=1}^{\infty} \langle T\xi_n, \eta_n \rangle \right|^2 \leq \left[\sum_{n=1}^{\infty} \|T\xi_n\|^2 \right] \left[\sum_{n=1}^{\infty} \|\eta_n\|^2 \right] \leq \|T\|^2 \left[\sum_{n=1}^{\infty} \|\xi_n\|^2 \right] \left[\sum_{n=1}^{\infty} \|\eta_n\|^2 \right].$$

Finally, ultraweakly and ultrastrongly closed subsets of $B(\mathcal{H})$ are also norm-closed.

One could wonder if the ultraweak (resp. ultrastrong) topology could coincide with the weak (resp. strong) in some case, and the following proposition provides a circumstance under which this is true; the proof is quite elementary.

Proposition 2.1. *Let $T \in B(\mathcal{H})$ and let $(T_\alpha)_{\alpha\in A}$ be a bounded net in $B(\mathcal{H})$. Then $T_\alpha \to T$ ultraweakly (resp. ultrastrongly) to $T \in B(\mathcal{H})$ if and only if $T_\alpha \to T$ weakly (resp. strongly). Hence the ultraweak (resp. ultrastrong) and weak (resp. strong) topology coincide on bounded subsets of $B(\mathcal{H})$.*

Proof. The "only if" implications are trivial. Supposing that $\|T_\alpha\| \le M$ for all $\alpha \in A$, then if $\xi = (\xi_n)_{n\ge1}$ and $\eta = (\eta_n)_{n\ge1}$ are square-summable sequences in $\mathcal{H}$, then for all $N \ge 1$ and $\alpha \in A$ we find

$$\begin{aligned}\left|\sum_{n=1}^{\infty}\langle (T_\alpha - T)\xi_n, \eta_n\rangle\right| &\le \sum_{n=1}^{N} |\langle (T_\alpha - T)\xi_n, \eta_n\rangle| + (M + \|T\|)\sum_{n=N+1}^{\infty} \|\xi_n\|\|\eta_n\| \\ &\le \sum_{n=1}^{N} |\langle (T_\alpha - T)\xi_n, \eta_n\rangle| + (M + \|T\|)\left[\sum_{n=N+1}^{\infty}\|\xi_n\|^2\right]^{1/2}\left[\sum_{n=N+1}^{\infty}\|\eta_n\|^2\right]^{1/2}\end{aligned}$$

by using the Cauchy-Schwarz inequality. For any $\varepsilon > 0$ then by first choosing an appropriate N to make the second term arbitrarily small, then one can pick an $\alpha \in A$ such that the first term becomes arbitrarily small as well, hence implying that $T_\alpha \to T$ ultraweakly if $T_\alpha \to T$ weakly. Similarly since we have

$$\sum_{n=1}^{\infty}\|(T_\alpha - T)\xi_n\|^2 \le \sum_{n=1}^{N}\|(T_\alpha - T)\xi_n\|^2 + (M + \|T\|)\sum_{n=N+1}^{\infty}\|\xi_n\|^2$$

for all $N \ge 1$ and $\alpha \in A$, we see that $T_\alpha \to T$ ultrastrongly if $T_\alpha \to T$ strongly. □

Throughout the project, we will mainly be considering and using the ultraweak topology. For the following proposition, though, an understanding of the ultrastrong topology is absolutely essential.

Proposition 2.2. *Let $\omega\colon B(\mathcal{H}) \to \mathbb{C}$ be a linear functional. Then the following are equivalent:*

(i) *ω is ultraweakly continuous.*
(ii) *ω is ultrastrongly continuous.*
(iii) *There exist sequences $(\xi_n)_{n\ge1}$ and $(\eta_n)_{n\ge1}$ of $\mathcal{H}$ with $\sum_{n=1}^{\infty}\|\xi_n\|^2 < \infty$ and $\sum_{n=1}^{\infty}\|\eta_n\|^2 < \infty$ such that*

$$\omega(T) = \sum_{n=1}^{\infty}\langle T\xi_n, \eta_n\rangle.$$

Proof. (iii) ⇒ (i) ⇒ (ii) is clear, so we only need prove (ii) ⇒ (iii). Suppose that ω is an ultrastrongly continuous linear functional on $B(\mathcal{H})$. Then there exist square-summable sequences $(\xi_n'^1)_{n\ge1}, \dots, (\xi_n'^m)_{n\ge1}$ in $\mathcal{H}$ and $C > 0$ such that

$$|\omega(T)| \le C\sum_{i=1}^{m}\left[\sum_{n=1}^{\infty}\|T\xi_n'^i\|^2\right]^{1/2}$$

for all $T \in B(\mathcal{H})$ [13, Proposition 5.15]. Defining a sequence $(\xi_n)_{n\ge1}$ by

$$\xi_{(i-1)m+j} = \xi_i'^j, \quad i \in \mathbb{N},\ 1 \le j \le m,$$

then $\sum_{n=1}^{\infty}\|\xi_n\|^2 = \sum_{j=1}^{\infty}\sum_{i=1}^{m}\|\xi_i'^j\|^2 < \infty$, so $(\xi_n)_{n\ge1} \in \mathcal{H}^{\mathbb{N}}$ and

$$|\omega(T)| \le C\left[\sum_{n=1}^{\infty}\|T\xi_n^i\|^2\right]^{1/2}.$$

Defining $\mathcal{K}_0 = \{(T\xi_n)_{n\ge1} \in \mathcal{H}^{\mathbb{N}} \mid T \in B(\mathcal{H})\}$ and let $\mathcal{K}$ denote the norm closure of $\mathcal{K}_0$ in $\mathcal{H}^{\mathbb{N}}$. The linear functional $\varphi\colon \mathcal{K}_0 \to \mathbb{C}$ given by

$$\varphi((T\xi_n)_{n\ge1}) = \omega(T)$$

is then well-defined, and since φ is also bounded above by C, φ extends to a bounded linear functional on the Hilbert space $\mathcal{K}$ by Proposition A.1. By the Riesz representation theorem [14, Theorem 2.3.1], there exists $(\eta_n)_{n\geq 1} \in \mathcal{K}$ such that

$$\omega(T) = \langle (T\xi_n)_{n\geq 1}, (\eta_n)_{n\geq 1}\rangle = \sum_{n=1}^{\infty} \langle T\xi_n, \eta_n\rangle,$$

proving (iii). $\square$

With this in mind, great things will happen. Read on.

2.2 The predual of an ultraweakly closed subspace

For $\xi, \eta \in \mathcal{H}$ define a linear functional $\omega_{\xi,\eta}$ on $B(\mathcal{H})$ by

$$\omega_{\xi,\eta}(T) = \langle T\xi, \eta\rangle.$$

Obviously, $\omega_{\xi,\eta} \in B(\mathcal{H})^*$ with $\|\omega_{\xi,\eta}\| = \|\xi\|\|\eta\|$. For $\xi \in \mathcal{H}$, we also define $\omega_\xi = \omega_{\xi,\xi}$. If ξ is a unit vector, then ω_ξ is a state on $B(\mathcal{H})$; it is called a *vector state*.

Definition 2.2. The subspace of $B(\mathcal{H})^*$ spanned by the bounded linear functionals $\omega_{\xi,\eta}$ is denoted by $B(\mathcal{H})_\sim$, and the norm closure of $B(\mathcal{H})_\sim$ in $B(\mathcal{H})^*$ is denoted by $B(\mathcal{H})_*$.

It is clear from the outset that $B(\mathcal{H})_\sim$ is the set of weakly continuous linear functionals on $B(\mathcal{H})$. Proposition 2.2 would make one expect that $B(\mathcal{H})_*$ would then be the set of ultraweakly continuous linear functionals on $B(\mathcal{H})$, and indeed this is true, but the proof is not as obvious as one would think.

Recall that the set of finite rank operators is the linear span of the rank one operators $E_{\xi,\eta}\colon \mathcal{H} \to \mathcal{H}$, $\xi, \eta \in \mathcal{H}$, given by

$$E_{\xi,\eta}(x) = \langle x, \xi\rangle\eta, \quad x \in \mathcal{H}.$$

Indeed, if $T \in B(\mathcal{H})$ has finite rank n, then $T(\mathcal{H})$ is a Hilbert space with a finite orthonormal basis $(\eta_i)_{i=1}^n$. Hence

$$T\xi = \sum_{i=1}^{n} \langle T\xi, \eta_i\rangle\eta_i$$

for all $\xi \in \mathcal{H}$. Putting $\xi_i = T^*\eta_i$ for $i = 1, \dots, n$ yields that $T = \sum_{i=1}^n E_{\xi_i,\eta_i}$. On the other hand, if T is a finite linear combinations of E_{ξ_i,η_i}'s, then $T(\mathcal{H})$ is contained in the span of $(\eta_i)_{i=1}^n$, so T has finite-dimensional image. Operators of the form $E_{\xi,\eta}$ are called *elementary operators*.

We now turn to the first big proof of this chapter – it is like a Christmas present you do not think you want when in fact you really need it.

Theorem 2.3. *For any weakly continuous linear functional ω on $B(\mathcal{H})$, there exist orthonormal sets $(e_i)_{i=1}^n$, $(e_i')_{i=1}^n$ in $\mathcal{H}$ and non-negative numbers λ_i, $i = 1, \dots, n$, such that*

$$\omega = \sum_{i=1}^{n} \lambda_i \omega_{e_i,e_i'}, \quad \|\omega\| = \sum_{i=1}^{n} \lambda_i.$$

Proof. We already know that ω has the form $\omega = \sum_{i=1}^p \omega_{\xi_i,\eta_i}$ from Proposition 0.7. Assume that $\xi_1', \dots, \xi_q'$ and $\eta_1', \dots, \eta_q'$ are elements of $\mathcal{H}$ such that the two finite rank operators $\mathcal{H} \to \mathcal{H}$ given by $x \mapsto \sum_{i=1}^p \langle x, \eta_i\rangle\xi_i$ and $x \mapsto \sum_{j=1}^q \langle x, \eta_i'\rangle\xi_i'$ for $x \in \mathcal{H}$ are in fact the same operator. Then

$$\begin{aligned}\sum_{i=1}^{p} \langle E_{\xi,\eta}\xi_i, \eta_i\rangle &= \sum_{i=1}^{p} \langle \xi_i, \xi\rangle\langle\eta, \eta_i\rangle = \left\langle \sum_{i=1}^{p} \langle \eta, \eta_i\rangle\xi_i, \xi \right\rangle = \left\langle \sum_{i=1}^{q} \langle \eta, \eta_i'\rangle\xi_i', \xi \right\rangle = \sum_{i=1}^{q} \langle \xi_i', \xi\rangle\langle\eta, \eta_i'\rangle \\ &= \sum_{i=1}^{q} \langle E_{\xi,\eta}\xi_i', \eta_i'\rangle\end{aligned}$$

for all $\xi, \eta \in \mathcal{H}$, so the functionals $\sum_{i=1}^p \omega_{\xi_i,\eta_i}$ and $\sum_{i=1}^q \omega_{\xi_i',\eta_i'}$ agree on all finite rank operators by linearity. If P_1 and P_2 denotes the finite rank orthogonal projections onto the linear spans of

$\eta_1, \ldots, \eta_p, \eta'_1, \ldots, \eta'_q$ and $\xi_1, \ldots, \xi_p, \xi'_1, \ldots, \xi'_q$ respectively, then $P_1 T P_2$ is finite rank for all $T \in B(\mathcal{H})$, yielding

$$\sum_{i=1}^{p} \langle T\xi_i, \eta_i \rangle = \sum_{i=1}^{p} \langle TP_2\xi_i, P_1\eta_i \rangle = \sum_{i=1}^{p} \langle P_1TP_2\xi_i, \eta_i \rangle = \sum_{i=1}^{q} \langle P_1TP_2\xi'_i, \eta'_i \rangle = \sum_{i=1}^{q} \langle TP_2\xi'_i, P_1\eta'_i \rangle$$
$$= \sum_{i=1}^{q} \langle T\xi'_i, \eta'_i \rangle.$$

Hence $\sum_{i=1}^{p} \omega_{\xi_i,\eta_i} = \sum_{i=1}^{q} \omega_{\xi'_i,\eta'_i}$.

Consider now the finite rank operator $x \mapsto \sum_{i=1}^{p} \langle x, \eta_i \rangle \xi_i$; let us call it A. Note now that A^*A is positive and has finite rank and hence finite spectrum $\sigma(A^*A) = \{\mu_1, \ldots, \mu_k\}$ by Lemma A.15, consisting of non-negative numbers only. Hence $A^*A = \sum_{j=1}^{k} \mu_j P_j$, where $P_j = \chi_j(A^*A)$ for $j = 1, \ldots, k$, χ_j denoting the characteristic function of the one-point set $\{\mu_j\}$. By uniqueness of the square root, we see that

$$|A| = (A^*A)^{1/2} = \sum_{j=1}^{k} \mu_j^{1/2} P_j$$

as it is positive with second power equal to A^*A. Because $(P_j)_{j=1}^{k}$ is a set of orthogonal finite rank projections in $B(\mathcal{H})$, then by taking orthonormal bases for $P_j(\mathcal{H})$ for $j = 1, \ldots, k$ and putting them into one set $e_1, \ldots, e_n$, we then obtain an orthonormal set $(e_i)_{i=1}^{n}$. For $i = 1, \ldots, n$, let μ'_i be the number μ_j such that $e_i \in P_j(\mathcal{H})$. Then

$$|A|\xi = \sum_{j=1}^{k} \mu_j^{1/2} P_j \xi = \sum_{i=1}^{n} \mu_i'^{1/2} \langle \xi, e_i \rangle e_i, \quad \xi \in \mathcal{H}.$$

Let $A = U|A|$ be the polar decomposition of A with U being the partial isometry with initial space $|A|(\mathcal{H})$. Note that $\langle Ue_i, Ue_j \rangle = \langle e_i, U^*Ue_j \rangle = \langle e_i, e_j \rangle$ for all $i, j = 1, \ldots, n$, as $e_j \in |A|(\mathcal{H})$. By defining $e'_i = Ue_i$ for all $i = 1, \ldots, n$, then $(e'_i)_{i=1}^{n}$ is an orthonormal set and

$$A\xi = U|A|\xi = \sum_{j=1}^{k} \mu_j^{1/2} P_j \xi = \sum_{i=1}^{n} \mu_i'^{1/2} \langle \xi, e_i \rangle e'_i, \quad \xi \in \mathcal{H}.$$

Defining $\lambda_i = \mu_i'^{1/2}$ for $i = 1, \ldots, n$, then by what we first proved it follows that

$$\omega = \sum_{i=1}^{p} \omega_{\xi_i,\eta_i} = \sum_{i=1}^{n} \lambda_i \omega_{e_i,e'_i}.$$

It is clear that $\|\omega\| \leq \sum_{i=1}^{n} \lambda_n$. If we put $T = \sum_{i=1}^{n} E_{e_i,e'_i}$, then note that

$$\|T\xi\|^2 = \left\| \sum_{i=1}^{n} E_{e_i,e'_i} \xi \right\|^2 = \left\| \sum_{i=1}^{n} \langle \xi, e_i \rangle e'_i \right\|^2 = \sum_{i=1}^{n} |\langle \xi, e_i \rangle|^2 \leq \|\xi\|^2$$

from Bessel's inequality [13, Theorem 5.26], so $\|T\| \leq 1$; as $\omega(T) = \sum_{i=1}^{n} \lambda_i \langle E_{e_i,e'_i} e_i, e'_i \rangle = \sum_{i=1}^{n} \lambda_i$, equality follows. □

Corollary 2.4. *$B(\mathcal{H})_*$ is the set of ultraweakly continuous linear functionals on $B(\mathcal{H})$.*

Proof. It is clear from Proposition 2.2 that all ultraweakly continuous linear functionals are contained in $B(\mathcal{H})$. For the converse inclusion, let $\omega \in B(\mathcal{H})_*$ and let $(\omega_n)_{k \geq 1}$ be a sequence of weakly continuous functionals such that $\|\omega_n - \omega\| \leq 2^{-n-2}$ for all $n \geq 1$. Then $\|\omega_n - \omega_{n-1}\| \leq 2^{-n-2} + 2^{-n-1} < 2^{-n}$ for all $n \geq 2$ and $\|\omega_1\| \leq \|\omega\| + 2^{-3}$. Defining $\varphi_1 = \omega_1$ and $\varphi_n = \omega_n - \omega_{n-1}$ for all $n \geq 2$, we obtain that $\varphi_n \in B(\mathcal{H})_\sim$ for all $n \geq 1$ with

$$\omega = \sum_{n=1}^{\infty} \varphi_n.$$

The preceding theorem yields non-negative numbers λ_i^n, $i = 1, \dots, k_n$ and orthonormal sets $e_1^n, \dots, e_{k_n}^n$ and $e_1'^n, \dots, e_{k_n}'^n$ such that

$$\varphi_n = \sum_{i=1}^{k_n} \lambda_i^n \omega_{e_i^n, e_i'^n}$$

for all $n \geq 1$. Note that $\sum_{i=1}^{k_n} \lambda_i^n < 2^{-n}$ for all $n \geq 2$ and $\sum_{i=1}^{k_1} \lambda_i^1 < \|\omega\| + 2^{-3}$. Hence

$$\omega = \sum_{n=1}^{\infty} \sum_{i=1}^{k_n} \omega_{(\lambda_i^n)^{1/2} e_i^n, (\lambda_i^n)^{1/2} e_i'^n},$$

with

$$\sum_{n=1}^{\infty} \sum_{i=1}^{k_n} \|(\lambda_i^n)^{1/2} e_i^n\|^2 = \sum_{n=1}^{\infty} \sum_{i=1}^{k_n} \lambda_i^n \leq \|\omega\| + 2^{-3} + \sum_{n=2}^{\infty} 2^{-n} < \infty$$

and $\sum_{n=1}^{\infty} \sum_{i=1}^{k_n} \|(\lambda_i^n)^{1/2} e_i'^n\|^2 < \infty$ seen analoguously. Proposition 2.2 then yields that ω is ultraweakly continuous and this proves the statement. $\square$

What we have obtained now is remarkable; the set of ultraweakly continuous linear functionals is in fact a Banach space. To what use, one may ask, and one would be silenced very quickly in view of the next result.

Proposition 2.5. *The Banach space $B(\mathcal{H})$ is isometrically isomorphic to the dual of the Banach space $B(\mathcal{H})_*$ by the natural mapping Λ given by evaluation, i.e.*

$$\Lambda \colon B(\mathcal{H}) \to (B(\mathcal{H})_*)^*, \quad \Lambda(T)(\omega) = \omega(T), \quad T \in B(\mathcal{H}),\ \omega \in B(\mathcal{H})_*.$$

Proof. It is clear that Λ is well-defined and linear. For $T \in B(\mathcal{H})$ and $\omega \in B(\mathcal{H})_*$, we have

$$|\Lambda(T)(\omega)| = |\omega(T)| \leq |\omega\| \|T\|,$$

proving $\|\Lambda(T)\| \leq \|T\|$. To prove that Λ is actually an isometry, note that $\omega_{\xi,\eta} \in B(\mathcal{H})_*$ for all $\xi, \eta \in \mathcal{H}$, whereupon we obtain from Lemma 1.24 that

$$\begin{aligned}
\|T\| &= \sup\{|\langle T\xi, \eta\rangle| \mid \xi, \eta \in \mathcal{H}, \|\xi\| = 1, \|\eta\| = 1\} \\
&= \sup\{|\omega_{\xi,\eta}(T)| \mid \xi, \eta \in \mathcal{H}, \|\xi\| = 1, \|\eta\| = 1\} \\
&\leq \sup\{|\Lambda(T)(\omega_{\xi,\eta})\| \mid \xi, \eta \in \mathcal{H}, \|\omega_{\xi,\eta}\| = 1\} \\
&\leq \|\Lambda(T)\|.
\end{aligned}$$

For $\varphi \in (B(\mathcal{H})_*)^*$, note that by defining

$$\tilde{\varphi}(\xi, \eta) = \varphi(\omega_{\xi,\eta}), \quad \xi, \eta \in \mathcal{H},$$

we obtain a bounded sesquilinear form on $\mathcal{H}$. By the Riesz representation theorem [14, Theorem 2.4.1], we obtain a unique operator $T \in B(\mathcal{H})$ such that

$$\varphi(\omega_{\xi,\eta}) = \langle T\xi, \eta\rangle, \quad \xi, \eta \in \mathcal{H}.$$

As $\langle T\xi, \eta\rangle = \omega_{\xi,\eta}(T) = \Lambda(T)(\omega_{\xi,\eta})$, it follows that $\Lambda(T)$ and φ coincide on $B(\mathcal{H})_\sim$. By continuity, it follows that $\Lambda(T) = \varphi$. $\square$

Note that in the above proof we only used the definition of $B(\mathcal{H})_*$, that is, we did not use Corollary 2.4. We shall see now that the reason for this identification comes from the fact that we are working with an ultraweakly closed subspace of $B(\mathcal{H})$, which the following results will make clear. For an ultraweakly closed subspace $\mathscr{M}$ of $B(\mathcal{H})$, let $\mathscr{M}^\perp$ denote the closed subspace of ultraweakly continuous linear functionals ω on $B(\mathcal{H})$ such that $\omega(\mathscr{M}) = \{0\}$.

Lemma 2.6. *If $\mathscr{M}$ is an ultraweakly closed subspace of $B(\mathcal{H})$, then*

$$\mathscr{M} = \mathscr{M}^{\perp\perp} := \{T \in B(\mathcal{H}) \mid \omega(T) = 0 \text{ for all } \omega \in \mathscr{M}^\perp\}.$$

Proof. That $\mathscr{M} \subseteq \mathscr{M}^{\perp\perp}$ is clear. If $T \in B(\mathcal{H}) \setminus \mathscr{M}$, then it is a consequence of the Hahn-Banach separation theorem [14, Corollary 1.2.13] that there exists an ultraweakly continuous linear functional ω on $B(\mathcal{H})$ with $\omega(T) \neq 0$ and $\omega(\mathscr{M}) = \{0\}$. □

Theorem 2.7. *Let $\mathscr{M}$ be an ultraweakly closed subspace of $B(\mathcal{H})$ and define*

$$\mathscr{M}_\sim := \{\varphi|_{\mathscr{M}} \mid \varphi \in B(\mathcal{H})_\sim\}, \quad \mathscr{M}_* = \{\varphi|_{\mathscr{M}} \mid \varphi \in B(\mathcal{H})_*\}.$$

Then $\mathscr{M}_\sim, \mathscr{M}_* \subseteq \mathscr{M}^*$ and

(i) *$\mathscr{M}_\sim$ consists of all weakly continuous linear functionals on $\mathscr{M}$.*

(ii) *$\mathscr{M}_*$ consists of all ultraweakly continuous linear functionals on $\mathscr{M}$.*

(iii) *$\mathscr{M}_*$ is a norm-closed subspace of $\mathscr{M}^*$ and is the norm closure of $\mathscr{M}_\sim$.*

(iv) *for any $\omega \in \mathscr{M}_*$ and any $\varepsilon > 0$ there exists $\varphi \in B(\mathcal{H})_*$ such that $\omega = \varphi|_{\mathscr{M}}$ and $\|\varphi\| \leq \|\omega\| + \varepsilon$.*

(v) *$\mathscr{M}$ is isometrically isomorphic to $(\mathscr{M}_*)^*$ through the canonical identification*

$$\Lambda_{\mathscr{M}} \colon \mathscr{M} \to (\mathscr{M}_*)^*, \quad \Lambda_{\mathscr{M}}(T)(\omega) = \omega(T), \quad T \in \mathscr{M},\ \omega \in \mathscr{M}_*.$$

Proof. (i) and (ii) follow from the Hahn-Banach extension theorem for locally convex topological vector spaces [14, Theorem 1.2.14].

(iii) Define a surjection $\tilde{\Omega}\colon B(\mathcal{H})_* \to \mathscr{M}_*$ by $\tilde{\Omega}(\varphi) = \varphi|_{\mathscr{M}}$. $\tilde{\Omega}$ is linear of norm ≤ 1 with kernel $\mathscr{M}^\perp$. Hence it induces a linear map $\Omega\colon B(\mathcal{H})_*/\mathscr{M}^\perp \to \mathscr{M}_*$ defined by

$$\Omega(\varphi + \mathscr{M}^\perp) = \varphi|_{\mathscr{M}}.$$

Note that $B(\mathcal{H})_*/\mathscr{M}^\perp$ is a Banach space; hence we can prove that $\mathscr{M}_*$ is closed in $\mathscr{M}^*$ by proving that Ω is in fact isometric. Letting $\varphi \in B(\mathcal{H})_*$, then for all $\psi \in \varphi + \mathscr{M}^\perp$ we have

$$\|\Omega(\varphi + \mathscr{M}^\perp)\| = \|\Omega(\psi + \mathscr{M}^\perp)\| = \|\psi|_{\mathscr{M}}\| \leq \|\psi\|,$$

so by taking the infimum over all ψ, we obtain $\|\Omega(\varphi + \mathscr{M}^\perp)\| \leq \|\varphi + \mathscr{M}^\perp\|$. Therefore Ω is contractive. Take $\varphi \in B(\mathcal{H})_* \setminus \mathscr{M}^\perp$, i.e. such that

$$\delta := \inf_{\omega \in \mathscr{M}^\perp} \|\varphi + \omega\| = \|\varphi + \mathscr{M}^\perp\| > 0.$$

The Hahn-Banach theorem [13, Theorem 5.8] then provides a bounded linear functional Φ on $B(\mathcal{H})_*$ satisfying $\Phi(\varphi) = \delta$, $\|\Phi\| = 1$ and $\Phi(\mathscr{M}^\perp) = \{0\}$; using Proposition 2.5 we then obtain $T \in B(\mathcal{H})$ such that $\varphi(T) = \Phi(\varphi) = \delta$, $\|T\| = 1$ and $\omega(T) = 0$ for all $\omega \in \mathscr{M}^\perp$. Hence $T \in \mathscr{M}^{\perp\perp} = \mathscr{M}$, so

$$\|\Omega(\varphi + \mathscr{M}^\perp)\| = \|\varphi|_{\mathscr{M}}\| \geq |\varphi(T)| = \delta = \|\varphi + \mathscr{M}^\perp\|.$$

Hence Ω is an isometric isomorphism, proving that $\mathscr{M}_*$ is closed.

Since $B(\mathcal{H})_\sim$ is norm-dense in $B(\mathcal{H})_*$ and the bounded linear map $\varphi \mapsto \varphi|_{\mathscr{M}}$ maps $B(\mathcal{H})_\sim$ to $\mathscr{M}_\sim$ and $B(\mathcal{H})_*$ to $\mathscr{M}_*$, it follows that $\mathscr{M}_\sim$ is norm-dense in $\mathscr{M}_*$. As $\mathscr{M}_*$ is closed, we finally obtain (iii).

(iv) For any $\omega \in \mathscr{M}_*$, then, on the grounds of what we just found, there exists $\varphi_0 \in B(\mathcal{H})_*$ such that $\varphi_0|_{\mathscr{M}} = \omega$ and $\|\omega\| = \|\varphi_0 + \mathscr{M}^\perp\|$. Hence for any $\varepsilon > 0$ there exists $\varphi_1 \in \mathscr{M}^\perp$ such that $\|\varphi_0 + \varphi_1\| \leq \|\omega\| + \varepsilon$. By defining $\varphi = \varphi_0 + \varphi_1 \in B(\mathcal{H})_*$, then $\varphi|_{\mathscr{M}} = \varphi_0|_{\mathscr{M}} = \omega$ and $\|\varphi\| \leq \|\omega\| + \varepsilon$.

(v) $\Lambda_{\mathscr{M}}$ is clearly well-defined, linear and contractive. To see that $\Lambda_{\mathscr{M}}$ is actually an isometry, let $T \in \mathscr{M}$ and $\varepsilon > 0$. For any $\varepsilon > 0$, take $\varphi \in B(\mathcal{H})_*$ with $\|\varphi\| \leq 1$ such that $|\Lambda(T)(\varphi)| \geq \|\Lambda(T)\| - \varepsilon$, where Λ denotes the canonical identification $B(\mathcal{H}) \to (B(\mathcal{H})_*)^*$. Then

$$\|\Lambda_{\mathscr{M}}(T)\| \geq |\Lambda_{\mathscr{M}}(T)(\varphi|_{\mathscr{M}})| = |\varphi(T)| = |\Lambda(T)(\varphi)\| \geq \|\Lambda(T)\| - \varepsilon = \|T\| - \varepsilon,$$

so $\|\Lambda_{\mathscr{M}}(T)\| \geq \|T\|$. Hence $\Lambda_{\mathscr{M}}$ is an isometry. Finally, let $\varphi \in (\mathscr{M}_*)^*$. Define a linear functional on $B(\mathcal{H})_*$ by

$$\Phi(\alpha) = \varphi(\alpha|_{\mathscr{M}}), \quad \alpha \in B(\mathcal{H})_*.$$

Then Φ is bounded and $\Phi(\mathscr{M}^{\perp}) = \{0\}$. Hence there exists $T \in B(\mathcal{H})$ such that

$$\alpha(T) = \varphi(\alpha|_{\mathscr{M}}), \quad \alpha \in B(\mathcal{H})_*.$$

In fact $T \in \mathscr{M}$ since $\alpha(T) = 0$ for all $\alpha \in \mathscr{M}^{\perp}$, implying $T \in \mathscr{M}^{\perp\perp}$. Therefore

$$\varphi(\alpha|_{\mathscr{M}}) = \alpha|_{\mathscr{M}}(T) = \Lambda_{\mathscr{M}}(T)(\alpha|_{\mathscr{M}}), \quad \alpha \in B(\mathcal{H})_*,$$

so $\Lambda_{\mathscr{M}}(T) = \varphi$ for some $T \in \mathscr{M}$. Hence $\Lambda_{\mathscr{M}}$ is surjective. □

All the magic tricks we have been developing so far requires a celebration of a kind, and why not start out with a definition?

Definition 2.3. For an ultraweakly closed subspace $\mathscr{M}$ of $B(\mathcal{H})$, the Banach space $\mathscr{M}_*$ from Theorem 2.7 is called the *predual* of $\mathscr{M}$, and it consists of all ultraweakly continuous linear functionals on $\mathscr{M}$.

The reason for the term is exactly because of statement (v) of Theorem 2.7; $\mathscr{M}_*$ is the predual of $\mathscr{M}$ in the sense that $\mathscr{M}$ can be identified with the dual space of $\mathscr{M}_*$. The canonical identification $\Lambda_{\mathscr{M}}\colon \mathscr{M} \to (\mathscr{M}_*)^*$ will be denoted by Λ if no illogical confusion can occur (one never knows, though).

A consequence of identifying an ultraweakly closed subspace $\mathscr{M}$ of $B(\mathcal{H})$ with a dual space is that we can compare the ultraweak topology on $\mathscr{M}$ with a well-known topology and obtain some very nice results.

Corollary 2.8. *Let $\mathscr{M}$ be an ultraweakly closed subspace of $B(\mathcal{H})$. Then the canonical identification $\Lambda\colon \mathscr{M} \to (\mathscr{M}_*)^*$ is an ultraweak-to-weak* homeomorphism.*

Proof. This follows the fact that $\mathscr{M}_* = \{\varphi|_{\mathscr{M}} \mid \varphi \in B(\mathcal{H})_*\}$, Proposition 2.2 and Corollary 2.4. □

In particular, for a net $(T_\alpha)_{\alpha\in A}$ in $\mathscr{M}$ and $T \in \mathscr{M}$, then $T_\alpha \to T$ ultraweakly if and only if we have $\omega(T_\alpha) \to \omega(T)$ for all $\omega \in \mathscr{M}_*$. We will use this fact a lot throughout the project, so the reader is advised to keep it in mind.

Corollary 2.9. *Let $\mathscr{M}$ be an ultraweakly closed subspace of $B(\mathcal{H})$. Then $(\mathscr{M})_1$ is ultraweakly compact. In particular, $(B(\mathcal{H}))_1$ is ultraweakly compact.*

Proof. Follows from Corollary 2.8 and Alaoglu's theorem [13, Theorem 5.18]. □

For our work to really have an influence, we reach into our analyst's hat and find a rabbit in form of the *Krein-Šmulian theorem* [6, Theorem V.12.1], namely that if $\mathfrak{X}$ is a Banach space and $\mathscr{S} \subseteq \mathfrak{X}^*$ is a convex subset of its dual space, then $\mathscr{S}$ is weak*-closed if and only if $\mathscr{S} \cap (\mathfrak{X}^*)_r$ is weak*-closed for all $r > 0$. This immediately leads to the following corollary.

Corollary 2.10. *Let $\mathscr{S}$ be a convex subset of $B(\mathcal{H})$. Then $\mathscr{S}$ is ultraweakly closed if and only if $\mathscr{S} \cap (B(\mathcal{H}))_r$ is ultraweakly closed for all $r > 0$.*

Proof. Since the canonical identification $\Lambda\colon B(\mathcal{H}) \to (B(\mathcal{H})_*)^*$ was an ultraweak-to-weak* homeomorphism and $\Lambda(\mathscr{S})$ is weak*-closed if and only if $\Lambda(\mathscr{S}) \cap ((B(\mathcal{H})_*)^*)_r = \Lambda(\mathscr{S} \cap (B(\mathcal{H}))_r)$ is weak*-closed for all $r > 0$ by the Krein-Šmulian theorem, the result follows. □

We can then finally summarize where our knowledge of preduals has taken us so far, by considering convex subsets of $B(\mathcal{H})$; since *-subalgebras are convex, one can imagine what good this final theorem of this section will do.

Theorem 2.11. *Let $\mathscr{S}$ be a convex subset of $B(\mathcal{H})$. Then the following are equivalent:*

(i) *$\mathscr{S}$ is ultraweakly closed.*
(ii) *$\mathscr{S}$ is ultrastrongly closed.*
(iii) *$(\mathscr{S})_r$ is weakly closed for all $r > 0$.*
(iv) *$(\mathscr{S})_r$ is strongly closed for all $r > 0$.*
(v) *$(\mathscr{S})_r$ is ultraweakly closed for all $r > 0$.*
(vi) *$(\mathscr{S})_r$ is ultrastrongly closed for all $r > 0$.*

Proof. Since the set of ultraweakly continuous linear functionals coincides with the set of ultrastrongly continuous linear functionals by Proposition 2.2, (ii) $\Leftrightarrow$ (i) follows from Theorem A.7. Since $(\mathscr{S})_r$ is convex, the same theorem and Proposition 0.7 yields (iv) $\Rightarrow$ (iii) and (vi) $\Rightarrow$ (v). The implications (iii) $\Rightarrow$ (iv) and (v) $\Rightarrow$ (vi) are trivial. Proposition 2.1 yields (i) $\Rightarrow$ (iii), (ii) $\Rightarrow$ (iv), (v) $\Rightarrow$ (iii) and (vi) $\Rightarrow$ (iv), and (v) $\Leftrightarrow$ (i) is just the statement of Corollary 2.10. $\square$

Corollary 2.12. *Let $\mathscr{M}$ be an ultraweakly closed subspace of $B(\mathcal{H})$, $\omega \colon \mathscr{M} \to \mathbb{C}$ be a linear functional and $r_0 > 0$. Then the following are equivalent:*

(i) *ω is ultraweakly continuous.*
(ii) *ω is ultrastrongly continuous.*
(iii) *(resp.* (iii.a)*) ω is weakly continuous on $(\mathscr{M})_r$ for all $r > 0$ (resp. $r = r_0$).*
(iv) *(resp.* (iv.a)*) ω is strongly continuous on $(\mathscr{M})_r$ for all $r > 0$ (resp. $r = r_0$).*
(v) *(resp.* (v.a)*) ω is ultraweakly continuous on $(\mathscr{M})_r$ for all $r > 0$ (resp. $r = r_0$).*
(vi) *(resp.* (vi.a)*) ω is ultrastrongly continuous on $(\mathscr{M})_r$ for all $r > 0$ (resp. $r = r_0$).*
(vii) *There exist sequences $(\xi_n)_{n\geq 1}$ and $(\eta_n)_{n\geq 1}$ of $\mathcal{H}$ with $\sum_{n=1}^{\infty} \|\xi_n\|^2 < \infty$ and $\sum_{n=1}^{\infty} \|\eta_n\|^2 < \infty$ such that*

$$\omega(T) = \sum_{n=1}^{\infty} \langle T\xi_n, \eta_n \rangle, \quad T \in \mathscr{M}.$$

Proof. The following implications are clear: (i) $\Rightarrow$ (v), (ii) $\Rightarrow$ (vi), (iii) $\Rightarrow$ (v) $\Rightarrow$ (vi), (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v), $(x) \Rightarrow (x.a)$ for $x \in \{\text{iii}, \text{iv}, \text{v}, \text{vi}\}$, (iii.a) $\Rightarrow$ (iv.a) $\Rightarrow$ (vi.a) and (iii.a) $\Rightarrow$ (v.a) $\Rightarrow$ (vi.a). Furthermore, Proposition 2.1 yields (i) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (iv) and Proposition 2.2 gives us (vii) $\Leftrightarrow$ (i) $\Leftrightarrow$ (ii), so it suffices only to prove (vi.a) $\Rightarrow$ (i). If ω satisfies (vi.a), let $\mathscr{S} = \ker \omega \subseteq \mathscr{M}$ and let $s > 0$. If $T \in B(\mathcal{H})$ and $T_\alpha \to T$ ultrastrongly for some net $(T_\alpha)_{\alpha \in A}$ in $(\mathscr{S})_s$, then $\frac{r_0}{s} T_\alpha \in (\mathscr{M})_{r_0}$ for all $\alpha \in A$. As $(\mathscr{M})_{r_0}$ is ultrastrongly closed by Theorem 2.11, we see that $\frac{r_0}{s} T \in (\mathscr{M})_{r_0}$, so $T \in (\mathscr{M})_s$. Moreover, $\omega(T_\alpha) \to \omega(T)$, so $T \in \mathscr{S} \cap (\mathscr{M})_s = (\mathscr{S})_s$. Hence $(\mathscr{S})_s$ is ultraweakly closed for all $s > 0$. Because $\mathscr{S}$ is convex, $\mathscr{S}$ is ultraweakly closed by Theorem 2.11, so ω is ultraweakly continuous [14, Corollary 1.2.5]. $\square$

We do not know it yet, but the two results above are more than enough to show how lovely von Neumann algebras really are.

2.3 Intermezzo 1: The central support

We will now deviate intermittently from what has been going on so far, in order to introduce some extremely relevant concepts relating to von Neumann algebras, as well as introduce some notation that proves very helpful in the next sections. $\mathcal{H}$ is once again a fixed Hilbert space.

Let $T \in B(\mathcal{H})$ and let P be the projection onto $\overline{T^*(\mathcal{H})}$, so that $1_{\mathcal{H}} - P$ is the projection onto $\ker T$. Then $T(1_{\mathcal{H}} - P) = 0$, so $T = TP$. Conversely, if $P_1 \in B(\mathcal{H})$ is a projection satisfying $TP_1 = T$, then for $T(1_{\mathcal{H}} - P_1)\xi = 0$ for all $\xi \in \mathcal{H}$. Hence $1_{\mathcal{H}} - P_1 \leq 1_{\mathcal{H}} - P$ or $P \leq P_1$, so P is the smallest projection in $B(\mathcal{H})$ satisfying $T = TP$; it is called the *right support of* T and is denoted by $S_r(T)$. If Q is the projection onto $\overline{T(\mathcal{H})}$, then $1_{\mathcal{H}} - Q$ is the projection onto $\ker T^*$. Hence $T^*(1_{\mathcal{H}} - Q) = 0$, so $(1_{\mathcal{H}} - Q)T = 0$ or $T = QT$. Similarly, one shows that Q is the smallest projection satisfying $T = QT$; it is called the *left support of* T and is denoted by $S_l(T)$. It is clear that $S_r(T) = S_l(T^*)$, and if T is contained in a von Neumann algebra $\mathscr{M}$, $\mathscr{M}$ contains $S_r(T)$ and $S_l(T)$ as well [31, Corollary 17.6].

Definition 2.4. For any von Neumann algebra $\mathscr{M}$, we define the *center* $\mathcal{Z}(\mathscr{M})$ of $\mathscr{M}$ by

$$\mathcal{Z}(\mathscr{M}) = \mathscr{M} \cap \mathscr{M}',$$

i.e. $\mathcal{Z}(\mathscr{M})$ consists of all operators of $\mathscr{M}$ that commute with everything in $\mathscr{M}$. Any projection in $\mathcal{Z}(\mathscr{M})$ is called *a central projection of* $\mathscr{M}$.

Let $\mathscr{M}$ be a von Neumann algebra and let $T \in \mathscr{M}$. If R is any central projection of $\mathscr{M}$ that majorizes the right support of T, $RT = TR = TS_r(T)R = T$, so R majorizes the left support of T as well. Taking the infimum over all such projections, we obtain a unique central projection C_T of $\mathscr{M}$ [31, Proposition 24.1] that is the smallest central projection that majorizes the left and right supports of

T. We have $C_T = C_{T^*}$ as well as $C_T T = TC_T = T$. If Q is a central projection of $\mathscr{M}$ such that $QT = 0$, then because QC_T is a projection and $T - QC_T T = T$, it follows that $1_{\mathcal{H}} - QC_T \geq S_l(T)$ and $1_{\mathcal{H}} - QC_T \geq S_r(T)$. Hence $1_{\mathcal{H}} - QC_T \geq C_T$ and $QC_T \leq 1_{\mathcal{H}} - C_T$, so $QC_T = 0$. C_T is called the *central support of* T. Let us summarize:

Definition 2.5. For any operator T in a von Neumann algebra $\mathscr{M} \subseteq B(\mathcal{H})$, the *central support of* T is the smallest central projection P of $\mathscr{M}$ such that

$$\overline{T(\mathcal{H})} \subseteq P(\mathcal{H}).$$

The central support of T is denoted by C_T, and it satisfies $C_T T = TC_T = T$. Furthermore, if $QT = 0$ for some central projection Q, then Q and C_T are orthogonal.

We will now, before finding reasons that the notion of a central support is useful, introduce the aforementioned relevant notation.

Definition 2.6. Let $\mathscr{M}$ be a subset of $B(\mathcal{H})$ and let $\mathfrak{X}$ be a subset of $\mathcal{H}$. We denote the closure of the linear span of

$$\mathscr{M}\mathfrak{X} = \{T\xi \,|\, T \in \mathscr{S},\ \xi \in \mathfrak{X}\} \subseteq \mathcal{H}$$

by $[\mathscr{M}\mathfrak{X}]$; $[\mathscr{M}\mathfrak{X}]$ is called the *essential subspace of* $\mathscr{M}$ *with respect to* $\mathfrak{X}$. If $\mathfrak{X} = \{\xi\}$, we write $[\mathscr{M}\xi]$ instead of $[\mathscr{M}\mathfrak{X}]$.

First follow some facts about essential subspaces.

Lemma 2.13. *Let $\mathscr{M}$ and $\mathscr{N}$ be subsets of $B(\mathcal{H})$ that are closed under multiplication and furthermore satisfy $\mathscr{M} \subseteq \mathscr{N}$. If it holds for subsets $\mathfrak{X}$ and $\mathfrak{Y}$ of $\mathcal{H}$ that $\mathfrak{X} \subseteq [\mathscr{N}\mathfrak{Y}]$, then $[\mathscr{M}\mathfrak{X}] \subseteq [\mathscr{N}\mathfrak{Y}]$. As a consequence, $[\mathscr{M}[\mathscr{M}\mathfrak{X}]] \subseteq [\mathscr{M}\mathfrak{X}]$ for any subset $\mathfrak{X} \subseteq \mathcal{H}$.*

Proof. For non-zero $T \in \mathscr{M}$ and $\xi \in \mathfrak{X}$, we have $\xi \in [\mathscr{N}\mathfrak{Y}]$. For any given $\varepsilon > 0$, we can take $T_1, \ldots, T_n \in \mathscr{N}$ and $\xi_1, \ldots, \xi_n \in \mathfrak{Y}$ such that $\|\xi - \sum_{i=1}^n T_i\xi_i\| < \frac{\varepsilon}{\|T\|}$, whence

$$\|T\xi - \sum_{i=1}^{n} TT_i\xi_i\| \leq \varepsilon.$$

As $TT_i \in \mathscr{N}$ for all $i = 1, \ldots, n$, it follows that $T\xi \in [\mathscr{N}\mathfrak{Y}]$. Hence we can conclude $[\mathscr{M}\mathfrak{X}] \subseteq [\mathscr{N}\mathfrak{Y}]$. $\square$

Lemma 2.14. *Let $\mathscr{M}$ be a self-adjoint subset of $B(\mathcal{H})$ that is closed under multiplication and contains the identity operator $1_{\mathcal{H}}$, and let $\mathfrak{X}$ be a subset of $\mathcal{H}$. Then $[\mathscr{M}\mathfrak{X}]$ is the smallest among all closed subspaces $\mathfrak{Y}$ of $\mathcal{H}$ such that $\mathfrak{X} \subseteq \mathfrak{Y}$ and the projection onto $\mathfrak{Y}$ belongs to the von Neumann algebra $\mathscr{M}'$.*

Proof. Since $\mathscr{M}$ is unital, $\mathfrak{X} \subseteq [\mathscr{M}\mathfrak{X}]$. If P is the projection onto $[\mathscr{M}\mathfrak{X}]$, then for all $T \in \mathscr{M}$ and $\xi \in \mathcal{H}$, we have $TP\xi = PTP\xi$ by Lemma 2.13. Hence $PTP = TP$ for all $T \in \mathscr{M}$. Since $\mathscr{M}$ is self-adjoint it also holds for $T \in \mathscr{M}$ that $PT^*P = T^*P$ and hence $PT = PTP$, so we see that $PT = TP$ and $P \in \mathscr{M}'$. If $\mathfrak{Y}$ is a closed subspace such that $\mathfrak{X} \subseteq \mathfrak{Y}$ and the projection Q onto $\mathfrak{Y}$ belongs to $\mathscr{M}'$, for all $T \in \mathscr{M}$ and $\xi \in \mathfrak{X}$, we have $T\xi = TQ\xi = QT\xi \in \mathfrak{Y}$, so $[\mathscr{M}\mathfrak{X}] \subseteq \mathfrak{Y}$. $\square$

The next proposition is quite surprising – just take a look at it and wonder for a moment.

Proposition 2.15. *Let $\mathscr{M} \subseteq B(\mathcal{H})$ be a von Neumann algebra and let $\mathfrak{X}$ be a subset of $\mathcal{H}$. Let $\mathcal{B} = [\mathscr{M}\mathfrak{X}]$. Then*

$$[\mathscr{M}'\mathcal{B}] = [\mathcal{Z}(\mathscr{M})'\mathfrak{X}].$$

Proof. Since $\mathscr{M} \subseteq \mathcal{Z}(\mathscr{M})'$, we have that the linear span of $\mathscr{M}\mathfrak{X}$ is contained in the linear span of $\mathcal{Z}(\mathscr{M})'\mathfrak{X}$, so $\mathcal{B} \subseteq [\mathcal{Z}(\mathscr{M})'\mathfrak{X}]$. The inclusion $\mathscr{M}' \subseteq \mathcal{Z}(\mathscr{M})'$ then yields $[\mathscr{M}'\mathcal{B}] \subseteq [\mathcal{Z}(\mathscr{M})'\mathfrak{X}]$ by Lemma 2.13. Let P be the projection onto $[\mathscr{M}'\mathcal{B}]$. We will show that $P \in \mathcal{Z}(\mathscr{M})$. By Lemma 2.14, we have $P \in (\mathscr{M}')' = \mathscr{M}$, so we only need to show that $P \in \mathscr{M}'$. For $T \in \mathscr{M}$, $T' \in \mathscr{M}'$ and $\xi \in \mathcal{B}$, we have

$$T(T'\xi) = T'(T\xi) \in T'(\mathcal{B}) \subseteq [\mathscr{M}'\mathcal{B}],$$

as $T\xi \in [\mathscr{M}\mathfrak{X}] = \mathcal{B}$ by Lemma 2.13. Hence T maps the linear span of $\mathscr{M}'\mathcal{B}$ into $[\mathscr{M}'\mathcal{B}]$, so by continuity of T, T maps $[\mathscr{M}'\mathcal{B}]$ into $[\mathscr{M}'\mathcal{B}]$. Therefore $PTP = TP$ for all $T \in \mathscr{M}$. In the same manner as in the previous proof, we see that $PT = TP$, so $P \in \mathcal{Z}(\mathscr{M})$. Hence for $T \in \mathcal{Z}(\mathscr{M})'$ and $\eta \in \mathfrak{X} \subseteq [\mathscr{M}'\mathcal{B}]$, we have $T\eta = TP\eta = PT\eta \in [\mathscr{M}'\mathcal{B}]$ and hence $[\mathcal{Z}(\mathscr{M})'\mathfrak{X}] \subseteq [\mathscr{M}'\mathcal{B}]$ by taking norm closures. $\square$

Why would one prove the above statement, one could again ask. The reason is the following, providing a gateway to introducing a new von Neumann algebra specimen as well as proving some greatly needed facts about it.

Corollary 2.16. *Let $\mathfrak{X}$ be a closed linear subspace of $\mathcal{H}$ such that the projection P onto $\mathfrak{X}$ is contained in a von Neumann algebra $\mathscr{M}$. Then the projection onto $[\mathscr{M}\mathfrak{X}]$ is the central support of P.*

Proof. Let Q be the projection onto $[\mathscr{M}\mathfrak{X}]$. Since $\mathfrak{X} \subseteq [\mathscr{M}\mathfrak{X}]$, we have $P = QP = PQ$. Since P commutes with all operators in $\mathscr{M}'$, then for any $T \in \mathscr{M}'$ and $\xi \in \mathfrak{X}$, we have $T\xi = TP\xi = PT\xi \in \mathfrak{X}$. Hence $[\mathscr{M}'\mathfrak{X}] = \mathfrak{X}$, so by Proposition 2.15, we have $[\mathscr{M}\mathfrak{X}] = [(\mathscr{M}')'\mathfrak{X}] = [\mathcal{Z}(\mathscr{M}')'\mathfrak{X}] = [\mathcal{Z}(\mathscr{M})'\mathfrak{X}]$. By Lemma 2.14, $[\mathscr{M}\mathfrak{X}]$ is the smallest among all closed subspaces $\mathfrak{Y}$ of $\mathcal{H}$ containing $\mathfrak{X}$ such that the projection onto $\mathfrak{Y}$ is in $\mathcal{Z}(\mathscr{M})$. Hence Q is a central projection that majorizes the left and right supports of P. If R is another central projection in $\mathscr{M}$ that majorizes the left and right supports of P, then for all $\xi \in \mathfrak{X}$, we have

$$\xi = P\xi = PS_r(P)\xi = PRS_r(P)\xi = RS_r(P)\xi \in R(\mathcal{H}),$$

so $\mathfrak{X} \subseteq R(\mathcal{H})$. Therefore $[\mathscr{M}\mathfrak{X}] \subseteq R(\mathcal{H})$ and $Q \leq R$, so Q is the central support of P. □

Without further ado, let us construct a new von Neumann algebra.

2.4 Intermezzo 2: The reduced von Neumann algebra

For a non-zero projection $P \in B(\mathcal{H})$ and a subset $\mathscr{S}$ of $B(\mathcal{H})$, then $PT = T$ for all $T \in P\mathscr{S}$. Defining

$$\mathscr{S}_P = \{T|_{P(\mathcal{H})} \,|\, T \in P\mathscr{S}\},$$

it is then clear that the restrictions in $\mathscr{S}_P$ map into $P(\mathcal{H})$, yielding that $\mathscr{S}_P$ is a subset of bounded linear operators on the Hilbert space $P(\mathcal{H})$. If $\mathscr{M}$ is a *-subalgebra of $B(\mathcal{H})$, then $P\mathscr{M}$ and $\mathscr{M}_P$ are *-subalgebras as well.

Proposition 2.17 (The reduced von Neumann algebra)**.** *If $\mathscr{M} \subseteq B(\mathcal{H})$ is a von Neumann algebra and $P \in \mathscr{M}$ is a non-zero projection with $\mathfrak{X} = P(\mathcal{H})$, then the following hold:*

(i) *$\mathscr{M}_P$ and $(\mathscr{M}')_P$ are von Neumann algebras on $\mathfrak{X}$.*
(ii) *the restriction map $P\mathscr{M}P \to \mathscr{M}_P$ given by $T \mapsto T|_{\mathfrak{X}}$ is a *-isomorphism.*
(iii) *$(\mathscr{M}_P)' = (\mathscr{M}')_P$, allowing for the name $\mathscr{M}'_P$ for these equal von Neumann algebras.*
(iv) *if $\mathcal{A} \subseteq B(\mathcal{H})$ is a closed set under multiplication and the adjoint operation that generates $\mathscr{M}$, i.e. $\mathscr{M} = \mathcal{A}''$, then $\mathcal{A}_P$ generates $\mathscr{M}_P$.*
(v) *if $\mathcal{B} \subseteq B(\mathcal{H})$ is a self-adjoint subset generating $\mathscr{M}'$, then $\mathcal{B}_P$ generates $(\mathscr{M}_P)'$.*
(vi) *$B(\mathcal{H})_P = B(P(\mathcal{H}))$.*

$\mathscr{M}_P$ is called the reduced von Neumann algebra *or* corner algebra of $\mathscr{M}$ associated to P.

Proof. (ii) is straightforward; the proof of Lemma A.13 can easily be adjusted to work in this case. In fact, (ii) holds for *any* $P \in B(\mathcal{H})$ and *any* C^*-subalgebra $\mathscr{M} \subseteq B(\mathcal{H})$, as $\mathscr{M}_P$ is also a C^*-algebra in this case. (vi) is also immediate, just by extending any operator on $\mathfrak{X}$ to $\mathcal{H}$ by defining it to be 0 on the orthogonal complement $\mathfrak{X}^\perp$.

It is clear that everything in $\mathscr{M}_P$ commutes with everything in $(\mathscr{M}')_P$, as $P \in \mathscr{M}$. If $T \in B(\mathfrak{X})$ commutes with all $PSP|_{\mathfrak{X}}$ where $S \in \mathcal{B}$, then by defining $T_1 = TP \in B(\mathcal{H})$ we see that

$$T_1 \in \mathcal{B}' = \mathcal{B}''' = \mathscr{M}'' = \mathscr{M},$$

as $\mathcal{B}'$ is a von Neumann algebra whence $T = PT_1|_{\mathfrak{X}} \in \mathscr{M}_P$. Thus $(\mathcal{B}_P)' \subseteq \mathscr{M}_P$. If we for a moment set $\mathcal{B} = \mathscr{M}'$, then we see that $\mathscr{M}_P = ((\mathscr{M}')_P)'$, so $\mathscr{M}_P$ is a von Neumann algebra. Returning to arbitrary self-adjoint subsets $\mathcal{B} \subseteq B(\mathcal{H})$, we have

$$(\mathcal{B}_P)'' \supseteq (\mathscr{M}_P)' = ((\mathscr{M}')_P)'' \supseteq (\mathscr{M}')_P \supseteq \mathcal{B}_P,$$

so $(\mathcal{B}_P)'' = (\mathscr{M}_P)'$, and hence we obtain (v).

Assume now that $1_{\mathcal{H}} \in \mathcal{A}$. We will show that $(\mathcal{A}_P)' \subseteq (\mathscr{M}')_P$. Once we show this, we will have

$$(\mathcal{A}_P)'' \supseteq ((\mathscr{M}')_P)' = \mathscr{M}_P \supseteq \mathcal{A}_P,$$

so that $(\mathcal{A}_P)'' = \mathscr{M}_P$ and hence (iv) holds; we will have

$$(\mathscr{M}')_P \subseteq ((\mathscr{M}')_P)'' = (\mathscr{M}_P)' = (\mathcal{A}_P)' \subseteq (\mathscr{M}')_P,$$

so that $(\mathscr{M})'_P$ is a von Neumann algebra and $(\mathscr{M}')_P = (\mathscr{M}_P)'$, giving us (i) and (iii), almost completing the proof. We therefore have to show that for each $T \in (\mathcal{A}_P)'$ that $T \in (\mathscr{M}')_P$. It suffices to show the result for unitary operators by [31, Theorem 10.6], since $(\mathcal{A}_P)'$ is a von Neumann algebra.

First of all, note that by Corollary 2.16, C_P is the projection onto $[\mathscr{M}\mathfrak{X}]$. Additionally, we have $\mathfrak{X} \subseteq [\mathcal{A}\mathfrak{X}]$ since $1_{\mathcal{H}} \in \mathcal{A}$. If Q is the projection onto $[\mathcal{A}\mathfrak{X}]$, then $Q \in \mathcal{A}' = \mathscr{M}'$ by Lemma 2.14. Hence by the same lemma, we must have $[\mathscr{M}\mathfrak{X}] = [\mathcal{A}\mathfrak{X}]$, so $Q = C_P$.

Let $U \in (\mathcal{A}_P)'$ be unitary and let $\xi_1, \dots, \xi_n \in \mathfrak{X}$ and $T_1, \dots, T_n \in \mathcal{A}$. Defining $T_{ji} = PT_j^* T_i P|_{\mathfrak{X}}$ for all $i, j = 1, \dots, n$, we have $T_{ji} \in \mathcal{A}_P$, so U commutes with all T_{ji}. Hence

$$\begin{aligned}\left\| \sum_{i=1}^n T_i U \xi_i \right\|^2 &= \sum_{i,j=1}^n \langle T_i P U \xi_i, T_j P U \xi_j \rangle = \sum_{i,j=1}^n \langle P T_j^* T_i P U \xi_i, U \xi_j \rangle = \sum_{i,j=1}^n \langle T_{ji} U \xi_i, U \xi_j \rangle \\ &= \sum_{i,j=1}^n \langle T_{ji} \xi_i, \xi_j \rangle = \left\| \sum_{i=1}^n T_i \xi_i \right\|^2.\end{aligned}$$

By Proposition A.1, there exists a unique isometric linear operator $S \colon [\mathcal{A}\mathfrak{X}] \to [\mathcal{A}\mathfrak{X}]$ satisfying

$$S\left(\sum_{i=1}^n T_i \xi_i\right) = \sum_{i=1}^n T_i U \xi_i, \quad T_1, \dots, T_n \in \mathcal{A},\ \xi_1, \dots, \xi_n \in \mathfrak{X},$$

which we can extend to an operator $S \in B(\mathcal{H})$ by defining it to be zero on the orthogonal complement of $[\mathcal{A}\mathfrak{X}]$. In addition, $SQ = QS = S$. For any $T \in \mathcal{A}$, $T_1, \dots, T_n \in \mathcal{A}$, and $\xi_1, \dots, \xi_n \in \mathfrak{X}$, we have

$$ST\left(\sum_{i=1}^n T_i \xi_i\right) = S\left(\sum_{i=1}^n T T_i \xi_i\right) = \sum_{i=1}^n T T_i U \xi_i = T\left(\sum_{i=1}^n T_i U \xi_i\right) = TS\left(\sum_{i=1}^n T_i U \xi_i\right),$$

so ST and TS agree on $[\mathcal{A}\mathfrak{X}]$ by continuity. Hence for any $\xi \in \mathcal{H}$, we have

$$TS\xi = TSQ\xi = STQ\xi = SQT\xi = ST\xi,$$

since Q projects onto $[\mathcal{A}\mathfrak{X}]$ and Q is central. Therefore $S \in \mathcal{A}' = \mathscr{M}'$. Moreover, $S\xi = U\xi$ for all $x \in \mathfrak{X}$, so $U = PSP|_{\mathfrak{X}} \in (\mathscr{M}')_P$.

If $1_{\mathcal{H}} \notin \mathcal{A}$ then by putting $\mathcal{A}_1 = \mathcal{A} \cup \{1_{\mathcal{H}}\}$, we have $((\mathcal{A}_1)_P)' \subseteq (\mathscr{M}')_P$ on the grounds of what we have just proved. As $(\mathcal{A}_1)_P = \mathcal{A}_P \cup \{1_{\mathfrak{X}}\}$, then by assuming that $T \in (\mathcal{A}_P)'$, then T clearly commutes with everything in $(\mathcal{A}_1)_P$, so we see that $(\mathcal{A}_P)' \subseteq (\mathscr{M}')_P$. Hence it follows from the case of $1_{\mathcal{H}} \in \mathcal{A}$ that (i), (iii) and (iv) hold for the case $1_{\mathcal{H}} \notin \mathcal{A}$. □

Let $\mathscr{M}$ be a von Neumann algebra and $P \in \mathscr{M}$ a projection. The map $\mathscr{M}' \to P\mathscr{M}'P$ given by $T' \mapsto PT'P = PT$ is then a surjective *-homomorphism. Combining this with the isomorphism of (ii) in the above proposition, we obtain a surjective *-homomorphism $\mathscr{M}' \to (\mathscr{M})'_P$ given by $T' \mapsto PT'|_{\mathfrak{X}}$. Suppose that $C_P = 1_{\mathscr{M}}$. If $\mathfrak{X} = P(\mathcal{H})$, then by Corollary 2.16, $\mathcal{H} = [\mathscr{M}\mathfrak{X}]$. If $T' \in \mathscr{M}'$ is such that $PT'|_{\mathfrak{X}} = 0$, then $T'(\mathfrak{X}) = PT'(\mathfrak{X}) = \{0\}$, so $T'T(\mathfrak{X}) = TT'(\mathfrak{X}) = \{0\}$ for all $T \in \mathscr{M}$. Since $\mathcal{H} = [\mathscr{M}\mathfrak{X}]$, T' is 0 on a dense subset of $\mathcal{H}$, and therefore $T' = 0$. We can therefore conclude the following:

Proposition 2.18. *If $\mathscr{M}$ is a von Neumann algebra and $P \in \mathscr{M}$ is a projection with $C_P = 1$, then $\mathscr{M}'$ and $(\mathscr{M})'_P$ are *-isomorphic.*

To finish off this section with a bang, we will investigate what can be derived from the simple notion of reduced von Neumann algebras, with no extra strings attached. The first important fact is this very handy isomorphism theorem.

Proposition 2.19. *Let $\mathscr{M} \subseteq B(\mathcal{H})$ be a von Neumann algebra and let $(P_\alpha)_{\alpha\in A}$ any family of non-zero pairwise orthogonal projections in $\mathscr{M}'$. If $\sum_{\alpha\in A} P_\alpha$ converges strongly to the identity, then $\mathscr{M}$ is *-isomorphic to the direct sum $\bigoplus_{\alpha\in A} \mathscr{M}_{P_\alpha}$ by the *-isomorphism $\Omega\colon \mathscr{M} \to \bigoplus_{\alpha\in A} \mathscr{M}_{P_\alpha}$ given by*

$$\Omega(T) = (P_\alpha T|_{P_\alpha(\mathcal{H})})_{\alpha\in A}.$$

Proof. Note first that $\mathcal{H}$ is isomorphic to the Hilbert space $\bigoplus_{\alpha\in A} \mathcal{H}_\alpha$ where $\mathcal{H}_\alpha = P_\alpha(\mathcal{H})$ by the unitary operator

$$U\colon \mathcal{H} \ni \xi \mapsto (P_\alpha \xi)_{\alpha\in A}.$$

The only nontrivial hurdle to overcome in a proof of this is to prove that U is surjective. If $(\xi_\alpha)_{\alpha\in A}$, then $(\sum_{\alpha\in F} \xi_\alpha)_{F\subseteq A}$ is a Cauchy net where the F are finite subsets of A. To see this, let $S \subseteq A$ be a finite subset such that

$$\sum_{\alpha\in A\setminus S} \|\xi_\alpha\|^2 < \varepsilon^2.$$

Indeed, for finite subsets F and G of A with $S \subseteq F$ and $S \subseteq G$, then we have

$$\begin{aligned}
\left\| \sum_{\alpha\in F} \xi_\alpha - \sum_{\alpha\in G} \xi_\alpha \right\|^2 &= \left\| \sum_{\alpha\in F\setminus G} \xi_\alpha - \sum_{\alpha\in G\setminus F} \xi_\alpha \right\|^2 \\
&= \sum_{\alpha\in F\setminus G} \|\xi_\alpha\|^2 + \sum_{\alpha\in G\setminus F} \|\xi_\alpha\|^2 \\
&= \sum_{\alpha\in F\cup G} \|\xi_\alpha\|^2 - \sum_{\alpha\in F\cap G} \|\xi_\alpha\|^2 \\
&\le \sum_{\alpha\in A} \|\xi_\alpha\|^2 - \sum_{\alpha\in S} \|\xi_\alpha\|^2 \\
&< \varepsilon^2
\end{aligned}$$

as the ξ_α's are orthogonal. Hence there exists $\xi \in \mathcal{H}$ such that $\xi = \lim_F \sum_{\alpha\in F} \xi_\alpha = \sum_{\alpha\in A} \xi_\alpha$. For $\alpha \in A$, note that $P_\alpha \xi = \sum_{\beta\in A} P_\alpha \xi_\beta = P_\alpha \xi_\alpha = \xi_\alpha$, hence proving surjectivity of U and that

$$U^{-1}((\xi_\alpha)_{\alpha\in A}) = \sum_{\alpha\in A} \xi_\alpha.$$

Going abruptly back to Ω, it is not hard to see that it is unital, linear and multiplicative. For $T \in \mathscr{M}$, then for $(\xi_\alpha)_{\alpha\in A}, (\eta_\alpha)_{\alpha\in A} \in \bigoplus_{\alpha\in A} \mathcal{H}_\alpha$ we have

$$\begin{aligned}
\langle (P_\alpha T^*|_{\mathcal{H}_\alpha})_{\alpha\in A} (\xi_\alpha)_{\alpha\in A}, (\eta_\alpha)_{\alpha\in A} \rangle &= \sum_{\alpha\in A} \langle T^* \xi_\alpha, \eta_\alpha \rangle \\
&= \sum_{\alpha\in A} \langle \xi_\alpha, T\eta_\alpha \rangle \\
&= \langle (\xi_\alpha)_{\alpha\in A}, (P_\alpha T|_{\mathcal{H}_\alpha})_{\alpha\in A} (\eta_\alpha)_{\alpha\in A} \rangle,
\end{aligned}$$

so Ω preserves involutions. If $P_\alpha S|_{\mathcal{H}_\alpha} = P_\alpha T|_{\mathcal{H}_\alpha}$ for $S, T \in \mathscr{M}$ and all $\alpha \in A$, then

$$S\xi = \sum_{\alpha\in A} P_\alpha S P_\alpha \xi = \sum_{\alpha\in A} P_\alpha T P_\alpha \xi = T\xi$$

for all $\xi \in \mathcal{H}$, so $S = T$. Let $T' = (T_\alpha)_{\alpha\in A} \in \bigoplus_{\alpha\in A} \mathscr{M}_{P_\alpha}$, and define $T\colon \mathcal{H} \to \mathcal{H}$ by $T = U^{-1} T' U$. T is bounded and linear, and for any $S \in \mathscr{M}'$, we see that $P_\alpha S|_{\mathcal{H}_\alpha} \subseteq \mathscr{M}'_{P_\alpha}$ for all $\alpha \in A$, so for all $\xi \in \mathcal{H}$ we have

$$TS\xi = \sum_{\alpha\in A} T_\alpha P_\alpha S\xi = \sum_{\alpha\in A} T_\alpha P_\alpha S P_\alpha \xi = \sum_{\alpha\in A} P_\alpha S T_\alpha P_\alpha \xi = S\left(\sum_{\alpha\in A} T_\alpha P_\alpha \xi\right) = ST\xi.$$

Hence $T \in \mathscr{M}'' = \mathscr{M}$ and as

$$\Omega(T)\xi = (P_\alpha U^{-1} T' U \xi_\alpha)_{\alpha\in A} = (P_\alpha \sum_{\beta\in A} T_\alpha \xi_\alpha)_{\alpha\in A} = (T_\alpha \xi_\alpha)_{\alpha\in A} = T'\xi$$

for $\xi = (\xi_\alpha)_{\alpha\in A}$, we find that Ω is surjective. $\square$

Before introducing the next result, concerning tensor products of reduced von Neumann algebras, some discussion is required. For Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ and closed subspaces $\mathcal{H}_0 \subseteq \mathcal{H}$ and $\mathcal{K}_0 \subseteq \mathcal{K}$, then it is immediate that $\mathcal{H}_0 \otimes \mathcal{K}_0$ is isomorphic to the norm closure of $\mathcal{H}_0 \odot \mathcal{K}_0$, the latter being considered as a subspace of $\mathcal{H} \otimes \mathcal{K}$. Hence we can consider $\mathcal{H}_0 \otimes \mathcal{K}_0$ as a closed subspace of $\mathcal{H} \otimes \mathcal{K}$. Hence if $P_1 \in B(\mathcal{H})$ and $P_2 \in B(\mathcal{K})$ are projections, then for any $\omega = \sum_{i=1}^n \xi_i \otimes \eta_i \in \mathcal{H} \odot \mathcal{K}$, we have

$$(P_1 \otimes P_2)\omega = \sum_{i=1}^n P_1\xi_i \otimes P_2\eta_i \in P_1(\mathcal{H}) \otimes P_2(\mathcal{K}),$$

Hence $(P_1 \otimes P_2)(\mathcal{H} \otimes \mathcal{K}) \subseteq P_1(\mathcal{H}) \otimes P_2(\mathcal{K})$ since such ω are dense in $\mathcal{H} \otimes \mathcal{K}$. The reverse inclusion is clear, so we have $(P_1 \otimes P_2)(\mathcal{H} \otimes \mathcal{K}) = P_1(\mathcal{H}) \otimes P_2(\mathcal{K})$ as subsets of $\mathcal{H} \otimes \mathcal{K}$.

Proposition 2.20. *Let $\mathscr{M} \subseteq B(\mathcal{H})$ and $\mathscr{N} \subseteq B(\mathcal{K})$ be von Neumann algebras. If P_1 and P_2 are projections of $\mathscr{M}$ and $\mathscr{N}$, respectively, then*

$$(\mathscr{M} \mathbin{\overline{\otimes}} \mathscr{N})_{P_1 \otimes P_2} = \mathscr{M}_{P_1} \mathbin{\overline{\otimes}} \mathscr{N}_{P_2}$$

as subsets of $B(P_1(\mathcal{H}) \otimes P_2(\mathcal{K}))$.

Proof. $\mathscr{M} \mathbin{\overline{\otimes}} \mathscr{N}$ is generated by operators of the form $\sum_{i=1}^n S_i \otimes T_i$ where $S_i \in \mathscr{M}$ and $T_i \in \mathscr{N}$ for all $i = 1, \ldots, n$. Hence by Proposition 2.17, $(\mathscr{M} \mathbin{\overline{\otimes}} \mathscr{N})_{P_1 \otimes P_2}$ is generated by operators of the form

$$(P_1 \otimes P_2)\left(\sum_{i=1}^n S_i \otimes T_i\right)(P_1 \otimes P_2)|_{(P_1 \otimes P_2)(\mathcal{H} \otimes \mathcal{K})} = \sum_{i=1}^n (P_1 S_i P_1)|_{P_1(\mathcal{H})} \otimes (P_2 T_i P_2)|_{P_2(\mathcal{K})} \in \mathscr{M}_{P_1} \mathbin{\overline{\otimes}} \mathscr{N}_{P_2},$$

so $(\mathscr{M} \mathbin{\overline{\otimes}} \mathscr{N})_{P_1 \otimes P_2} \subseteq \mathscr{M}_{P_1} \mathbin{\overline{\otimes}} \mathscr{N}_{P_2}$. The reverse inclusion is obtained by going backwards. □

The big deal about reduced von Neumann algebras is of course that we do not often know a lot about a given von Neumann algebra, but reducing it by sometimes more than one appropriate projection might yield a great deal of knowledge one would not be able to acquire at the outset. The third and final intermezzo is a great example of this.

2.5 Closure properties of von Neumann algebras

Von Neumann algebras are usually defined by means of the strong or weak operator topology. In this section we shall not only see what von Neumann algebras have to do with the topologies defined in this chapter, but prove a more flexible version of a well-known theorem for von Neumann algebras, namely von Neumann's bicommutant theorem (originally stated in Theorem 0.9). The *only* fact that we will take for granted here is that commutants of self-adjoint sets are always weakly closed, the proof of which is easy but can be found in [31, Proposition 18.1].

Before progressing, let us take a deep breath and introduce three relevant concepts. The third will be the center of focus in this section, while the first two will be the focal point in the next couple of sections, including the third intermezzo.

Definition 2.7. Let $\mathscr{M}$ be a *-subalgebra of $B(\mathcal{H})$ and let $\mathfrak{X} \subseteq \mathcal{H}$. $\mathfrak{X}$ is *cyclic for $\mathscr{M}$* if $[\mathscr{M}\mathfrak{X}] = \mathcal{H}$. $\mathfrak{X}$ is *separating* for $\mathscr{M}$ if for any $T \in \mathscr{M}$, $T\xi = 0$ for all $\xi \in \mathfrak{X}$ implies $T = 0$. If $\mathfrak{X} = \{\xi\}$, we say that ξ is a cyclic or separating vector for $\mathscr{M}$. Note that if $\mathscr{M}$ is a von Neumann algebra, then a cyclic or separating vector for $\mathscr{M}$ must necessarily be non-zero.

The connection between the two above concepts is the following.

Proposition 2.21. *Let $\mathscr{M} \subseteq B(\mathcal{H})$ be a *-subalgebra and $\mathfrak{X} \subseteq \mathcal{H}$ a subset. Then the following conditions are equivalent:*

(i) *$\mathfrak{X}$ is cyclic for $\mathscr{M}$.*
(ii) *$\mathfrak{X}$ is separating for $\mathscr{M}'$.*

Proof. Assume that $\mathfrak{X}$ is cyclic for $\mathscr{M}$ and that $T \in \mathscr{M}'$ satisfies $T\xi = 0$ for all $\xi \in \mathfrak{X}$. For any $S \in \mathscr{M}$ and $\xi \in \mathfrak{X}$, we then have $T(S\xi) = ST\xi = 0$, so $T[\mathscr{M}\mathfrak{X}] = \{0\}$, implying $T = 0$. If $\mathfrak{X}$ is separating for $\mathscr{M}'$, let P denote the orthogonal projection onto $[\mathscr{M}\mathfrak{X}]$. Then $P \in \mathscr{M}'$ by Lemma 2.14 and $(1_{\mathcal{H}} - P)\mathfrak{X} = \{0\}$, so $1_{\mathcal{H}} - P = 0$ and hence $[\mathscr{M}\mathfrak{X}] = \mathcal{H}$. □

The weakest (as in largest) possible cyclic subset is of course $\mathcal{H}$ itself, and a special name is in order if $\mathcal{H}$ is cyclic for some *-subalgebra of $B(\mathcal{H})$.

Definition 2.8. If $\mathscr{M}$ is a *-subalgebra of $B(\mathcal{H})$, $\mathscr{M}$ is said to be *nondegenerate* on $\mathcal{H}$ if $\mathcal{H}$ is cyclic for $\mathscr{M}$, i.e. if $[\mathscr{M}\mathcal{H}] = \mathcal{H}$.

Note that a *-subalgebra of $B(\mathcal{H})$ is nondegenerate at once if it contains the identity operator.

Lemma 2.22. *Let $\mathscr{M}$ be a *-subalgebra of $B(\mathcal{H})$. Then for $\xi \in \mathcal{H}$, the following are equivalent:*

(i) $\xi \in [\mathscr{M}\mathcal{H}]^\perp$.
(ii) $T\xi = 0$ *for all* $T \in \mathscr{M}$.

Proof. If $T\xi = 0$ for all $T \in \mathscr{M}$, then for all $\eta \in \mathcal{H}$ and $T \in \mathscr{M}$ we have

$$\langle \xi, T\eta \rangle = \langle T^*\xi, \eta \rangle = 0$$

since $\mathscr{M}$ is self-adjoint, so $\xi \in [\mathscr{M}\mathcal{H}]^\perp$. If $\xi \in [\mathscr{M}\mathcal{H}]^\perp$, we see that

$$\|T\xi\|^2 = \langle \xi, T^*T\xi \rangle = 0$$

and hence $T\xi = 0$ for all $T \in \mathscr{M}$. ☐

Corollary 2.23. *Let $\mathscr{M}$ be a *-subalgebra of $B(\mathcal{H})$. Then the following are equivalent:*

(i) $\mathscr{M}$ *is nondegenerate.*
(ii) *For any non-zero $\xi \in \mathcal{H}$, there exists $T \in \mathscr{M}$ such that $T\xi \neq 0$.*

Proof. $\mathscr{M}$ is nondegenerate if and only $[\mathscr{M}\mathcal{H}]^\perp = \{0\}$. ☐

Lemma 2.24. *Let $\mathscr{M}$ be a *-subalgebra of $B(\mathcal{H})$ and let $\mathcal{B} = \{x \in \mathcal{H} \mid Tx = 0 \text{ for all } T \in \mathscr{M}\}$. Then $[\mathscr{M}\mathcal{H}]$ and $\mathcal{B}$ are orthogonal complements in $\mathcal{H}$ and if P denotes the orthogonal projection onto $[\mathscr{M}\mathcal{H}]$, then $T = TP = PT$ for all $T \in \mathscr{M}$.*

Proof. By Lemma 2.22, $[\mathscr{M}\mathcal{H}]^\perp = \mathcal{B}$. If Q denotes the orthogonal projection onto $\mathcal{B}$, then for all $T \in \mathscr{M}$ we have $TQ = 0$, so $T = T(1_\mathcal{H} - Q) = TP$. This implies $T^* = T^*P$ for $T \in \mathscr{M}$, so by taking adjoints, we see that $T = PT$. ☐

The next two lemmas concern properties of nondegenerate *-subalgebras.

Lemma 2.25. *Let $\mathscr{M}$ be a nondegenerate *-subalgebra of $B(\mathcal{H})$. Then $\xi \in [\mathscr{M}\xi]$ for all $\xi \in \mathcal{H}$.*

Proof. Let P be the orthogonal projection onto $[\mathscr{M}\xi]$. Then for all $T \in \mathscr{M}$, we have $PTP = TP$ and by taking adjoints, we see that $PT^* = PT^*P$. Because $\mathscr{M}$ is a self-adjoint set, we see that

$$PT = TP = PTP$$

for all $T \in \mathscr{M}$, i.e. $P \in \mathscr{M}'$. If $\xi' = P\xi$ and $\xi'' = (1_\mathcal{H} - P)\xi$ then $\xi = \xi' + \xi''$, but since

$$T\xi' = TP\xi = PT\xi = T\xi$$

for all $T \in \mathscr{M}$, then $T\xi'' = 0$ for all $T \in \mathscr{M}$. Hence $\xi'' = 0$ by Corollary 2.23, so $\xi = \xi' \in [\mathscr{M}\xi]$. ☐

Lemma 2.26. *Let $\mathscr{M}$ be a nondegenerate *-subalgebra of $B(\mathcal{H})$. Then for any $\xi \in \mathcal{H}$, $S \in \mathscr{M}''$ and $\varepsilon > 0$, there exists $T \in \mathscr{M}$ such that*

$$\|(S - T)\xi\| < \varepsilon.$$

In particular, the strong operator closure of $\mathscr{M}$ contains the identity operator.

Proof. Let P be the orthogonal projection onto $[\mathscr{M}\xi]$. We saw above that $P \in \mathscr{M}'$, so P commutes with everything in $\mathscr{M}''$. Therefore $PSP = SP$, so $S\xi \in S[\mathscr{M}\xi] \subseteq [\mathscr{M}\xi]$ by the above lemma, yielding the result. ☐

In proofs of the "simpler" version of the bicommutant theorem, one has a stronger version of the above lemma in the case where the *-subalgebra is unital, so one might wonder if the following holds:

Lemma 2.27. *Let $\mathscr{M}$ be a nondegenerate *-subalgebra of $B(\mathcal{H})$. Then for any $\xi_1, \dots, \xi_n \in \mathcal{H}$, $S \in \mathscr{M}''$ and $\varepsilon > 0$, there exists $T \in \mathscr{M}$ such that*

$$\|(S - T)\xi_i\| < \varepsilon, \quad i = 1, \dots, n.$$

In fact it does hold, and the following somewhat messy lemma will help us out a lot.

Lemma 2.28. *Let $\mathscr{M}$ be a nondegenerate *-subalgebra of $B(\mathcal{H})$ and let I be a non-empty set. Moreover, let $\Delta\colon B(\mathcal{H}) \to B(\mathcal{H}^I)$ be given by*

$$\Delta(T)(\xi_i)_{i\in I} = (T\xi_i)_{i\in I}, \quad (\xi_i)_{i\in I} \in \mathcal{H}^I.$$

Then Δ is a unital *-homomorphism, $\Delta(\mathscr{M})$ is a nondegenerate *-subalgebra of $B(\mathcal{H}^I)$ and we have an inclusion $\Delta(\mathscr{M}'') \subseteq \Delta(\mathscr{M})''$.

Proof. First of all Δ is well-defined: we indeed have $\Delta(T) \in B(\mathcal{H}^I)$ with $\|\Delta(T)\| \le \|T\|$ for all $T \in B(\mathcal{H})$. It is straightforward to check that Δ is unital, linear and multiplicative; furthermore for all $(\xi_i)_{i\in I}, (\eta_i)_{i\in I} \in \mathcal{H}^I$, we note that

$$\langle \Delta(T)(\xi_i)_{i\in I}, (\eta_i)_{i\in I}\rangle = \sum_{i\in I} \langle T\xi_i, \eta_i\rangle = \sum_{i\in I}\langle \xi_i, T^*\eta_i\rangle = \langle (\xi_i)_{i\in I}, \Delta(T^*)(\eta_i)_{i\in I}\rangle,$$

so $\Delta(T)^* = \Delta(T^*)$ for all $T \in B(\mathcal{H})$. Hence Δ is a *-homomorphism. Additionally, if $\xi = (\xi_i)_{i\in I} \in \mathcal{H}^I$ is non-zero, then there exists $i \in I$ such that $\xi_i \neq 0$. Because $\mathscr{M}$ is nondegenerate, Corollary 2.23 yields $T \in \mathscr{M}$ such that $T\xi_i \neq 0$, hence implying $\Delta(T)\xi \neq 0$, so $\Delta(\mathscr{M})$ is nondegenerate by the same corollary. Finally, the equation (1.2) in the proof of Proposition 1.33(ii), p. 17, provides the last inclusion. □

Proof of Lemma 2.27. Define $\Delta\colon B(\mathcal{H}) \to B(\mathcal{H}^n)$ by $\Delta(T)(\eta_1, \dots, \eta_n) = (T\eta_1, \dots, T\eta_n)$. Then by Lemma 2.28, $\Delta(\mathscr{M})$ is a nondegenerate *-subalgebra of $B(\mathcal{H}^n)$ with $\Delta(\mathscr{M}'') \subseteq \Delta(\mathscr{M})''$. Apply Lemma 2.26 to the *-algebra $\Delta(\mathscr{M})$ to obtain $T \in \mathscr{M}$ such that

$$\|\Delta(S) - \Delta(T)(\xi_1, \dots, \xi_n)\| < \varepsilon,$$

immediately yielding what we want. □

All of this leads to this *really* important density theorem.

Theorem 2.29 (The von Neumann density theorem)**.** *Let $\mathscr{M}$ be a nondegenerate *-subalgebra of $B(\mathcal{H})$. Then*

$$\overline{\mathscr{M}} = \mathscr{M}''$$

where $\overline{\mathscr{M}}$ denotes the closure of $\mathscr{M}$ in any one of the weak, strong, ultraweak or ultrastrong operator topologies.

Proof. Since the ultrastrong closure of $\mathscr{M}$ is contained in any of the closures of $\mathscr{M}$ in the above topologies and $\mathscr{M}''$ is weakly closed, it suffices to show that $\mathscr{M}''$ is contained in the ultrastrong closure of $\mathscr{M}$. Define $\Delta\colon B(\mathcal{H}) \to B(\mathcal{H}^{\mathbb{N}})$ by

$$\Delta(T)(\xi_n)_{n\ge1} = (T\xi_n)_{n\ge1}.$$

Then $\Delta(\mathscr{M})$ is a nondegenerate *-subalgebra by Lemma 2.28. Let $T \in \mathscr{M}''$ and

$$\xi^1 = (\xi^1_n)_{n\ge1}, \; \dots, \; \xi^m = (\xi^m_n)_{n\ge1}$$

be sequences in $\mathcal{H}$ satisfying $\sum_{n=1}^\infty \|\xi^i_n\|^2 < \infty$ for all $i = 1, \dots, n$, i.e. $\xi^i \in \mathcal{H}^{\mathbb{N}}$ for all $i = 1, \dots, n$. For any $\varepsilon > 0$, then by applying Lemma 2.27 to the *-subalgebra $\Delta(\mathscr{M})$ of $B(\mathcal{H}^{\mathbb{N}})$, there exists $S \in \mathscr{M}$ such that

$$\varepsilon > \|(\Delta(S) - \Delta(T))(\xi^i_n)_{n\ge1}\| = \left[\sum_{n=1}^\infty \|(S-T)\xi^i_n\|^2\right]^{1/2}$$

for all $i = 1, \dots, n$. From this we infer that every ultrastrong neighbourhood of T contains elements of $\mathscr{M}$, so it follows that T is contained in the ultrastrong closure of $\mathscr{M}$. Hence the result follows. □

If the preceding result could in any way be likened to the Sage's blessing of the earth in the first part of Igor Stravinsky's *Le Sacre du Printemps*, then the following corollary is the ecstatic dancing of the tribes therein.

Theorem 2.30 (The von Neumann bicommutant theorem). *Let $\mathscr{M}$ be a nondegenerate *-subalgebra of $B(\mathcal{H})$. Then the following are equivalent:*

(i) $\mathscr{M} = \mathscr{M}''$.
(ii) *(resp.* (ii.a)*) $\mathscr{M}$ (resp. $(\mathscr{M})_1$) is weakly closed.*
(iii) *(resp.* (iii.a)*) $\mathscr{M}$ (resp. $(\mathscr{M})_1$) is strongly closed.*
(iv) *(resp.* (iv.a)*) $\mathscr{M}$ (resp. $(\mathscr{M})_1$) is ultraweakly closed.*
(v) *(resp.* (v.a)*) $\mathscr{M}$ (resp. $(\mathscr{M})_1$) is ultrastrongly closed.*

If any of the above conditions hold, $\mathscr{M}$ is a von Neumann algebra.

Proof. The implications (iv) $\Leftrightarrow$ (v), (iv) $\Rightarrow$ (iv.a), (v) $\Rightarrow$ (v.a) and (ii.a) $\Leftrightarrow$ (iii.a) $\Leftrightarrow$ (iv.a) $\Leftrightarrow$ (v.a) follow immediately from Theorem 2.11. If $(\mathscr{M})_1$ is ultraweakly closed, then $(\mathscr{M})_1 \subseteq (B(\mathcal{H}))_1$ is also ultraweakly compact by Corollary 2.9. Hence $(\mathscr{M})_r$ is ultraweakly compact for all $r > 0$ (since the map $T \mapsto rT$ is ultraweakly-to-ultraweakly continuous), so Theorem 2.11 yields that $\mathscr{M}$ is ultraweakly closed. Hence the conditions (ii.a), (iii.a), (iv), (iv.a), (v) and (v.a) are equivalent. Finally, the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (v) are trivial, so it suffices to show (v) $\Rightarrow$ (i), but this follows immediately from von Neumann's density theorem. $\square$

Hence von Neumann algebras are closed in any of the operator topologies defined, and in order to check whether a nondegenerate *-subalgebra of $B(\mathcal{H})$ is a von Neumann algebra, one only needs consider its closed unit ball.

The above theorem is perhaps the greatest testament to how powerful the von Neumann density theorem really is. The next result is another great offshoot of that theorem, comparable to the effect of buying a new deodorant.

Lemma 2.31. *Let $\mathscr{M}$ be an ultraweakly closed *-subalgebra of $B(\mathcal{H})$. Then the orthogonal projection P onto $[\mathscr{M}\mathcal{H}]$ belongs to $\mathscr{M}$ and majorizes any other projection in $\mathscr{M}$.*

Proof. Let $\mathfrak{X} = [\mathscr{M}\mathcal{H}]$. Lemma 2.24 told us that $PT = TP = T$ for all $T \in \mathscr{M}$, so $P\mathscr{M}P = \mathscr{M}$. As $P|_{\mathfrak{X}}$ is the identity operator on $\mathfrak{X}$, then $\mathscr{M}_P$ is a unital and hence nondegenerate *-subalgebra on $\mathfrak{X}$. As the map $\mathscr{M} \to \mathscr{M}_P$ given by $T \mapsto T|_{\mathfrak{X}}$ is obviously ultraweakly-to-ultraweakly continuous and a surjective isometry by the proof of Proposition 2.17(ii), and $(\mathscr{M})_1$ is ultraweakly compact by Corollary 2.9, it follows that $(\mathscr{M}_P)_1$ is ultraweakly compact and hence $\mathscr{M}_P$ is ultraweakly closed by von Neumann's bicommutant theorem. Therefore $1_{\mathfrak{X}} \in \mathscr{M}_P$ by the von Neumann density theorem, so there is $Q \in \mathscr{M}$ such that $Q|_{\mathfrak{X}} = 1_{\mathfrak{X}}$. For $\xi \in \mathcal{H}$, write $\xi = \xi_1 + \xi_2$ with $\xi_1 \in \mathfrak{X}$ and $\xi_2 \in \mathfrak{X}^\perp$; then

$$Q\xi = Q\xi_1 + Q\xi_2 = \xi_1 + QP\xi_2 = \xi_1 = P\xi,$$

so $P = Q \in \mathscr{M}$ and therefore P is the greatest projection of $\mathscr{M}$. $\square$

Proposition 2.32. *Let $\mathscr{M} \subseteq B(\mathcal{H})$ be a von Neumann algebra and let $\mathfrak{J}$ be an ultraweakly closed left ideal of $\mathscr{M}$. Then $\mathscr{M}$ is weakly closed, and there exists a unique projection $P \in \mathscr{M}$ such that $\mathfrak{J} = \{T \in \mathscr{M} \mid T = TP\}$. If $\mathfrak{J}$ is two-sided, then P is a central projection.*

Proof. Let $\mathfrak{K} = \mathfrak{J} \cap \mathfrak{J}^*$; then $\mathfrak{K}$ is an ultraweakly closed *-subalgebra of $\mathscr{M}$. By virtue of Lemma 2.31, let P be the greatest projection of $\mathfrak{K}$, and let

$$\mathfrak{J}_0 = \{T \in \mathscr{M} \mid T = TP\}.$$

We know that $P \in \mathfrak{J}$, so if $T \in \mathfrak{J}_0$, then obviously $T \in \mathfrak{J}$ since $\mathfrak{J}$ is a left ideal. On the other hand, if $T \in \mathfrak{J}$ with polar decomposition $T = U|T|$, then $|T| = U^*T \in \mathfrak{J}$. Hence $|T| \in \mathfrak{K}$, so $|T| = |T|P$ and therefore

$$TP = U|T|P = U|T| = T,$$

implying $\mathfrak{J} = \mathfrak{J}_0$. If Q is a projection of $\mathscr{M}$ such that $\mathfrak{J} = \{T \in \mathscr{M} \mid T = TQ\}$, then $P = PQ$ and $Q = QP$, so $P = P^* = (PQ)^* = QP = Q$. If $\mathfrak{J}$ is two-sided, then for any $T \in \mathscr{M}$, $PT \in \mathfrak{J}$, so $PT = PTP$. Using this for T and T^*, we see that

$$PT = PTP = (PT^*P)^* = (PT^*)^* = TP,$$

so P is central. □

The last result of this chapter is Kaplansky's density theorem, generalizing [31, Theorem 19.5].

Theorem 2.33 (Kaplansky's density theorem). *Let $\mathscr{M}$ be a *-subalgebra of $B(\mathcal{H})$, and let $\mathscr{N}$ be the strong (or weak) operator closure of $\mathscr{M}$ and let $r > 0$. Then*

(i) *$(\mathscr{M})_r$ is strongly dense in $(\mathscr{N})_r$;*
(ii) *if T is a self-adjoint operator in $(\mathscr{N})_r$, T is contained in the strong operator closure of the set of self-adjoint operators in $(\mathscr{M})_r$;*
(iii) *if $\mathscr{N}$ is a von Neumann algebra and T is a positive operator in $(\mathscr{N})_r$, T is contained in the strong operator closure of the set of positive operators in $(\mathscr{M})_r$.*

Proof. The proof contained in [31, Theorem 19.5] (or [14, Theorem 5.3.5] for that matter) does not require $\mathscr{M}$ to be a C^*-algebra, and the replacement of $(\mathscr{M})_1$ by $(\mathscr{M})_r$ requires the proof therein to change the *strongly continuous real function vanishing at infinity* by

$$t \mapsto \begin{cases} t & |t| \le r \\ \frac{r^2}{t} & |t| > r. \end{cases}$$

For (iii), if we assume that $T \in (\mathscr{N})_r$ and $T \ge 0$, then since $\mathscr{N}$ is a unital C^*-algebra, there exists $S \in \mathscr{N}_{\mathrm{sa}}$ with $\|S\| \le r^{1/2}$ such that $S^2 = T$. Hence by (ii) there exists a net of self-adjoint operators $(S_\alpha)_{\alpha\in A}$ in $(\mathscr{M})_{r^{1/2}}$ converging strongly to S. Therefore $S_\alpha^2 \to T$ strongly, as the net $(S_\alpha)_{\alpha\in A}$ is bounded, so by setting $T_\alpha = S_\alpha^2$, then because T_α is positive and $\|T_\alpha\| \le r$ for all $\alpha \in A$, we have found a net of positive operators in $\mathscr{M}$ converging strongly to T, bounded by r. □

Of all the sections in this chapter, this one might just be the most beautiful one. Everything is short and sweet, not too complicated, and yet it cannot be overstated how much power so many of the results have. Much of this project indeed relies extensively on von Neumann's and Kaplansky's density theorems and we shall start exploiting them shortly, but first we will head into completely different terrain.

2.6 The Jordan decomposition

This section covers the Jordan decomposition, a way of decomposing any linear functional on a C^*-algebra into a finite linear combination of positive linear functionals. Recall that a linear functional $\varphi\colon \mathcal{A} \to \mathbb{C}$ on a C^*-algebra $\mathcal{A}$ is *Hermitian* if it satisfies $\varphi(x^*) = \overline{\varphi(x)}$ for all $x \in \mathcal{A}$.

Theorem 2.34 (The Jordan decomposition). *For any C^*-algebra $\mathcal{A}$, then any bounded Hermitian linear functional $\omega\colon \mathcal{A} \to \mathbb{C}$ is the difference of two positive linear functionals ω_+ and ω_- on $\mathcal{A}$ such that*

$$\omega = \omega_+ - \omega_-, \quad \|\omega\| = \|\omega_+\| + \|\omega_-\|.$$

In particular, every element in $\mathcal{A}^*$ is a linear combination of at most four states on $\mathcal{A}$.

Proof. Assume first that $\mathcal{A}$ is unital and let $S(\mathcal{A})$ denote the state space of $\mathcal{A}$. $S(\mathcal{A})$ is convex and compact in the weak* topology on $\mathcal{A}^*$ [31, Proposition 13.8] since $\mathcal{A}$ is unital. If $S = S(\mathcal{A}) \cup (-S(\mathcal{A}))$, then note that because $S(\mathcal{A})$ is convex, then the convex hull of S is given by

$$\mathrm{conv}S = \{\lambda\omega_1 - \mu\omega_2 \mid \omega_1, \omega_2 \in S(\mathcal{A}),\ \lambda + \mu = 1,\ \lambda, \mu \ge 0\}.$$

Hence conv$S$ is the image of the map $S(\mathcal{A}) \times S(\mathcal{A}) \times [0,1] \to \mathcal{A}^*$ given by

$$(\omega_1, \omega_2, \lambda) \mapsto \lambda\omega_1 - (1-\lambda)\omega_2.$$

Since this map is continuous if $\mathcal{A}^*$ and $S(\mathcal{A})$ are given the weak* topology and the product is given the product topology, it follows since $S(\mathcal{A}) \times S(\mathcal{A}) \times [0,1]$ is compact that conv$S$ is weak*-compact.

Assume now that ω is a bounded Hermitian linear functional on $\mathcal{A}$; we can furtherly assume that ω is non-zero. Let $\omega' = \|\omega\|^{-1}\omega$, so that $\|\omega'\| = 1$ and ω' is Hermitian. We claim that $\omega' \in \mathrm{conv}S$ and prove it by assuming for contradiction that $\omega' \notin \mathrm{conv}S$. Since $\{\omega'\}$ and $\mathrm{conv}S$ are convex subsets of $\mathcal{A}^*$, then by the Hahn-Banach separation theorem and Lemma A.3 we can find $x \in \mathcal{A}$ and $\mu \in \mathbb{R}$ such that

$$\mathrm{Re}\gamma(x) \leq \mu \leq \mathrm{Re}\omega'(x)$$

for all $\gamma \in \mathrm{conv}S$. If $x = x_1 + ix_2$ is the decomposition of x into self-adjoint elements, then for $y = x_1$ we have

$$\gamma(y) = \mathrm{Re}\gamma(x) \leq \mu \leq \mathrm{Re}\omega'(x) = \omega'(y)$$

for all $\gamma \in \mathrm{conv}S$, since they are Hermitian along with ω'. Moreover, since y is self-adjoint, we have $\|y\| = |\varphi(y)|$ for some $\varphi \in S(\mathcal{A})$ or $\|y\| = \varphi(y)$ for some $\varphi \in S$. Hence $\omega'(y) \leq \|y\| \leq \mu$, contradicting the above inequality. Hence $\omega' \in \mathrm{conv}S$, so $\omega' = \lambda\omega_1 + \mu\omega_2$ for $\lambda, \mu \geq 0$, $\lambda + \mu = 1$ and $\omega_1, \omega_2 \in S(\mathcal{A})$. Then

$$\|\omega'\| = 1 = \lambda + \mu = \lambda\|\omega_1\| + \mu\|\omega_2\| = \|\lambda\omega_1\| + \|\mu\omega_2\|.$$

Hence by putting $\omega_+ = \lambda\|\omega\|\omega_1$ and $\omega_- = \mu\|\omega\|\omega_2$, we obtain the desired decomposition.

Supposing now that $\mathcal{A}$ is not unital, let $\tilde{\mathcal{A}}$ denote the unitalisation of $\mathcal{A}$. If ω is a bounded Hermitian linear functional on $\mathcal{A}$, then the linear functional $\tilde{\omega}\colon \tilde{\mathcal{A}} \to \mathbb{C}$ given by $\tilde{\omega}(a + \lambda 1_{\tilde{\mathcal{A}}}) = \omega(a)$ for $a \in \mathcal{A}$ and $\lambda \in \mathbb{C}$ is bounded and Hermitian, so by virtue of what we just proved there exist positive linear functionals $\tilde{\omega}_+, \tilde{\omega}_-$ on $\tilde{\mathcal{A}}$ such that $\tilde{\omega} = \tilde{\omega}_+ - \tilde{\omega}_-$. By restricting $\tilde{\omega}_+$ and $\tilde{\omega}_-$ to $\mathcal{A}$, we obtain the desired decomposition of ω.

Finally, let $\omega \in \mathcal{A}^*$. By defining the linear functionals ω_1, ω_2 on $\mathcal{A}$ by

$$\omega_1(a) = \frac{\omega(a) + \overline{\omega(a^*)}}{2}, \quad \omega_2(a) = \frac{\omega(a) - \overline{\omega(a^*)}}{2i}, \quad a \in \mathcal{A},$$

it is readily seen that ω_1 and ω_2 are Hermitian and that $\omega = \omega_1 + i\omega_2$. Since both ω_1 and ω_2 can be written as linear combinations of at most two states, the last statement follows. □

We will need an equivalent condition to the equality of norms above later:

Proposition 2.35. *Let $\mathcal{A}$ be a unital C^*-algebra and let φ_+ and φ_- be positive linear functionals on $\mathcal{A}$. Then the following are equivalent:*

(i) $\|\varphi_+ - \varphi_-\| = \|\varphi_+\| + \|\varphi_-\|$.
(ii) *For any $\varepsilon > 0$ there is a positive element $a \in (\mathcal{A})_1$ such that $\varphi_+(1_\mathcal{A} - a) < \varepsilon$ and $\varphi_-(a) < \varepsilon$.*

Proof. Suppose first that $\|\varphi_+ - \varphi_-\| = \|\varphi_+\| + \|\varphi_-\|$ and let $\varepsilon > 0$. Since $\varphi_+ - \varphi_-$ is Hermitian, then by [31, Proposition 13.3] and [31, Theorem 13.5] there exists $b \in (\mathcal{A}_{\mathrm{sa}})_1$ such that

$$\varphi_+(b) - \varphi_-(b) + \varepsilon \geq \|\varphi_+ - \varphi_-\| = \|\varphi_+\| + \|\varphi_-\| = \varphi_+(1_\mathcal{A}) + \varphi_-(1_\mathcal{A}).$$

Hence $\varphi_+(1_\mathcal{A} - b) + \varphi_-(1_\mathcal{A} + b) \leq \varepsilon$. Since $0 \leq 1_\mathcal{A} - b \leq 2$ and $0 \leq 1_\mathcal{A} + b \leq 2$ by the continuous functional calculus, define $a = \frac{1}{2}(1_\mathcal{A} + b)$ so that $1_\mathcal{A} - a = \frac{1}{2}(1_\mathcal{A} - b)$. Then a is positive with $0 \leq a \leq 1$. Then

$$\varphi_+(1_\mathcal{A} - a) + \varphi_-(a) = \frac{1}{2}(\varphi_+(1_\mathcal{A} - b) + \varphi_-(1_\mathcal{A} + b)) < \varepsilon.$$

Since $\varphi_+(1_\mathcal{A} - a) \geq 0$ and $\varphi_-(a) \geq 0$, we must have $\varphi_+(1_\mathcal{A} - a) < \varepsilon$ and $\varphi_-(a) < \varepsilon$, so (ii) follows.

Suppose that (ii) holds and note that $\|\varphi_+ - \varphi_-\| \leq \|\varphi_+\| + \|\varphi_-\|$. For any $\varepsilon > 0$, take a positive element $a \in (\mathcal{A})_1$ such that $\varphi_+(1_\mathcal{A} - a) < \varepsilon$ and $\varphi_-(a) < \varepsilon$, and note that $\sigma(2a - 1_\mathcal{A}) \subseteq [-1, 1]$, so

$$\begin{aligned}
\|\varphi_+\| + \|\varphi_-\| &= \varphi_+(1_\mathcal{A}) + \varphi_-(1_\mathcal{A}) \\
&\leq \varphi_+(1_\mathcal{A}) + \varphi_-(1_\mathcal{A}) + (4\varepsilon - 2\varphi_+(1_\mathcal{A} - a) - 2\varphi_-(a)) \\
&= \varphi_+(2a - 1_\mathcal{A}) + \varphi_-(1_\mathcal{A} - 2a) + 4\varepsilon \\
&= (\varphi_+ - \varphi_-)(2a - 1_\mathcal{A}) + 4\varepsilon \\
&\leq \|\varphi_+ - \varphi_-\| + 4\varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we have $\|\varphi_+\| + \|\varphi_-\| \leq \|\varphi_+ - \varphi_-\|$. □

2.7 Normal linear functionals

We now turn towards a *pseudo-new* type of linear functional, the notion of which can be expanded to positive maps of von Neumann algebras. The reason that the word *pseudo-new* is emphasized is that the whole chapter will now start resembling the movie *Sleuth* directed by Joseph L. Mankiewicz. If you have not seen that movie, stop reading *right now* and spend the next two and a half hours having a blast with the sheer euphoria of being filmically manipulated so effectively; then return to the project.

Movie-related obsessions and pretensions aside, here comes a definition.

Definition 2.9. Let $\mathscr{M}$ be a von Neumann algebra and $\omega \in \mathscr{M}^*$. We say that ω is *normal* if it holds for any bounded increasing net $(T_\alpha)_{\alpha\in A}$ of self-adjoint operators in $\mathscr{M}$ that $(\omega(T_\alpha))_{\alpha\in A}$ converges to $\omega(\sup_{\alpha\in A} T_\alpha)$ (see Theorem 0.8). We denote the space of normal linear functionals on $\mathscr{M}$ by $\mathscr{M}_n$.

As we shall see, the notation of $\mathscr{M}_n$ will be completely expendable in a few pages or so.

Before we go any further, we introduce some useful notation. $\mathscr{M}^*$ can be canonically equipped with a Banach $\mathscr{M}$-bimodule structure by defining

$$(T\cdot\omega)(S) = \omega(ST), \quad (\omega\cdot T)(S) = \omega(TS), \quad \omega\in\mathscr{M}_n,\ S,T\in\mathscr{M},$$

by means of the inequalities $\|T\cdot\omega\| \le \|T\|\|\omega\|$ and $\|\omega\cdot T\| \le \|T\|\|\omega\|$. Furthermore, for any $\omega\in\mathscr{M}^*$, we define $\omega^*\in\mathscr{M}^*$ by

$$\omega^*(T) = \overline{\omega(T^*)}.$$

Note that the equation $\omega = \omega^*$ just states that ω is Hermitian.

The next lemma proves useful things concerning normal functionals.

Lemma 2.36. *Let $\mathscr{M}\subseteq B(\mathcal{H})$ be a von Neumann algebra. Then:*

(i) *$\mathscr{M}_n$ is a norm-closed subspace of $\mathscr{M}^*$.*
(ii) *For any $\omega\in\mathscr{M}_n$, $\omega^*\in\mathscr{M}_n$.*
(iii) *For any $\omega\in\mathscr{M}_n$ and $T\in\mathscr{M}$, we have that $T\cdot\omega$ and $\omega\cdot T$ belong to $\mathscr{M}_n$.*
(iv) *For any $\omega\in\mathscr{M}_n$ with $\omega=\omega^*$, then the Jordan decomposition of ω yields* normal *positive linear functionals ω_+ and ω_- such that*

$$\omega = \omega_+ - \omega_-, \quad \|\omega\| = \|\omega_+\| + \|\omega_-\|.$$

Proof. (i) It is easily verified that $\mathscr{M}_n$ is a subspace of $\mathscr{M}^*$. Let $(\omega_n)_{n\ge1}$ be a sequence of $\mathscr{M}_n$ converging in norm to ω and let $(T_\alpha)_{\alpha\in A}$ be a bounded increasing net of self-adjoint operators in $\mathscr{M}$ with supremum and strong operator limit T. Then $\|T_\alpha - T\|\le\lambda$ for some $\lambda>0$ and all $\alpha\in\mathcal{A}$. Let $\varepsilon>0$ and pick $n\ge1$ such that $\|\omega-\omega_n\| < \frac{\varepsilon}{2\lambda}$ and $\alpha_0\in A$ such that $|\omega_n(T-T_\alpha)| < \frac{\varepsilon}{2}$ for all $\alpha\ge\alpha_0$. Then

$$\|\omega(T-T_\alpha)\| \le \|\omega-\omega_n\|\|T-T_\alpha\| + |\omega_n(T-T_\alpha)| < \varepsilon$$

for all $\alpha\ge\alpha_0$ so that $\omega\in\mathscr{M}_n$.

(ii) Since the definition of normality only mentions self-adjoint operators, we trivially obtain $\omega^*\in\mathscr{M}_n$.

(iii) If $T\in\mathscr{M}$ and $(S_\alpha)_{\alpha\in A}$ is a bounded increasing net of self-adjoint operators converging strongly to S, then $(T^*S_\alpha T)_{\alpha\in A}$ is a bounded increasing net of self-adjoint operators converging strongly to T^*ST. Hence $T\cdot\omega\cdot T^*\in\mathscr{M}_n$. Because

$$ST = \frac14\sum_{n=0}^{3} i^n (T+i^n 1_{\mathcal{H}})^* S (T+i^n 1_{\mathcal{H}})$$

for any operators S and T, we have

$$T\cdot\omega = \frac14\sum_{n=0}^{3} i^n (T+i^n 1_{\mathcal{H}})\cdot\omega\cdot(T+i^n 1_{\mathcal{H}})^* \in\mathscr{M}_n$$

for all $T \in \mathscr{M}$. Because $\mathscr{M}_n$ is *-invariant, $\omega \cdot T = (T^* \cdot \omega)^* \in \mathscr{M}_n$.

(iv) The Jordan decomposition itself (Proposition 2.34) yields that ω decomposes in the way described above. Thus it will suffice to prove that ω_+ and ω_- are normal. Since $\|\omega_+ - \omega_-\| = \|\omega_+\| + \|\omega_-\|$, it follows from Proposition 2.35 that for any $\varepsilon > 0$ there exists a positive operator T such that $0 \leq T \leq 1$, $\omega_+(1_{\mathcal{H}} - T) < \varepsilon$ and $\omega_-(T) < \varepsilon$. For all $S \in \mathscr{M}$ we then have

$$\begin{aligned}
|\omega_+(S) - \omega(TS)| &\leq |\omega_+(S) - \omega_+(TS) + \omega_-(TS)| \\
&\leq |\omega_+((1_{\mathcal{H}} - T)S)| + |\omega_-(TS)| \\
&\leq |\omega_+((1_{\mathcal{H}} - T)^{1/2}(1_{\mathcal{H}} - T)^{1/2}S)| + |\omega_-(T^{1/2}T^{1/2}S)| \\
&\leq (\omega_+(1_{\mathcal{H}} - T)\omega_+(S^*(1_{\mathcal{H}} - T)S)^{1/2} + (\omega_-(T)\omega_-(S^*TS)^{1/2} \\
&\leq \omega_+(1_{\mathcal{H}} - T)^{1/2}\|\omega_+\|^{1/2}\|S\| + \omega_-(T)^{1/2}\|\omega_-\|^{1/2}\|S\| \\
&\leq \varepsilon^{1/2}(\|\omega_+\|^{1/2} + \|\omega_-\|^{1/2})\|S\|.
\end{aligned}$$

Hence $\|\omega_+ - \omega \cdot T\| \leq \varepsilon^{1/2}(\|\omega_+\|^{1/2} + \|\omega_-\|^{1/2})$. Since $\omega \cdot T$ and $\mathscr{M}_n$ is norm-closed, it follows that ω_+ and hence ω_- belong to $\mathscr{M}_n$. $\square$

The proof of the next lemma is somewhat confusing, but do not think that the statement itself is unimportant. It is not only elegant, but it also becomes extremely useful later on.

Lemma 2.37. *Let $\mathscr{M} \subseteq B(\mathcal{H})$ be a von Neumann algebra and ω a normal state on $\mathscr{M}$. Then there is a family $(P_i)_{i \in I}$ of non-zero mutually orthogonal projections in $\mathscr{M}$ with $\sum_{i \in I} P_i = 1_{\mathcal{H}}$ such that $P_i \cdot \omega$ is weakly continuous for all $i \in I$.*

Proof. By Zorn's lemma we can find a maximal family $(P_i)_{i \in I}$ of non-zero mutually orthogonal projections in $\mathscr{M}$ such that $P_i\omega$ is weakly continuous for all $i \in I$. We claim that $P = \sum_{i \in I} P_i = 1_{\mathcal{H}}$; suppose not. Choose a unit vector $\xi \in P(\mathcal{H})^\perp$ and define a linear functional $\psi \colon \mathscr{M} \to \mathbb{C}$ by $\psi(T) = 2\langle T\xi, \xi\rangle$. Using Zorn's lemma once more, we can find a maximal family $(Q_j)_{j \in J}$ of mutually orthogonal projections in $\mathscr{M}$ such that $\omega(Q_j) \geq \psi(Q_j)$ and $Q_j \leq 1 - P$ for all $j \in J$, and put $Q = \sum_{j \in J} Q_j$. Since ω and ψ are normal, $\omega(Q) \geq \psi(Q)$. We must have $1 - P - Q \neq 0$, since otherwise

$$2 = \psi(1 - P) = \psi(Q) \leq \omega(Q) \leq 1.$$

Put $P_1 = 1 - P - Q$. If E is a projection majorized by P_1, then $(1 - P)E = (1 - P)P_1E = P_1E = E$, so $E \leq 1 - P$. Moreover, $EQ_j = E(1 - P - Q)Q_j = E(Q_j - Q_j) = 0$ for all $j \in J$. Hence $\omega(E) < \psi(E)$ since the family $(Q_j)_{j \in J}$ was maximal. Since any positive element of $\mathscr{M}$ can be approximated in norm by positive finite linear combinations of projections (cf. [31, Exercise 20.2]), it follows that $\omega(P_1TP_1) \leq \psi(P_1TP_1)$ for any positive $T \in \mathscr{M}$. Hence

$$|\omega(TP_1)|^2 \leq \omega(1_{\mathcal{H}})\omega(P_1T^*TP_1) \leq \psi(P_1T^*TP_1) = 2\|TP_1\xi\|^2, \quad T \in \mathscr{M},$$

so $P_1 \cdot \omega$ is strongly continuous and hence weakly continuous by Proposition 0.7. Since $P_1 \leq 1 - P$, we have $P_1P_i = QP_i = Q(1 - P)P_i = 0$, contradicting maximality of the family $(P_i)_{i \in I}$. Hence $\sum_{i \in I} P_i = 1_{\mathcal{H}}$, so we are done. $\square$

We are almost ready for the big result, but we must first take a detour. The more relevant result of the next two is for now the last one, but it requires the first result, the proof of which is somewhat unelegant (in the opinion of the author), but extremely essential in the last section of the next chapter.

Proposition 2.38. *Let $\mathcal{A}$ be a C^*-algebra and let $\pi \colon \mathcal{A} \to B(\mathcal{H})$ be a representation with $\mathscr{M} = \pi(\mathcal{A})$. If $\xi \in \mathcal{H}$ and ω_ξ denotes the vector functional associated with ξ, then the following holds:*

(i) *If $T \in \mathscr{M}'$ is self-adjoint and $0 \leq T \leq 1_{\mathcal{H}}$, then the functional $\theta_T \colon \mathcal{A} \to \mathbb{C}$ given by*

$$\theta_T(a) = \omega_{T\xi}(\pi(a)) = \langle \pi(a)T\xi, T\xi\rangle = \langle \pi(a)\xi, T^2\xi\rangle, \quad a \in \mathcal{A},$$

is a positive linear functional that is dominated by $\omega_\xi \circ \pi$.

(ii) *Any positive linear functional on $\mathcal{A}$ that is dominated by $\omega_\xi \circ \pi$ is of the form θ_T for some self-adjoint $T \in \mathscr{M}'$ such that $0 \leq T \leq 1_{\mathcal{H}}$.*

Proof. It is obvious that θ_T is positive for any $T \in \mathscr{M}'$ and for $a \in \mathcal{A}$, we have

$$\theta_T(a^*a) = \langle \pi(a)T\xi, \pi(a)T\xi \rangle = \|\pi(a)T\xi\|^2 = \|T\pi(a)\xi\|^2 \leq \|\pi(a)\xi\|^2 = \omega_\xi(\pi(a^*a)),$$

so θ_T is dominated by $\omega_\xi \circ \pi$, hence proving (i).

Let $\varphi \in \mathcal{A}^*$ such that $0 \leq \varphi \leq \omega_\xi \circ \pi$. It then follows for $a, b \in \mathcal{A}$ that

$$|\varphi(b^*a)|^2 \leq \varphi(a^*a)\varphi(b^*b) \leq \|\pi(a)\xi\|^2 \|\pi(b)\xi\|^2$$

by Proposition 0.1. Note that if $c, d \in \mathcal{A}$ satisfy $\pi(a)\xi = \pi(c)\xi$ and $\pi(b)\xi = \pi(d)\xi$, then

$$|\varphi(b^*a - d^*c)| \leq |\varphi(b^*(a-c))| + |\varphi((b-d)^*c)| \leq \|\pi(a-c)\xi\|\|\pi(b)\xi\| + \|\pi(c)\xi\|\|\pi(b-d)\xi\| = 0,$$

so that the map $\Phi_0 \colon \mathscr{M}\xi \times \mathscr{M}\xi \to \mathbb{C}$ given by

$$\Phi_0(\pi(a)\xi, \pi(b)\xi) = \varphi(b^*a), \quad a, b \in \mathcal{A}$$

is then a well-defined sesquilinear form of norm less than or equal to 1 on the subspace $\mathscr{M}\xi$ of $\mathcal{H}$. Moreover, it is Hermitian, i.e.

$$\Phi_0(S\xi, T\xi) = \overline{\Phi_0(T\xi, S\xi)}, \quad S, T \in \mathscr{M},$$

and positive, i.e. $\Phi_0(T\xi, T\xi) \geq 0$ for $T \in \mathscr{M}$, because φ is positive and hence Hermitian. By Corollary A.2 Φ_0 thus extends to a bounded, Hermitian and positive sesquilinear form Φ on $[\mathscr{M}\xi]$ of norm ≤ 1. As $[\mathscr{M}\xi]$ is a Hilbert space, it follows from the Riesz representation theorem [14, Theorem 2.4.1] that there exists an operator $T_0 \in B([\mathscr{M}\xi])$ with $\|T_0\| \leq 1$ such that

$$\varphi(b^*a) = \langle T_0\pi(a)\xi, \pi(b)\xi \rangle, \quad a, b \in \mathcal{A}.$$

It is clear that T_0 is positive, and by extending T_0 to $\mathcal{H}$ by defining it to be 0 on the orthogonal complement, then the resultant operator, which we will still denote by T_0, stays positive. Moreover, for $a, b, c \in \mathcal{A}$, we have

$$\langle T_0\pi(c)\pi(a)\xi, \pi(b)\xi \rangle = \varphi(b^*ca) = \varphi((c^*b)^*a) = \langle T_0\pi(a)\xi, \pi(c)^*\pi(b)\xi \rangle = \langle \pi(c)T_0\pi(a)\xi, \pi(b)\xi \rangle.$$

Hence $\pi(c)T_0 = T_0\pi(c)$ on $[\mathscr{M}\xi]$ for all $c \in \mathcal{A}$. Additionally, for $\eta \in [\mathscr{M}\xi]^\perp$ we have

$$\langle \pi(c)\eta, \pi(a)\xi \rangle = \langle \eta, \pi(c^*a)\xi \rangle = 0,$$

so $\pi(c)\eta \in [\mathscr{M}\xi]^\perp$. This implies $\pi(c)T_0 = T_0\pi(c)$ on $\mathcal{H}$ for all $c \in \mathcal{A}$, so $T_0 \in \mathscr{M}'$. Defining $T = (T_0)^{1/2} \in \mathscr{M}'$, then $0 \leq T \leq 1_\mathcal{H}$. Letting $(e_\alpha)_{\alpha \in A}$ be an approximate identity for $\mathcal{A}$, we have for all $a \in \mathcal{A}$ that

$$\varphi(e_\alpha a) = \langle T_0\pi(a)\xi, \pi(e_\alpha)\xi \rangle = \langle \pi(ae_\alpha)T\xi, T\xi \rangle = \omega_{T\xi}(\pi(ae_\alpha)) \to \omega_{T\xi}(\pi(a))$$

and $\varphi(e_\alpha a) \to \varphi(a)$; hence $\varphi = \omega_{T\xi} \circ \pi$, completing the proof. □

Corollary 2.39. *Let $\mathscr{M}$ be a C^*-subalgebra of $B(\mathcal{H})$ and let $\xi, \eta \in \mathcal{H}$. If $\omega_{\xi,\eta} \colon \mathscr{M} \to \mathbb{C}$ is a positive linear functional on $\mathscr{M}$, there exists $\zeta \in \mathcal{H}$ such that $\omega_{\xi,\eta} = \omega_\zeta$ on $\mathscr{M}$.*

Proof. Let $T \in \mathscr{M}$ be positive. Then $\omega_{\xi,\eta}(T) = \langle T\xi, \eta \rangle = \langle \xi, T\eta \rangle = \langle T\eta, \xi \rangle = \omega_{\eta,\xi}(T)$ by the assumption that $\omega_{\xi,\eta}$ was positive, we find since

$$\langle T(\xi + \eta), \xi + \eta \rangle - \langle\langle T(\xi - \eta), \xi - \eta \rangle = 2\langle T\xi, \eta \rangle + 2\langle T\eta, \xi \rangle$$

that

$$4\omega_{\xi,\eta}(T) = 2\omega_{\xi,\eta}(T) + 2\omega_{\eta,\xi}(T) = \omega_{\xi+\eta}(T) - \omega_{\xi-\eta}(T) \leq \omega_{\xi+\eta}(T).$$

Thus $\omega_{\xi,\eta} \leq \omega_{\frac{1}{2}\xi + \frac{1}{2}\xi}$. We can then apply Proposition 2.38 with π equal to the identity $\mathscr{M} \to B(\mathcal{H})$. □

After seemingly going in wildly differing directions up until now, we nonetheless combine all of the above into one big lump of greatness. The next theorem can take your breath away if you are not prepared.

Theorem 2.40. *Let $\mathscr{M} \subseteq B(\mathcal{H})$ be a von Neumann algebra and ω a positive linear functional on $\mathscr{M}$. Then the following are equivalent:*

(i) *ω is normal.*
(ii) *ω is ultraweakly continuous.*
(iii) *There exists a sequence $(\xi_n)_{n\geq 1}$ of $\mathcal{H}$ satisfying $\sum_{n=1}^{\infty} \|\xi_n\|^2 < \infty$ such that*

$$\omega(T) = \sum_{n=1}^{\infty} \langle T\xi_n, \xi_n \rangle, \quad T \in \mathscr{M}.$$

Moreover, every ultraweakly continuous linear functional on $\mathscr{M}$ is a linear combination of four normal states and

$$\mathscr{M}_n = \mathscr{M}_*.$$

Proof. If ω is a (not necessarily positive) ultraweakly continuous linear functional, then ω is obviously normal by Proposition 2.1, since strong operator limits are also weak operator limits; hence $\mathscr{M}_* \subseteq \mathscr{M}_n$ and (ii) implies (i). (iii) clearly implies (ii) by Proposition 2.12.

Assume now that ω is a normal state. By Lemma 2.37 there is a family $(P_i)_{i\in I}$ of non-zero mutually orthogonal projections in $\mathscr{M}$ with $\sum_{i\in I} P_i = 1_{\mathcal{H}}$ such that $P_i \cdot \omega$ is weakly continuous for all $i \in I$. For finite subsets $F \subseteq I$, let $P_F = \sum_{i \in F} P_i$. Then for $T \in (\mathscr{M})_1$, we have $T^*T \in (\mathscr{M})_1$ as well, so for any finite subset $F \subseteq I$ we find that

$$|\omega(T(1_{\mathcal{H}} - P_F))| \leq \omega(T^*(1_{\mathcal{H}} - P_F)T)^{1/2}\omega(1_{\mathcal{H}} - P_F)^{1/2} \leq \omega(T^*T)^{1/2}\omega(1_{\mathcal{H}} - P_F)^{1/2} \leq \omega(1_{\mathcal{H}} - P_F)^{1/2}.$$

Therefore $\|\omega - P_F \cdot \omega\| \leq \omega(1_{\mathcal{H}} - P_F)^{1/2}$. As ω is normal we see that $\omega(P_F) \to \omega(1_{\mathcal{H}})$ or $\omega(1_{\mathcal{H}} - P_F) \to 0$. Hence ω is the norm-limit of the weakly and therefore ultraweakly continuous functionals $P_F \cdot \omega$, so ω is ultraweakly continuous. Since (i) $\Rightarrow$ (ii) then holds for states, it follows for any positive linear functional as well.

Defining the map $\Delta \colon \mathscr{M} \to B(\mathcal{H}^{\mathbb{N}})$ by $\Delta(T)(\xi_n)_{n\geq 1} = (T\xi_n)_{n\geq 1}$ for $T \in \mathscr{M}$ and $(\xi_n)_{n\geq 1} \in \mathcal{H}^{\mathbb{N}}$, then it is clear with help from Lemma 2.28 that $\Delta(\mathscr{M})$ is a unital *-subalgebra of $B(\mathcal{H}^{\mathbb{N}})$. Now, if ω is positive and ultraweakly continuous, then ω is of the form

$$\omega(T) = \sum_{n=1}^{\infty} \langle T\xi_n, \eta_n \rangle = \langle \Delta(T)\xi, \eta \rangle$$

for elements $\xi = (\xi_n)_{n\geq 1}$ and $\eta = (\eta_n)_{n\geq 1}$ in $\mathcal{H}^{\mathbb{N}}$. If $\Delta(T)$ is positive for some $T \in \mathscr{M}$, then for $\zeta \in \mathcal{H}$, define $\zeta_1 = \zeta$ and $\zeta_n = 0$ for $n \geq 2$. Then $\langle T\zeta, \zeta \rangle = \langle \Delta(T)(\zeta_n)_{n\geq 1}, (\zeta_n)_{n \geq 1} \rangle \geq 0$, so T is positive and

$$\langle \Delta(T)\xi, \eta \rangle = \omega(T) \geq 0.$$

Therefore, by Corollary 2.39, we find $\zeta = (\zeta_n)_{n\geq 1} \in \mathcal{H}^{\mathbb{N}}$ such that

$$\omega(T) = \langle \Delta(T)\xi, \eta \rangle = \langle \Delta(T)\zeta, \zeta \rangle = \sum_{n=1}^{\infty} \langle T\zeta_n, \zeta_n \rangle.$$

Hence we obtain (iii) from (ii). Since

$$4\langle T\xi, \eta \rangle = \sum_{n=0}^{3} i^n \langle T(\xi + i^n \eta), \xi + i^n \eta \rangle$$

for all $\xi, \eta \in \mathcal{H}$ and $T \in B(\mathcal{H})$, any ultraweakly continuous linear functional on $\mathscr{M}$ is a finite linear combination of four normal positive linear functionals; by scaling, each of the positive functionals can be assumed to be a state.

Finally, if $\omega \in \mathscr{M}_n$ is Hermitian, it follows from Lemma 2.36 that ω decomposes into normal positive linear functionals, so $\omega \in \mathscr{M}_*$ since each of these is then ultraweakly continuous. Finally, if $\omega \in \mathscr{M}_n$, then $\omega_1 = \frac{1}{2}(\omega + \omega^*)$ and $\omega_2 = \frac{1}{2i}(\omega - \omega^*)$ are Hermitian and normal by Lemma 2.36 and hence ultraweakly continuous. Therefore $\omega = \omega_1 + i\omega_2$ is ultraweakly continuous, so $\mathscr{M}_n = \mathscr{M}_*$. $\square$

The above theorem is the reason why we will completely obliviate the notation $\mathscr{M}_n$ ($\mathscr{M}_*$ looks nicer anyway). It is also the reason that normality of a linear functional is oftentimes defined as continuity with respect to the ultraweak operator topology. In this project however, we will stick to the fact that $\mathscr{M}_*$ consists of all ultraweakly continuous linear functionals on $\mathscr{M}$ and keep the above theorem in mind throughout; hence, whenever a linear functional on $\mathscr{M}$ is normal, it belongs to $\mathscr{M}_*$, and vice versa.

We might as well keep proving nice things about ultraweakly continuous linear functionals. Just like the polar decomposition yields that any operator T in a von Neumann algebra $\mathscr{M}$ decomposes into the product of a partial isometry and a positive operator, both contained in $\mathscr{M}$ [31, Theorem 18.9], we are about to prove that any $\omega \in \mathscr{M}_*$ decomposes in a similar way. It still requires some preparation, but not a lot.

Lemma 2.41. *Let $\mathcal{A}$ be a C^*-algebra and $\varphi \in \mathcal{A}^*$. If there exists $a \in \mathcal{A}_+$ with $\|a\| \leq 1$ such that $\varphi(a) = \|\varphi\|$, then φ is positive. For any C^*-subalgebra $\mathcal{B}$ of $\mathcal{A}$ and any positive linear functional $\psi \in \mathcal{B}^*$, there exists a positive linear functional $\varphi \in \mathcal{A}^*$ such that $\varphi|_{\mathcal{B}} = \psi$ and $\|\varphi\| = \|\psi\|$.*

Proof. Assume first that $\mathcal{A}$ is unital. For any $\theta \in \mathbb{R}$ and $\lambda \in \sigma(a)$, we have

$$|(1-e^{i\theta})\lambda + e^{i\theta}| = |\lambda + e^{i\theta}(1-\lambda)| \leq |\lambda| + |1-\lambda| = 1,$$

so we have $\sigma(a + e^{i\theta}(1_{\mathcal{A}} - a)) = (1-e^{i\theta})\sigma(a) + e^{i\theta} \subseteq (\mathbb{C})_1$, whereupon $\|a + e^{i\theta}(1_{\mathcal{A}} - a)\| \leq 1$ for all $\theta \in \mathbb{R}$. Choosing $\theta \in \mathbb{R}$ such that $e^{i\theta}\varphi(1_{\mathcal{A}} - a) \geq 0$, we then have

$$\|\varphi\| = \varphi(a) \leq \varphi(a) + e^{i\theta}\varphi(1_{\mathcal{A}} - a) = \varphi(a + e^{i\theta}(1_{\mathcal{A}} - a)) \leq \|\varphi\|$$

, so $\varphi(1_{\mathcal{A}} - a) = 0$. Therefore, $\varphi(1_{\mathcal{A}}) = \varphi(a) = \|\varphi\|$, so φ is positive by [31, Theorem 13.5].

If $\mathcal{A}$ is not unital and $\varphi \in \mathcal{A}^*$, then the Hahn-Banach extension $\tilde{\varphi}$ of φ to the unitization $\tilde{\mathcal{A}}$ satisfies the condition for the unital C^*-algebra $\tilde{\mathcal{A}}$, so by virtue of what we have just proved, $\tilde{\varphi}$ is positive, implying that φ is positive.

For the second statement, let $\mathcal{B}$ and ψ be as defined above. If $\mathcal{B}$ is unital, then for any Hahn-Banach extension φ of ψ to $\mathcal{A}$ [13, Theorem 5.7], we have

$$\varphi(1_{\mathcal{B}}) = \psi(1_{\mathcal{B}}) = \|\psi\| = \|\varphi\|$$

so the first statement yields that φ is positive. If $\mathcal{B}$ is non-unital, assume first that $\mathcal{A}$ has a unit and and put $\mathcal{B}_1 = \mathcal{B} + \mathbb{C}1_{\mathcal{A}}$. Extend ψ to a linear functional $\Psi \colon \mathcal{B}_1 \to \mathbb{C}$ by defining

$$\Psi(b + \lambda 1_{\mathcal{A}}) = \psi(b) + \lambda\|\psi\|.$$

As the map $\tilde{\mathcal{B}} \to \mathcal{A}$ given by $(b, \lambda) \mapsto b + \lambda 1_{\mathcal{A}}$ is an injective *-homomorphism, so that $\mathcal{B}_1$ is a unital C^*-algebra. Let $(f_\alpha)_{\alpha \in A}$ be an approximate identity for $\mathcal{B}$. Then for $b \in \mathcal{B}$ and $\lambda \in \mathbb{C}$, Proposition 0.3 yields

$$|\Psi(b + \lambda 1_{\mathcal{A}})| = |\psi(b) + \lambda\|\varphi\|| = |\lim_{\alpha \in A} \psi(bf_\alpha + \lambda f_\alpha)| \leq \|\psi\| \sup_{\alpha \in A} \|bf_\alpha + \lambda 1_{\mathcal{A}} f_\alpha\| \leq \|\psi\| \|b + \lambda 1_{\mathcal{A}}\|,$$

so that $\|\Psi\| \leq \|\psi\|$. Clearly we also have $\|\Psi\| \geq \|\psi\|$, so we conclude that $\Psi(1_{\mathcal{A}}) = \|\psi\| = \|\Psi\|$, proving that Ψ is positive. Letting φ be a Hahn-Banach extension of Ψ to $\mathcal{A}$, we see that φ extends ψ, that $\|\varphi\| = \|\psi\|$ and that $\varphi(1_{\mathcal{A}}) = \Psi(1_{\mathcal{A}}) = \|\Psi\| = \|\varphi\|$. Hence φ is positive. Finally if $\mathcal{A}$ is non-unital, then by replacing $\mathcal{A}$ with its unitization $\tilde{\mathcal{A}}$ and $\mathcal{B}_1$ with the subset $\{(b, \lambda) \,|\, b \in \mathcal{B},\ \lambda \in \mathbb{C}\}$ of $\tilde{\mathcal{A}}$ in the above argument, we obtain a positive linear functional φ on $\tilde{\mathcal{A}}$ with the wanted properties. Restricting to $\mathcal{A}$ yields the wanted linear functional. □

Lemma 2.42. *Let $\mathscr{M}$ be a von Neumann algebra and $\omega \in \mathscr{M}_*$. Then for any $T \in \mathscr{M}$, we have $T \cdot \omega \in \mathscr{M}_*$.*

Proof. This is obvious from Corollary 2.4 and Proposition 2.2. □

Lemma 2.43. *Let $\mathscr{M} \subseteq B(\mathcal{H})$ be a von Neumann algebra and let $\omega \in \mathscr{M}_*$. If a projection $P \in \mathscr{M}$ satisfies $\|P \cdot \omega\| = \|\omega\|$, then we have $P \cdot \omega = \omega$.*

Proof. We may assume that ω is non-zero and that $\|\omega\| = 1$ by scaling. Putting $Q = 1_{\mathcal{H}} - P$, we will show that $Q \cdot \omega = 0$. Assume for contradiction that $Q \cdot \omega \neq 0$. Then there is a $T \in (\mathscr{M})_1$ such that $\delta := (Q \cdot \omega)(T) > 0$. Since $P \cdot \omega \neq 0$ by assumption, then by the Hahn-Banach theorem [13, Theorem 5.8], there exists $\varphi \in (\mathscr{M}_*)^*$ such that $\varphi(P \cdot \omega) = \|P \cdot \omega\| = \|\omega\| = 1$ and $\|\varphi\| = 1$. Hence by Theorem 2.7, we obtain $S \in (\mathscr{M})_1$ such that $(P \cdot \omega)(S) = \varphi(P \cdot \omega) = 1$. From seeing that

$$\|SP + \delta TQ\|^2 = \|(SP + \delta TQ)(SP + \delta TQ)^*\| = \|SPS^* + \delta^2 TQT^*\| \leq 1 + \delta^2,$$

we find $\|SP + \delta TQ\| \leq (1 + \delta^2)^{1/2} < 1 + \delta^2$, since $\sqrt{x} < x$ for all $x > 1$. However, this implies

$$1 + \delta^2 = (P \cdot \omega)(S) + \delta(Q \cdot \omega)(T) = \omega(SP + \delta TQ) \leq \|\omega\|\|SP + \delta TQ\| < 1 + \delta^2,$$

a contradiction, so $Q \cdot \omega = 0$. Therefore, $\omega = P \cdot \omega + Q \cdot \omega = P \cdot \omega$. □

Now we are set.

Proposition 2.44. *If $\mathscr{M}$ is a von Neumann algebra and $\omega \in \mathscr{M}_*$, then there exists a partial isometry $U \in \mathscr{M}$ and a positive linear functional $\varphi \in \mathscr{M}_*$ such that $\omega = U \cdot \varphi$, $\varphi = U^* \cdot \omega$ and $\|\varphi\| = \|\omega\|$.*

Proof. We can assume that ω is non-zero. By the Hahn-Banach theorem [13, Theorem 5.8], there exists $\psi \in (\mathscr{M}_*)^*$ such that $\|\psi\| = 1$ and $\psi(\omega) = \|\omega\|$. By Theorem 2.7, there exists $S \in (\mathscr{M})_1$ such that $\omega(S) = \|\omega\|$. Let $S^* = U|S^*|$ be the polar decomposition of S^*. Then

$$\|\omega\| = \omega(S) = \omega(|S^*|U^*) = (U^* \cdot \omega)(|S^*|).$$

Define $\varphi = U^* \cdot \omega$. Since $\|SS^*\|^2 \leq 1$, then $|S^*|$ has norm less than or equal to 1; since

$$\|\varphi\| \leq \|U^*\|\|\omega\| \leq \|\omega\| = \varphi(|S^*|) \leq \|\varphi\|,$$

we have $\|\varphi\| = \varphi(|S^*|)$, so by Lemma 2.41, φ is positive. By [31, Theorem 18.9] we have that $U \in \mathscr{M}$, so $P = UU^*$ is a projection contained in $\mathscr{M}$. Note that $U \cdot \varphi = (UU^*) \cdot \omega = P \cdot \omega$ and

$$(SP)^* = P^*S^* = UU^*S^* = UU^*U|S^*| = U|S^*| = S^*,$$

so that $SP = S$. Therefore

$$\|P \cdot \omega\| \leq \|\omega\| = \omega(S) = \omega(SP) = (P \cdot \omega)(S) \leq \|P \cdot \omega\|,$$

so $P \cdot \omega = \omega$ by Lemma 2.43, and hence $\omega = U \cdot \varphi$. Finally, $\|\omega\| \leq \|U\|\|\varphi\| \leq \|U^*\|\|\omega\| \leq \|\omega\|$, as $\|U\| = \|U\| \leq 1$, so $\|\varphi\| = \|\omega\|$. □

The above expression of an ultraweakly continuous linear functional is called its *polar decomposition*.

2.8 Normal linear maps

One might have noticed that the definition of normality did not strictly depend on the fact that the map in question mapped into $\mathbb{C}$; as it is equally possible to taking supremums in von Neumann algebras, there might be a possibility of generalizing. (Of course there is.) This section requires some of the definitions encountered in Chapter 3, so if you, kind reader, are not familiar with the concept of a positive linear map, skip ahead to page 67 and spend 10 seconds reading the relevant definitions and the statement that positive maps on C^*-algebras are Hermitian (do not continue reading from there, since this chapter is still extremely relevant!). Here goes.

Definition 2.10. Let $\mathscr{M}$ and $\mathscr{N}$ be von Neumann algebras and let $\varphi\colon \mathscr{M} \to \mathscr{N}$ be a bounded positive linear map. φ is then called *normal* if it holds for any bounded increasing net $(T_\alpha)_{\alpha \in A}$ of self-adjoint operators in $\mathscr{M}$ that

$$\varphi\left(\sup_{\alpha \in A} T_\alpha\right) = \sup_{\alpha \in A} \varphi(T_\alpha).$$

One might expect that the above notion of normality has a connection to the one for linear functionals, and as it turns out that *tout est vraiment beau.*

Proposition 2.45. *Let $\mathscr{M}$ and $\mathscr{N}$ be von Neumann algebras and let $\varphi\colon \mathscr{M} \to \mathscr{N}$ be a bounded positive linear map. Then φ is normal if and only if $\omega \circ \varphi \in \mathscr{M}_*$ for all $\omega \in \mathscr{N}_*$.*

Proof. Let $(T_\alpha)_{\alpha\in A}$ be a bounded increasing net of self-adjoint operators in $\mathscr{M}$ with $T = \sup_{\alpha\in A} T_\alpha$. If φ is normal, then because $\varphi(T) = \sup_{\alpha\in A}\varphi(T_\alpha)$, we have $\omega(\varphi(T_\alpha)) \to \omega(\varphi(T))$ for all $\omega \in \mathscr{N}_*$. Thus $\omega\circ\varphi \in \mathscr{M}_*$ for all $\omega \in \mathscr{N}_*$ by Theorem 2.40. On the other hand, if $\omega(\varphi(T_\alpha)) \to \omega(\varphi(T))$ for all $\omega \in \mathscr{N}_*$, then $\varphi(T_\alpha) \to \varphi(T)$ ultraweakly and hence weakly. Since φ is positive, $(\varphi(T_\alpha))_{\alpha\in A}$ converges weakly to $S = \sup_{\alpha\in A}\varphi(T_\alpha)$, but then $S = \varphi(T)$. Hence φ is normal. $\square$

Corollary 2.46. *Let $\mathscr{M}$ and $\mathscr{N}$ be von Neumann algebras and let $\varphi\colon \mathscr{M} \to \mathscr{N}$ be a bounded positive linear map. Then φ is normal if and only if φ is ultraweakly-to-ultraweakly (or ultrastrongly-to-ultrastrongly) continuous.*

Proof. This is an immediate consequence of the preceding proposition. $\square$

It is normal (pun intended) that ultraweakly-to-ultraweakly continuous positive linear maps are called *normal*, and the above corollary is the reason why. We will adopt this convention throughout the project, so that normal maps of von Neumann algebras are the ultraweakly-to-ultraweakly continuous ones, also satisfying Definition 2.10.

The next two theorems are so useful that it hurts.

Theorem 2.47. *Let $\mathscr{M}$ be a von Neumann algebra and let $\pi\colon \mathscr{M} \to B(\mathcal{H})$ be a normal unital representation. Then $\pi(\mathscr{M})$ is a von Neumann algebra.*

Proof. We have that $(\mathscr{M})_1$ is ultraweakly compact by Corollary 2.9. As $(\pi(\mathscr{M}))_1 = \pi((\mathscr{M})_1)$ by Proposition 2.61 and π is normal, then $(\pi(\mathscr{M}))_1$ is ultraweakly compact and hence ultraweakly closed. Since $1_{\mathcal{H}} \in \pi(\mathscr{M})$, it follows from von Neumann's bicommutant theorem (Theorem 2.30) that $\pi(\mathscr{M})$ is a von Neumann algebra. $\square$

Proposition 2.48. *Let $\mathscr{M}$ and $\mathscr{N}$ be von Neumann algebras and let $\pi\colon \mathscr{M} \to \mathscr{N}$ be a *-isomorphism. Then π is a homeomorphism of the ultraweak and ultrastrong topologies on $\mathscr{M}$ and $\mathscr{N}$.*

Proof. As can be easily checked, π and π^{-1} are normal *-homomorphisms. $\square$

The last one in particular is just amazing; who would expect that a *-isomorphism, a strictly algebraic notion, is automatically ultraweakly-to-ultraweakly continuous?

Next follows a couple of examples of normal maps.

Proposition 2.49. *Let $\mathscr{M}$ be a von Neumann algebra and let $\varphi \in S(\mathscr{M})$ be a state. Let $(\mathcal{H}, \pi, \xi)$ be its associated GNS triple (see page viii). If φ is normal, then π is normal and $\pi(\mathscr{M})$ is a von Neumann algebra.*

Proof. Let $\pi^*\colon B(\mathcal{H})^* \to \mathscr{M}^*$ be the dual mapping of π. For each R, S and T in $\mathscr{M}$ we have

$$\pi^*(\omega_{\pi(R)\xi,\pi(S)\xi})(T) = \langle \pi(T)\pi(R)\xi, \pi(S)\xi\rangle = \langle \pi(S^*TR)\xi, \xi\rangle = \varphi(S^*TR) = (R\cdot\varphi\cdot S^*)(T).$$

Hence $\pi^*(\omega_{\pi(R)\xi,\pi(S)\xi}) = R\cdot\varphi\cdot S \in \mathscr{M}_*$ by Lemma 2.36 since φ is normal. Let $\eta \in \mathcal{H}$, $S \in \mathscr{M}$ and $\varepsilon > 0$. As $\pi(\mathscr{M})\xi$ is dense in $\mathcal{H}$, we can pick $R \in \mathscr{M}$ such that $\|\pi(R)\xi - \eta\| < \varepsilon$. Then as

$$\|\pi^*(\omega_{\eta,\pi(S)\xi}) - \pi^*(\omega_{\pi(R)\xi,\pi(S)\xi})\| \le \|\eta - \pi(R)\xi\|\|\pi(S)\xi\| < \varepsilon\|\pi(S)\xi\|,$$

we see that $\pi^*(\omega_{\eta,\pi(S)\xi})$ is contained in the norm closure of $\mathscr{M}_*$ and hence in $\mathscr{M}_*$. In a similar way, one proves that $\pi^*(\omega_{\eta,\chi}) \in \mathscr{M}_*$ for all $\eta, \chi \in \mathcal{H}$, so π^* maps all finite linear combinations of $\omega_{\eta,\chi}$ into $\mathscr{M}_*$. For any $\omega \in B(\mathcal{H})_*$, we have by Proposition 2.2 and Corollary 2.4 that $\omega = \sum_{n=1}^\infty \omega_{\eta_n,\chi_n}$, converging in norm, for square-summable sequences $(\eta_n)_{n\ge1}$ and $(\chi_n)_{n\ge1}$ in $\mathcal{H}$ where the series converges in norm. Because

$$\left\|\pi^*(\omega) - \pi^*\left(\sum_{n=1}^N \omega_{\eta_n,\chi_n}\right)\right\| \le \left\|\sum_{n=N+1}^\infty \omega_{\eta_n,\chi_n}\right\| \le \sum_{n=N+1}^\infty \|\eta_n\|\|\chi_n\| \to 0$$

for $N \to \infty$, we see that $\omega\circ\pi = \pi^*(\omega) \in \mathscr{M}_*$. By Proposition 2.45, π is normal and hence $\pi(\mathscr{M})$ is a von Neumann algebra by Proposition 2.47. $\square$

Proposition 2.50. *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces and let $\mathscr{M} \subseteq B(\mathcal{H})$ and $\mathscr{N} \subseteq B(\mathcal{K})$ be von Neumann algebras. The maps $\pi\colon \mathscr{M} \to B(\mathcal{H} \otimes \mathcal{K})$ and $\rho\colon \mathscr{N} \to B(\mathcal{H} \otimes \mathcal{K})$ given by $\pi(T) = T \otimes 1_{\mathcal{K}}$ and $\rho(S) = 1_{\mathcal{H}} \otimes T$ are normal *-homomorphisms.*

Proof. As $\pi\colon \mathscr{M} \to \mathscr{M} \otimes \mathbb{C}1_{\mathcal{K}}$ and $\rho\colon \mathscr{N} \to \mathbb{C}1_{\mathcal{H}} \otimes \mathscr{N}$ are *-isomorphisms, the result follows from Proposition 2.48. □

Corollary 2.51. *Let $\mathscr{M} \subseteq B(\mathcal{H})$ be a von Neumann algebra and let I be a non-empty set. The map $\Delta\colon \mathscr{M} \to \bigoplus_{i\in I} \mathscr{M}$ given by $\Delta(T)(\xi_i)_{i\in I} = (T\xi_i)_{i\in I}$ for $T \in \mathscr{M}$ is normal.*

Proof. Let $\mathcal{K} = \ell^2(I)$ with orthonormal basis $\{\delta_i \,|\, i \in I\}$ and define $U\colon \mathcal{H}^I \to \mathcal{H} \otimes \mathcal{K}$ by

$$U(\xi_i)_{i\in I} = \sum_{i\in I} \xi_i \otimes \delta_i.$$

We saw in Section 1.3 that U was an isometric isomorphism. Assume that $T_\alpha \to T$ ultraweakly in $\mathscr{M}$. Proposition 2.50 now yields that $T_\alpha \otimes 1_{\mathcal{K}} \to T \otimes 1_{\mathcal{K}}$ ultraweakly. As the map $\mathcal{H} \otimes \mathcal{K} \to \mathcal{H}^I$ given by $S \to U^{-1}SU$ is an ultraweak-to-ultraweak homeomorphism, the proof of Proposition 1.33(i) now tells us that $\Delta(T_\alpha) \to \Delta(T)$ ultraweakly. □

Proposition 2.52. *Let $(\mathscr{M}_i)_{i\in I}$ be a family of von Neumann algebras with $\mathscr{M}_i \subseteq B(\mathcal{H}_i)$ for each $i \in I$ and let $\mathscr{M} = \bigoplus_{i\in I} \mathscr{M}_i$. For any $i_0 \in I$, let $\vartheta_{i_0}\colon \mathscr{M}_{i_0} \to \mathscr{M}$ and $\theta_{i_0}\colon \mathscr{M} \to \mathscr{M}_{i_0}$ be the inclusion and projection respectively. Then ϑ_{i_0} and θ_{i_0} are normal *-homomorphisms. Similarly, if $J \subseteq I$ and $\vartheta_J\colon \bigoplus_{i\in J} \mathscr{M}_i \to \mathscr{M}$ and $\theta_J\colon \mathscr{M} \to \bigoplus_{i\in J} \mathscr{M}_i$ are the inclusions and projections respectively, then ϑ_J and θ_J are normal homomorphisms.*

Proof. It is easy to see that the maps in question are *-homomorphisms. Assume that $S_\alpha \to S$ ultraweakly in $\mathscr{M}_{i_0}$ and $T_\alpha \to T$ ultraweakly in $\mathscr{M}$. For any square-summable sequences $\xi = (\xi_n)_{n\geq 1}$ and $\eta = (\eta_n)_{n\geq 1}$ in $\bigoplus_{i\in I} \mathcal{H}_i$, let ξ'_n and η'_n be the i_0'th coordinate of ξ_n and η_n for all $n \geq 1$, and note that $(\xi'_n)_{n\geq 1}$ and $(\eta'_n)_{n\geq 1}$ are square-summable. Hence

$$\sum_{n=1}^{\infty} \langle (\vartheta(S_\alpha) - \vartheta(S))\xi_n, \eta_n\rangle = \sum_{n=1}^{\infty} \langle (S_\alpha - S)\xi'_n, \eta'_n\rangle \to 0,$$

so that $\vartheta(S_\alpha) \to \vartheta(S)$ ultraweakly. That the other maps are normal is shown in a similar manner. □

The next result combines some earlier results along with some new ones into a neat statement about normal *-epimorphisms.

Proposition 2.53. *Let $\mathscr{M}$ and $\mathscr{N}$ be von Neumann algebras and let $\varphi\colon \mathscr{M} \to \mathscr{N}$ be a normal surjective *-homomorphism. Then there exists a central projection $P \in \mathscr{M}$ such that $\ker\varphi \cong \mathscr{M}_P$, $\mathscr{N} \cong \mathscr{M}_{1_{\mathscr{M}}-P}$ and*

$$\mathscr{M} \cong \mathscr{M}_P \oplus \mathscr{M}_{1_{\mathscr{M}}-P}.$$

Proof. $\ker\varphi$ is an ultraweakly closed, two-sided ideal in $\mathscr{M}$. Hence by Proposition 2.32 there exists a central projection $P \in \mathscr{M}$ such that $\ker\varphi = \mathscr{M}P$, so $\ker\varphi$ is isomorphic to $\mathscr{M}_P$ by Proposition 2.17. Moreover, the *-subalgebra $\mathscr{M}(1_{\mathscr{M}} - P)$ is *-isomorphic to $\mathscr{N}$: the map $\tilde{\varphi}\colon \mathscr{M}(1_{\mathscr{M}} - P) \to \mathscr{N}$ given by $\tilde{\varphi}(T(1_{\mathscr{M}} - P)) = \varphi(T(1_{\mathscr{M}} - P)) = \varphi(T)$ is clearly a surjective *-homomorphism, and it is injective: indeed, if $\varphi(T) = 0$ for some $T \in \mathscr{M}(1_{\mathscr{M}} - P)$, then $T \in \ker\varphi$ and hence $T = SP$ for some $S \in \mathscr{M}$. As $T(1_{\mathscr{M}} - P) = T$, we have $T = SP = TP = T(1_{\mathcal{H}} - P)P = 0$. Therefore $\mathscr{N}$ is isomorphic to $\mathscr{M}_{1_{\mathscr{M}}-P}$ by Proposition 2.17, so $\mathscr{M} = \mathscr{M}_P \oplus \mathscr{M}_{1_{\mathscr{M}}-P}$ by Proposition 2.19. □

Regrettably, the next theorem is stated without proof; however, a thorough proof would kill the momentum of the chapter as it would require some results about Hilbert space tensor products that obviously hold, but nonetheless would take quite a bit of time to prove. The statement of the theorem is a little convoluted in itself though, so hopefully the reader will not miss a proof too much.

Theorem 2.54. *Let $\mathscr{M} \subseteq B(\mathcal{H})$ and $\mathscr{N} \subseteq B(\mathcal{K})$ be von Neumann algebras. If $\pi\colon \mathscr{M} \to \mathscr{N}$ is a normal surjective *-homomorphism, then there exists a Hilbert space $\mathcal{L}$, a projection $Q \in \mathscr{M}' \mathbin{\overline{\otimes}} B(\mathcal{L})$ and an isometric isomorphism $U\colon Q(\mathcal{H} \otimes \mathcal{L}) \to \mathcal{K}$ such that*

$$\pi(T) = U\left[Q(T \otimes 1_{\mathcal{L}})|_{Q(\mathcal{H}\otimes\mathcal{L})}\right] U^{-1}.$$

Proof. Omitted. See [24, Theorem IV.5.5]. □

The only reason we need the above result is the following corollary which we will come back to in a few chapters.

Corollary 2.55. *Let $\mathscr{M} \subseteq B(\mathcal{H})$ be a von Neumann algebra, and let $P \in \mathscr{M}$ be a projection with central support $C_P = 1_{\mathcal{H}}$. Then $\mathscr{M}$ is isomorphic to a reduced von Neumann algebra of $\mathscr{M}_P \overline{\otimes} B(\mathcal{K})$ for some Hilbert space $\mathcal{K}$.*

Proof. By Proposition 2.18, there exists a *-isomorphism $\varphi\colon (\mathscr{M})'_P \to \mathscr{M}'$. By Proposition 2.48, φ is normal, so Theorem 2.54 yields the existence of a Hilbert space $\mathcal{K}$, a projection

$$Q \in ((\mathscr{M})'_P)' \,\overline{\otimes}\, B(\mathcal{K}) = \mathscr{M}_P \,\overline{\otimes}\, B(\mathcal{K}) \subseteq B(P(\mathcal{H}) \otimes \mathcal{K})$$

and an isometric isomorphism $U\colon Q(P(\mathcal{H}) \otimes \mathcal{K}) \to \mathcal{H}$ such that

$$\varphi(T) = U(Q(T \otimes 1_{\mathcal{K}})|_{Q(P(\mathcal{H})\otimes\mathcal{K})})U^{-1}.$$

Hence

$$\mathscr{M}' = U((\mathscr{M}_P)' \otimes \mathbb{C}1_{\mathcal{K}})_Q U^{-1},$$

so $\mathscr{M} = U(\mathscr{M}_P \,\overline{\otimes}\, B(\mathcal{K}))_Q U^{-1}$ by Proposition 1.29. Hence U induces a spatial isomorphism between the von Neumann algebras $\mathscr{M}$ and $(\mathscr{M}_P \,\overline{\otimes}\, B(\mathcal{K}))_Q$. □

For the next chapter already, we will need a result on how we combine two normal *-homomorphisms into one normal *-homomorphism on the von Neumann algebra tensor product. We put it here for reference, without proof.

Proposition 2.56. *Let $\mathscr{M}_1$, $\mathscr{M}_2$, $\mathscr{N}_1$ and $\mathscr{N}_2$ be von Neumann algebras and let $\varphi_1\colon \mathscr{M}_1 \to \mathscr{N}_1$ and $\varphi_2\colon \mathscr{M}_2 \to \mathscr{N}_2$ be normal unital *-homomorphisms. Then there exists a unique normal unital *-homomorphism $\varphi\colon \mathscr{M}_1 \,\overline{\otimes}\, \mathscr{M}_2 \to \mathscr{N}_1 \,\overline{\otimes}\, \mathscr{N}_2$ such that*

$$\varphi(T_1 \otimes T_2) = \varphi_1(T_1) \otimes \varphi_2(T_2), \quad T_1 \in \mathscr{M}_1,\ T_2 \in \mathscr{M}_2.$$

Proof. See [10, Proposition I.4.5.2]. □

2.9 The predual of a direct sum of von Neumann algebras

Since we have only now started working with ultraweakly continuous linear functionals in a more developed manner, the next proposition has not had a place to be until now; since it does not fit in anywhere else, we put it here; we will need it in the next chapter. For a family $(\mathfrak{X}_i)_{i\in I}$ of Banach spaces, its ℓ^1-*direct sum*, denoted by $\bigoplus_{i\in I} \mathfrak{X}_i$, is the set $\{(x_i)_{i\in I} \mid x_i \in \mathfrak{X}_i \text{ for all } i \in I,\ \sum_{i\in I} \|x_i\| < \infty\}$ equipped with pointwise addition and multiplication. Under the norm $(x_i)_{i\in I} \mapsto \sum_{i\in I} \|x_i\|$, the direct sum is then a Banach space itself.

Proposition 2.57. *Let $(\mathscr{M}_i)_{i\in I}$ be a family of von Neumann algebras with $\mathscr{M}_i \subseteq B(\mathcal{H}_i)$ for some Hilbert space $\mathcal{H}_i$ for all $i \in I$, and let $\mathscr{M} = \bigoplus_{i\in I} \mathscr{M}_i$. For each $i \in I$, define $\rho_i\colon \mathscr{M}_i \to \mathscr{M}$ by $\rho_i(T) = (T_j)_{j\in I}$ where $T_j = T$ for $j = i$ and $T_j = 0$ for $j \neq i$. Then there is an isometric isomorphism $\Phi\colon \mathscr{M}_* \to \bigoplus_{i\in I}(\mathscr{M}_i)_*$ given by*

$$\Phi(\omega) = (\omega_i)_{i\in I}$$

where $\omega_i = \omega \circ \rho_i$ for each $i \in I$. Under this isomorphism, then for any $T = (T_i)_{i\in I} \in \mathscr{M}$ we have

$$\omega(T) = \sum_{i\in I} \omega_i(T_i).$$

Proof. Let $\iota_i\colon \mathcal{H}_i \to \mathcal{H}$ denote the injection related to the Hilbert space $\mathcal{H}_i$ (see page vii), and let $\mathcal{H} = \bigoplus_{i\in I} \mathcal{H}_i$. We will first prove that Φ is in fact well-defined. For any $T = (T_i)_{i\in I} \in \mathscr{M}$ and $\xi = (\xi_i)_{i\in I} \in \mathcal{H}$, note that

$$\left(\sum_{i\in I} \rho_i(T_i)\right)(\xi_i)_{i\in I} = \sum_{i\in I} \iota_i(T_i\xi_i) = T(\xi_i)_{i\in I},$$

so $\sum_{i\in I}\rho_i(T_i)=T$ where the sum is strongly and hence weakly convergent. For any finite subset $F\subseteq I$, note that

$$\left\|\left(\sum_{i\in F}\rho_i(T_i)\right)(\xi_i)_{i\in I}\right\|^2=\left\|\sum_{i\in F}\iota_i(T_i\xi_i)\right\|^2=\sum_{i\in F}\|T_i\xi_i\|^2\le\left(\sup_{i\in I}\|T_i\|\right)^2\|\xi\|^2,$$

so $\|\sum_{i\in F}\rho_i(T_i)\|\le\sup_{i\in I}\|T_i\|$. Hence by Proposition 2.1, $\sum_{i\in I}\rho_i(T_i)=T$ where the sum is ultraweakly convergent. Letting $\omega\in\mathscr{M}_*$ this immediately implies

$$\omega(T)=\omega\left(\sum_{i\in I}\rho_i(T_i)\right)=\sum_{i\in I}\omega_i(T_i).$$

Therefore,

$$\|\omega(T)\|\le\sum_{i\in I}\|\omega_i\|\|T_i\|\le\left(\sup_{i\in I}\|T_i\|\right)\sum_{i\in I}\|\omega_i\|=\|T\|\sum_{i\in I}\|\omega_i\|.$$

For a finite subset $F\subseteq I$, let $\varepsilon>0$, let λ be the cardinality of F and take operators $T_i\in(\mathscr{M}_i)_1$ for all $i\in F$ such that $0\ge\omega_i(T_i)\ge\|\omega_i\|-\frac{\varepsilon}{\lambda}$. Since $T=\sum_{i\in F}\rho_i(T_i)\in\mathscr{M}$ now satisfies $\|T\|\le 1$, we have

$$\sum_{i\in F}\|\omega_i\|\le\sum_{i\in F}\omega_i(T_i)+\varepsilon\le\omega(T)+\varepsilon\le\|\omega\|+\varepsilon.$$

Hence $\sum_{i\in F}\|\omega_i\|\le\|\omega\|$ for all finite subsets $F\subseteq I$, so $\sum_{i\in I}\|\omega_i\|\le\|\omega\|$.

If we can now prove that $\omega\circ\rho_i\in(\mathscr{M}_i)_*$ for all $\omega\in\mathscr{M}_*$ and $i\in I$, we will have proved that Φ is well-defined, but this is easy: assume that $T_\alpha\to T$ ultraweakly in $\mathscr{M}_i$. For any square-summable sequences $(\xi^n)_{n\ge1}$ and $(\eta^n)_{n\ge1}$ in $\mathcal{H}$ with $\xi^n=(\xi_i^n)_{i\in I}$ and $\eta^n=(\eta_i^n)_{i\in I}$, note that

$$\sum_{n=1}^\infty\|\xi_i^n\|^2\le\sum_{n=1}^\infty\sum_{i\in I}\|\xi_i^n\|^2=\sum_{n=1}^\infty\|\xi^n\|^2<\infty,$$

so that we have

$$\sum_{n=1}^\infty\langle(\rho_i(T_\alpha)-\rho_i(T))\xi^n,\eta^n\rangle=\sum_{n=1}^\infty\langle(T_\alpha-T)\xi_i^n,\eta_i^n\rangle,$$

whence $\rho_i(T_\alpha)\to\rho_i(T)$ ultraweakly. Hence $\omega_i(T_\alpha)\to\omega_i(T)$, so Φ is well-defined. Φ is also clearly linear, and on the grounds of what we have proved, we can also conclude that Φ is an isometry.

Finally we prove Φ is surjective. For any $(\varphi_i)_{i\in I}\in\bigoplus_{i\in I}(\mathscr{M}_i)_*$, define $\varphi\colon\mathscr{M}\to\mathbb{C}$ by

$$\varphi((T_i)_{i\in I})=\sum_{i\in I}\varphi_i(T_i).$$

Then φ is clearly well-defined, linear and bounded. To prove that φ is ultraweakly continuous, let $F\subseteq I$ be finite and define $\varphi_F(T)=\sum_{i\in F}\varphi_i(T_i)$ for $T=(T_i)_{i\in I}\in\mathscr{M}$. Let $(T_\alpha)_{\alpha\in A}$ be a net in $(\mathscr{M})_1$ with $T^\alpha=(T_i^\alpha)_{i\in I}$ for all α, let $T=(T_i)_{i\in I}\in(\mathscr{M})_1$ and assume that $T^\alpha\to T$ weakly. For $i\in I$ and $\xi,\eta\in\mathcal{H}_i$, note that

$$\langle T_i^\alpha\xi,\eta\rangle=\langle T^\alpha\iota_i(\xi),\iota_i(\eta)\rangle\to\langle T\iota_i(\xi),\iota_i(\eta)\rangle=\langle T_i\xi,\eta\rangle,$$

so $T_i^\alpha\to T_i$ weakly for all $i\in I$. As φ_i is weakly continuous on $(\mathscr{M}_i)_1$ by Corollary 2.12, we have $\varphi_i(T_i^\alpha)\to\varphi_i(T_i)$ for all $i\in I$. Hence $\varphi_F(T_\alpha)\to\varphi_F(T)$, so $\varphi_F\in\mathscr{M}_*$. For $\varepsilon>0$, let $T=(T_i)_{i\in I}\in(\mathscr{M})_1$ and choose a finite subset $F\subseteq I$ such that $\sum_{i\notin F}\|\varphi_i\|<\varepsilon$. Then

$$\|\varphi(T)-\varphi_F(T)\|\le\left\|\sum_{i\notin F}\varphi_i(T_i)\right\|\le\sum_{i\notin F}\|\varphi_i(T_i)\|\le\sum_{i\notin F}\|\varphi_i\|<\varepsilon,$$

so $\|\varphi-\varphi_F\|<\varepsilon$. Hence $\varphi_F\to\varphi$ in norm, so since $\mathscr{M}_*$ is norm-closed by Theorem 2.7, we have $\varphi\in\mathscr{M}_*$. Finally, it is clear that $\Phi(\varphi)=(\varphi_i)_{i\in I}$, so Φ is an isometric isomorphism. □

2.10 Intermezzo 3: σ-finite von Neumann algebras

As promised, this intermezzo will use the knowledge of the two previous intermezzos along with some ideas from the subsequent couple of sections.

Definition 2.11. A von Neumann algebra $\mathscr{M}$ is said to be *σ-finite* or *countably decomposable* if every family of non-zero pairwise orthogonal projections of $\mathscr{M}$ is countable.

The notion of σ-finiteness will be irrelevant for the moment, but will return with great vengeance in Chapter 5 because of a result proved in this section. The result itself requires us to know about equivalent conditions to σ-finiteness for von Neumann algebras which we will investigate immediately:

Proposition 2.58. *Let $\mathscr{M} \subseteq B(\mathcal{H})$ be a von Neumann algebra. Then the following are equivalent:*

(i) *$\mathscr{M}$ is σ-finite.*
(ii) *There exists a countable family of separating vectors for $\mathscr{M}$.*
(iii) *There exists a faithful normal state on $\mathscr{M}$.*
(iv) *$\mathscr{M}$ is *-isomorphic to a von Neumann algebra $\mathscr{N} \subseteq B(\mathcal{K})$ admitting a separating and cyclic unit vector.*

Proof. (i) $\Rightarrow$ (ii): Let $(\xi_\alpha)_{\alpha\in A}$ be a maximal family of non-zero vectors in $\mathcal{H}$ such that the subspaces $[\mathscr{M}'\xi_\alpha]$ and $[\mathscr{M}'\xi_\beta]$ are pairwise orthogonal for $\alpha \neq \beta$. Let P_α be the projection onto $[\mathscr{M}'\xi_\alpha]$ for all $\alpha \in A$. Assuming that $P = \sum_{\alpha\in A} P_\alpha < 1_{\mathcal{H}}$, then there exists a non-zero $\xi \in \mathcal{H}$ such that $P_\alpha\xi = 0$ for all $\alpha \in A$. For $\alpha \in A$ and any $T \in \mathscr{M}'$, we have $T^*[\mathscr{M}'\xi_\alpha] \subseteq [\mathscr{M}'\xi_\alpha]$, so

$$\langle T\xi, \eta\rangle = \langle \xi, T^*\eta\rangle = \langle P_\alpha\xi, T^*\eta\rangle = 0$$

for all $\eta \in [\mathscr{M}'\xi_\alpha]$, so $[\mathscr{M}'\xi]$ and $[\mathscr{M}'\xi_\alpha]$ are orthogonal subspaces for all $\alpha \in A$, contradicting maximality. Hence

$$\sum_{\alpha\in A} P_\alpha = 1_{\mathcal{H}}.$$

Hence for $\xi \in \mathcal{H}$ and $\varepsilon > 0$, there exists a finite subset $F \subseteq A$ such that $\|\xi - \sum_{\alpha\in F} P_\alpha\xi\| < \frac{\varepsilon}{2}$. Letting λ be the cardinality of F and choosing $T_\alpha \in \mathscr{M}'$ such that $\|T_\alpha\xi_\alpha - P_\alpha\xi\| < \frac{\varepsilon}{2\lambda}$ for $\alpha \in F$, it follows that

$$\left\|\xi - \sum_{\alpha\in F} T_\alpha\xi_\alpha\right\| \leq \left\|\xi - \sum_{\alpha\in F} P_\alpha\xi\right\| + \sum_{\alpha\in F} \|T_\alpha\xi_\alpha - P_\alpha\xi\| < \varepsilon,$$

so $(\xi_\alpha)_{\alpha\in A}$ is a set of cyclic vectors for $\mathscr{M}'$. By Proposition 2.21, $(\xi_\alpha)_{\alpha\in A}$ is a separating set for $\mathscr{M}$. By σ-finiteness, A is countable.

(ii) $\Rightarrow$ (iii): Let $(\xi_n)_{n\geq 1}$ be a sequence of separating vectors for $\mathscr{M}$. By scaling, we can assume that $\sum_{n=1}^\infty \|\xi_n\|^2 = 1$. Define $\omega\colon \mathscr{M} \to \mathbb{C}$ by

$$\omega(T) = \sum_{n=1}^\infty \langle T\xi_n, \xi_n\rangle.$$

ω is then a normal state, and moreover if $\omega(T^*T) = 0$ for $T \in \mathscr{M}$, then $\|T\xi_n\|^2 = 0$ for all $n \geq 1$, implying $T = 0$ since the sequence $(\xi_n)_{n\geq 1}$ was separating.

(iii) $\Rightarrow$ (iv): Let ω be a faithful normal state on $\mathscr{M}$, and let $(\mathcal{H}_\omega, \pi_\omega, \xi_\omega)$ be the corresponding GNS triple where π_ω maps $\mathscr{M}$ into $B(\mathcal{H}_\omega)$. Then by Proposition 2.49, $\pi_\omega(\mathscr{M})$ is a von Neumann algebra and ξ_ω is a cyclic vector for $\pi_\omega(\mathscr{M})$ by construction. Moreover, π_ω is injective because ω is faithful, so $\mathscr{M} \cong \pi_\omega(\mathscr{M})$. If $\pi_\omega(T)\xi_\omega = 0$ for $T \in \mathscr{M}$, then $\omega(T^*T) = \langle \pi_\omega(T^*T)\xi_\omega, \xi_\omega\rangle = \|\pi_\omega(T)\xi_\omega\|^2 = 0$, so $T = 0$ by faithfulness of ω, implying that ξ_ω is separating for $\pi_\omega(\mathscr{M})$ as well.

(iv) $\Rightarrow$ (i): Let $\pi\colon \mathscr{M} \to \mathscr{N}$ be the *-isomorphism connecting $\mathscr{M}$ to $\mathscr{N}$, let $\xi \in \mathcal{K}$ be the separating (and cyclic vector) for $\mathscr{N}$, and let $(P_\alpha)_{\alpha\in A}$ be a family of mutually orthogonal projections in $\mathscr{M}$. Put $P = \sum_{\alpha\in A} P_\alpha$. Then $(\pi(P_\alpha))_{\alpha\in A}$ is a family of mutually orthogonal projections in $\mathscr{N}$ and $\pi(P) = \sum_{\alpha\in A} \pi(P_\alpha)$ where the sum is strong operator convergent. Therefore

$$\sum_{\alpha\in A} \|\pi(P_\alpha)\xi\|^2 = \|\pi(P)\xi\|^2 < \infty$$

by [31, Corollary 17.4], so $\pi(P_\alpha)\xi$ is non-zero for only countably many $\alpha \in A$. Since ξ is separating, $\pi(P_\alpha)$ and hence P_α is non-zero for only countably many $\alpha \in A$, so $\mathscr{M}$ is σ-finite. $\square$

Because of its encompassingness, the next result is wonderfully surprising and surprisingly wonderful.

Proposition 2.59. *Let $\mathscr{M} \subseteq B(\mathcal{H})$ be a von Neumann algebra. Then*

$$\mathscr{M} \cong \bigoplus_{\alpha \in A} \mathscr{M}_\alpha$$

where $\mathscr{M}_\alpha$ is *-isomorphic to a reduced von Neumann algebra of $\mathscr{M}_{P_\alpha} \overline{\otimes} B(\mathcal{K}_\alpha)$, $\mathcal{K}_\alpha$ being a suitable Hilbert space and P_α is a projection in $\mathscr{M}$ such that $\mathscr{M}_{P_\alpha} \subseteq B(P_\alpha(\mathcal{H}))$ has a separating vector. Moreover, each $\mathscr{M}_{P_\alpha}$ is σ-finite.

Proof. Let $\xi \in \mathcal{H}$ and let P be the projection onto $[\mathscr{M}'\xi]$. Then by Lemma 2.14, $P \in \mathscr{M}$. For $\eta \in [\mathscr{M}'\xi]$, there exists $T \in \mathscr{M}'$ such that $\|T\xi - \eta\| < \varepsilon$, and hence $\|PT|_{P(\mathcal{H})}\xi - \eta\| < \varepsilon$, proving that ξ is a cyclic vector for the von Neumann algebra $(\mathscr{M}')_P \subseteq B(P(\mathcal{H}))$ and hence a separating vector for $((\mathscr{M}')_P)' = \mathscr{M}_P$ by Propositions 2.17 and 2.21.

Let C_P denote the central support of a projection $P \in \mathscr{M}$. Choose a maximal family $(\xi_\alpha)_{\alpha \in A}$ of non-zero vectors in $\mathcal{H}$ such that C_{P_α} and $C_{P_{\alpha'}}$ are orthogonal for $\alpha \neq \alpha'$, where P_α is the projection onto $[\mathscr{M}'\xi_\alpha]$ for $\alpha \in A$. Assume that $\sum_{\alpha \in A} C_{P_\alpha} < 1_{\mathcal{H}}$. Then there exists a non-zero vector in $\mathcal{H}$ such that $C_{P_\alpha}\xi = 0$ for all $\alpha \in A$. Letting P denote the orthogonal projection onto $[\mathscr{M}'\xi]$, then for $\eta \in \mathcal{H}$ and $T \in \mathscr{M}'$, we have

$$\langle T\xi, C_{P_\alpha}\eta\rangle = \langle C_{P_\alpha}\xi, T^*\eta\rangle = 0$$

for all $\alpha \in \mathcal{A}$, so C_{P_α} and P are orthogonal for all $\alpha \in A$. Hence C_{P_α} and C_P are orthogonal for all $\alpha \in A$, contradicting maximality.

Hence $\sum_{\alpha \in A} C_{P_\alpha} = 1_{\mathcal{H}}$, so

$$\mathscr{M} \cong \bigoplus_{\alpha \in A} \mathscr{M}_{C_{P_\alpha}}$$

by Proposition 2.19. Now $C_{P_\alpha}P_\alpha|_{C_{P_\alpha}(\mathcal{H})} = P_\alpha|_{C_{P_\alpha}(\mathcal{H})}$ is a projection in $\mathscr{M}_{C_{P_\alpha}}$ whose central support in $\mathscr{M}_{C_{P_\alpha}}$ is equal to the identity operator on $C_{P_\alpha}(\mathcal{H})$; indeed, this follows just by considering what the central support really is (see Definition 2.5). Corollary 2.55 then tells us that $\mathscr{M}_{C_{P_\alpha}}$ is isomorphic to a reduced von Neumann algebra of $\mathscr{M}_{P_\alpha} \overline{\otimes} B(\mathcal{K}_\alpha)$ for some Hilbert space $\mathcal{K}_\alpha$.

Finally $\mathscr{M}_{P_\alpha}$ has a separating vector ξ_α by construction seen in the first paragraph of the proof, so by Proposition 2.58, $\mathscr{M}_{P_\alpha}$ is σ-finite. $\square$

2.11 The universal enveloping von Neumann algebra

So far this chapter might have seemed more like a "take-your-daughter-to-work day" thing than a development of a single idea, say, the ultraweak topology. True, the digressions throughout we have made have been necessary but nonetheless, reading through the preceding 10 sections has probably not been the smoothest ride. We will make up for this by developing a concept, using *a lot* of the concepts developed and theorems proved throughout the chapter, and the results will be of great use for the remaining three chapters.

Knowing now that von Neumann algebras are really rigid concerning the various operator topologies and that one of them, namely the ultraweak topology, has a lot in common with the weak* topology, we can pass on to a very important application of what we have been proving up until now. It turns out that a lot of theorems for C^*-algebras need only be checked for von Neumann algebras because of a very special embedding of a C^*-algebra into a von Neumann algebra that we will be working towards finding from here onward. One might then consider that there could be a lot of von Neumann algebras allowing for such an embedding, and therefore it would be the best thing to find a von Neumann algebra that is possible to derive directly from the original C^*-algebra, without any reference to a specific Hilbert space. That is exactly what we will do: our specific von Neumann algebra will have the structure of a dual space related to the C^*-algebra.

To commence our search, we bring in another definition.

Definition 2.12. Let $\mathcal{A}$ be a C^*-algebra and let $\pi\colon \mathcal{A} \to B(\mathcal{H})$ be a representation of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$. Then π is said to be *nondegenerate* if $\pi(\mathcal{A})$ is nondegenerate. If π is nondegenerate, then π is called *universal* if it satisfies the following property: Given any other nondegenerate representation ρ of $\mathcal{A}$ on a Hilbert space $\mathcal{K}$, there exists a normal *-homomorphism ρ_0 of $\pi(\mathcal{A})''$ onto $\rho(\mathcal{A})''$ such that $\rho = \rho_0 \circ \pi$, i.e. such that the diagram

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "\pi"] \arrow[dr, "\rho"'] & & \pi(\mathcal{A})'' \arrow[dl, dotted, "\rho_0"] \\ & \rho(\mathcal{A})'' & \end{tikzcd}$$

commutes.

Note that the requirement above that $\rho = \rho_0 \circ \pi$ and that ρ_0 is normal implies that ρ_0 is actually a *-homomorphism; this follows since $\pi(\mathcal{A})$ is ultraweakly dense in $\pi(\mathcal{A})''$ by the von Neumann density theorem. For instance, multiplicativity follows from letting $x, y \in \pi(\mathcal{A})''$, whereupon there exist nets $(x_\alpha)_{\alpha\in A}$ and $(y_\beta)_{\beta\in B}$ such that $\pi(x_\alpha) \to x$ and $\pi(y_\beta) \to y$ ultraweakly. Hence

$$\begin{aligned} \rho_0(xy) &= \lim_{\alpha\in A}\lim_{\beta\in B}\rho_0(\pi(x_\alpha)\pi(y_\beta)) = \lim_{\alpha\in A}\lim_{\beta\in B}\rho_0(\pi(x_\alpha y_\beta)) \\ &= \lim_{\alpha\in A}\lim_{\beta\in B}\rho(x_\alpha y_\beta) = \lim_{\alpha\in A}\rho(x_\alpha)\lim_{\beta\in B}\rho(y_\beta) \\ &= \lim_{\alpha\in A}\rho_0(\pi(x_\alpha))\lim_{\beta\in B}\rho_0(\pi(y_\beta)) = \rho_0(x)\rho_0(y), \end{aligned}$$

as left and right multiplication are ultraweakly continuous operations.

The notion of a representation of course allows us to embed $\mathcal{A}$ in a von Neumann algebra, namely the von Neumann algebra generated by the image of the representation. Requiring that a representation be universal yields that this von Neumann algebra is essentially unique. Indeed, if $\pi_1\colon \mathcal{A} \to B(\mathcal{H}_1)$ and $\pi_2\colon \mathcal{A} \to B(\mathcal{H}_2)$ are universal representations of $\mathcal{A}$, there exist normal *-homomorphisms $\rho_1\colon \pi_1(\mathcal{A})'' \to \pi_2(\mathcal{A})''$ and $\rho_2\colon \pi_2(\mathcal{A})'' \to \pi_1(\mathcal{A})''$ such that $\pi_2 = \rho_1 \circ \pi_1$ and $\pi_1 = \rho_2 \circ \pi_2$. Hence

$$\rho_2 \circ \rho_1 \circ \pi_1 = \rho_2 \circ \pi_2 = \pi_1, \quad \rho_1 \circ \rho_2 \circ \pi_2 = \rho_1 \circ \pi_1 = \pi_2,$$

so $\rho_2 \circ \rho_1$ and $\rho_1 \circ \rho_2$ are the identity maps on $\pi_1(\mathcal{A})$ and $\pi_2(\mathcal{A})$ respectively. Because they are also normal, they are the identity maps on $\pi_1(\mathcal{A})''$ and $\pi_2(\mathcal{A})''$. Hence ρ_1 and ρ_2 are isomorphisms and inverses of each other, and $\pi_1(\mathcal{A})''$ and $\pi_2(\mathcal{A})''$ are isomorphic von Neumann algebras. Thus for a universal representation $\pi\colon \mathcal{A} \to B(\mathcal{H})$, we may speak of the *(universal) enveloping von Neumann algebra* $\pi(\mathcal{A})''$ of $\mathcal{A}$; it is unique up to isomorphism.

We now aim at proving that the universal enveloping von Neumann algebra is related to $\mathcal{A}$ in a very non-obvious way that nonetheless is very delicate. One might of course inquire first whether a C^*-algebra even *has* a universal representation. The GNS representation comes to our aid: for every $\varphi \in S(\mathcal{A})$, let $(\pi_\varphi, \mathcal{H}_\varphi, \xi_\varphi)$ denote the GNS triple corresponding to φ. Let $\mathcal{H} = \bigoplus_{\varphi\in S(\mathcal{A})}\mathcal{H}_\varphi$ and let

$$\pi = \bigoplus_{\varphi\in S(\mathcal{A})}\pi_\varphi\colon \mathcal{A} \to B(\mathcal{H}),$$

i.e.

$$\pi(x)(\eta_\varphi)_{\varphi\in S(\mathcal{A})} = (\pi_\varphi(x)\eta_\varphi)_{\varphi\in S(\mathcal{A})}, \quad x \in \mathcal{A},\ (\eta_\varphi)_{\varphi\in S(\mathcal{A})} \in \mathcal{H}.$$

Then we have shown that π is a faithful representation of $\mathcal{A}$ (see page viii). π is also nondegenerate; if $\mathcal{A}$ is unital, then this is clear as $\pi(1_{\mathcal{A}}) = 1_{\mathcal{H}}$. If $\mathcal{A}$ is non-unital, then there exists an approximate identity $(e_\alpha)_{\alpha\in A}$ of $\mathcal{A}$ that satisfies $\pi(e_\alpha)\eta \to \eta$ for all $\eta \in \mathcal{H}$ by Proposition 0.5. Hence $\eta \in [\pi(\mathcal{A})\mathcal{H}]$ for all $\eta \in \mathcal{H}$, so π is nondegenerate.

Before going any further, we will prove an essential fact about *-homomorphisms of C^*-algebras. The proof below does not require the C^*-algebras to be unital; the continuous functional calculus is involved, but its use revolves around a function that maps 0 to 0 and hence can be approximated by a polynomial with no constant term.

Proposition 2.60. *Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras and let $\varphi\colon \mathcal{A} \to \mathcal{B}$ be a *-homomorphism. Then:*

(i) *If $b \in \varphi(\mathcal{A})$ and $b \in \mathcal{B}_{\mathrm{sa}}$, then there exists $a \in \mathcal{A}_{\mathrm{sa}}$ such that $\varphi(a) = b$ and $\|a\| = \|b\|$.*
(ii) *If $b \in \varphi(\mathcal{A})$, then there exists $a \in \mathcal{A}$ such that $\varphi(a) = b$ and $\|a\| = \|b\|$.*

Proof. (i) Take $x \in \mathcal{A}$ such that $\varphi(x) = b$ and set $y = \frac{1}{2}(x + x^*)$. Then $y \in \mathcal{A}_{\mathrm{sa}}$ and $\varphi(y) = b$. Define a function $f\colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} -\|b\| & x \le -\|b\| \\ x & -\|b\| \le x \le \|b\| \\ \|b\| & x \ge \|b\|. \end{cases}$$

Then f is continuous and $f(b) = b$. Putting $a = f(y)$, then $a \in \mathcal{A}_{\mathrm{sa}}$, $\|a\| \le \|b\|$ and

$$\varphi(a) = \varphi(f(y)) = f(\varphi(y)) = f(b) = b.$$

As φ is contractive, it also follows that $\|b\| \le \|a\|$, and hence we are done.

(ii) Take $x \in \mathcal{A}$ such that $\varphi(x) = b$. Define $\tilde{b} \in M_2(\mathcal{B})$ by

$$\tilde{b} = \begin{pmatrix} 0 & b^* \\ b & 0 \end{pmatrix}.$$

Then $\tilde{b} = \tilde{b}^*$ and $\|\tilde{b}\| = \|b\|$. As $\varphi^{(2)}\colon M_2(\mathcal{A}) \to M_2(\mathcal{B})$ is a *-homomorphism, and

$$\varphi^{(2)} \begin{pmatrix} 0 & x^* \\ x & 0 \end{pmatrix} = b,$$

it follows from (i) that there exists $\tilde{a} \in M_2(\mathcal{A})_{\mathrm{sa}}$ such that $\varphi^{(2)}(\tilde{a}) = \tilde{b}$ and $\|\tilde{a}\| = \|\tilde{b}\|$. Write

$$\tilde{a} = \begin{pmatrix} \star & \star \\ a & \star \end{pmatrix},$$

so that $\varphi(a) = b$ and $\|a\| \le \|\tilde{a}\| = \|\tilde{b}\| = \|b\|$. As *-homomorphisms are contractive, it follows that $\|b\| \le \|a\|$ as well, and this proves the statement. $\square$

Corollary 2.61. *Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras and let $\varphi\colon \mathcal{A} \to \mathcal{B}$ be a *-homomorphism. Then for all $r > 0$, $\varphi((\mathcal{A})_r) = (\varphi(\mathcal{A}))_r$.*

Proof. If $a \in \mathcal{A}$ with $\|a\| \le r$, then $\|\varphi(a)\| \le \|a\| \le r$, so $\varphi((\mathcal{A})_r) \subseteq (\varphi(\mathcal{A}))_r$. On the other hand, for $b \in \varphi(\mathcal{A})$ with $\|b\| \le r$, then by Proposition 2.60 there exists $a \in \mathcal{A}$ with $\varphi(a) = b$ and $\|a\| \le r$. Hence $(\varphi(\mathcal{A}))_r \subseteq \varphi((\mathcal{A})_r)$. $\square$

If $\varphi\colon \mathcal{A} \to \mathcal{B}$ is a bounded linear map of normed spaces, then its *dual mapping* $\varphi^*\colon \mathcal{B}^* \to \mathcal{A}^*$ or the *adjoint map of* φ is given by

$$\varphi^*(\psi) = \psi \circ \varphi.$$

φ^* is linear and bounded by $\|\varphi\|$ and is weak*-to-weak* continuous. Indeed, if $\psi_\alpha \to \psi$ in the weak* topology in $\mathcal{B}^*$, then for all $x \in \mathcal{A}$,

$$\varphi^*(\psi_\alpha)(x) = \psi_\alpha(\varphi(x)) \to \psi(\varphi(x)) = \varphi^*(\psi)(x),$$

so $\varphi^*(\psi_\alpha) \to \varphi^*(\psi)$ in the weak* topology in $\mathcal{A}^*$. Moreover, for bounded linear maps $\varphi\colon \mathcal{A} \to \mathcal{B}$ and $\psi\colon \mathcal{B} \to \mathcal{C}$, then $(\varphi \circ \psi)^* = \psi^* \circ \varphi^*$. In particular, if a bounded linear map $\varphi\colon \mathcal{A} \to \mathcal{B}$ is bijective and the inverse $\varphi^{-1}\colon \mathcal{B} \to \mathcal{A}$ is bounded, then φ^* is bijective as well with $(\varphi^*)^{-1} = (\varphi^{-1})^*$; note that if $\mathcal{A}$ and $\mathcal{B}$ are Banach spaces, then boundedness of the inverse φ^{-1} follows from the Open Mapping Theorem [13, Theorem 5.10].

Theorem 2.62. *Let $\mathcal{A}$ be a C^*-algebra and $\pi\colon \mathcal{A} \to B(\mathcal{H})$ be a representation of $\mathcal{A}$. Let $\mathscr{M}$ denote the von Neumann algebra $\pi(\mathcal{A})''$. Then there is a unique weak*-to-ultraweakly continuous linear map $\tilde{\pi}\colon \mathcal{A}^{**} \to \mathscr{M}$ such that the diagram*

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "\pi"] \arrow[dr, "\iota"'] & & \mathscr{M} \\ & \mathcal{A}^{**} \arrow[ur, dotted, "!\tilde{\pi}"'] & \end{tikzcd}$$

commutes, where $\iota\colon \mathcal{A} \to \mathcal{A}^{**}$ denotes the natural inclusion. Moreover, $\tilde{\pi}$ maps the closed unit ball of $\mathcal{A}^{**}$ onto the closed unit ball of $\mathscr{M}$ and is therefore a surjection.

Proof. Let Ω denote the restriction of the adjoint linear map $\pi^*\colon \mathscr{M}^* \to \mathcal{A}^*$ to the predual Banach space $\mathscr{M}_* \subseteq \mathscr{M}^*$ of $\mathscr{M}$., i.e. $\Omega = \pi^*|_{\mathscr{M}_*}$. Taking the adjoint of Ω yields a linear map $\Omega^*\colon \mathcal{A}^{**} \to (\mathscr{M}_*)^*$; by composing with the inverse of the isometric isomorphism $\Lambda\colon \mathscr{M} \to (\mathscr{M}_*)^*$, we obtain a map $\Lambda^{-1} \circ \Omega^*\colon \mathcal{A}^{**} \to \mathscr{M}$. We claim that this is the wanted $\tilde{\pi}$. First and foremost, it *does* extend π to $\mathcal{A}^{**}$, as $\tilde{\pi} \circ \iota = \pi$: indeed for all $a \in \mathcal{A}$ and $\omega \in \mathscr{M}_*$ we have

$$(\Omega^*(\iota(a))(\omega) = \iota(a)(\Omega(\omega)) = \Omega(\omega)(a) = \omega(\pi(a)) = \Lambda(\pi(a))(\omega),$$

so $\tilde{\pi} \circ \iota = \Lambda^{-1} \circ \Omega^* \circ \iota = \pi$. As adjoints of bounded linear maps are weak*-to-weak* continuous and Λ^{-1} is weak*-to-ultraweakly continuous, it also follows that $\tilde{\pi}$ is weak*-to-ultraweakly continuous. Because $\iota(\mathcal{A})$ is weak*-dense in $\mathcal{A}^{**}$ by Goldstine's theorem [29, Theorem II.A.13], it follows that $\tilde{\pi}$ is the only weak*-to-ultraweakly continuous extension making the above diagram commute.

Finally, let $\mathscr{S} = \tilde{\pi}((\mathcal{A}^{**})_1)$. We claim that $\mathscr{S} = (\mathscr{M})_1$. Note that $(\iota(\mathcal{A}))_1$ is the weak* closure of $(\mathcal{A}^{**})_1$ by Goldstine's theorem, and that $\tilde{\pi}((\iota(\mathcal{A}))_1) = \pi((\mathcal{A})_1) = (\pi(\mathcal{A}))_1$ since ι is an isometry and π is a *-homomorphism, by using Corollary 2.61. As $\tilde{\pi}$ is weak*-to-ultraweakly continuous, it follows that $\mathscr{S}$ is contained in the ultraweak closure of $\pi(\mathcal{A})_1$. Since the ultraweak closure of $(\pi(\mathcal{A}))_1$ is equal to $(\mathscr{M})_1$ by Kaplansky's density theorem and Proposition 2.1, it then follows that $\mathscr{S} \subseteq (\mathscr{M})_1$. For the converse inclusion, note that

$$(\pi(\mathcal{A}))_1 = \pi((\mathcal{A})_1) = \tilde{\pi}((\iota(\mathcal{A}))_1) \subseteq \tilde{\pi}((\mathcal{A}^{**})_1) = \mathscr{S}$$

by Corollary 2.61. By Alaoglu's theorem [13, Theorem 5.18], $(\mathcal{A}^{**})_1$ is weak*-compact, so $\mathscr{S}$ is ultraweakly compact and therefore ultraweakly closed. Hence $\mathscr{S}$ contains the ultraweak closure of $(\pi(\mathcal{A}))_1$, which is $(\mathscr{M})_1$ by Kaplansky's density theorem and Proposition 2.1. Therefore $\mathscr{S} = (\mathscr{M})_1$. It clearly follows that $\tilde{\pi}$ is surjective. □

Theorem 2.63. *Let $\mathcal{A}$ be a C^*-algebra. There exists a universal representation π of $\mathcal{A}$ onto a Hilbert space $\mathcal{H}$ such that $\pi\colon \mathcal{A} \to \pi(\mathcal{A})''$ extends to a surjective isometry $\tilde{\pi}\colon \mathcal{A}^{**} \to \pi(\mathcal{A})''$ that is a weak*-to-ultraweak homeomorphism. Moreover, the predual of $\pi(\mathcal{A})''$ is isometrically isomorphic to $\mathcal{A}^*$.*

Proof. We will use the representation π and Hilbert space $\mathcal{H}$ defined above Proposition 2.60. By Theorem 2.62, there is a unique linear surjective map $\tilde{\pi}\colon \mathcal{A}^{**} \to \mathscr{M}$ where $\mathscr{M} = \pi(\mathcal{A})''$. Recalling its construction, $\tilde{\pi}$ was the composition of the isometric isomorphism $\Lambda^{-1}\colon (\mathscr{M}_*)^* \to \mathscr{M}$ and the conjugate of the map $\Omega\colon \mathscr{M}_* \to \mathcal{A}^*$ that itself was the restriction of the conjugate map $\pi^*\colon \mathscr{M}^* \to \mathcal{A}^*$. We will first show that Ω is a surjective isometry, yielding the second statement.

For all $\omega \in \mathscr{M}_*$, then

$$\|\Omega(\omega)\| = \|\omega \circ \pi\| = \sup\{|\omega(\pi(x))| \,|\, x \in \mathcal{A},\ \|x\| \leq 1\}.$$

π is an injective *-homomorphism and hence an isometry, so

$$\sup\{|\omega(\pi(x))| \,|\, x \in \mathcal{A},\ \|x\| \leq 1\} = \sup\{|\omega(y)| \,|\, y \in \pi(\mathcal{A}),\ \|y\| \leq 1\} \leq \sup\{|\omega(y)| \,|\, y \in \mathscr{M},\ \|y\| \leq 1\}.$$

In fact, the opposite inequality also holds. Since $\pi(\mathcal{A})$ is nondegenerate, it is ultraweakly dense in $\mathscr{M}$ by von Neumann's density theorem. Hence for any $y \in (\mathscr{M})_1$ there is a net $(y_\alpha)_{\alpha \in A}$ with $\|y_\alpha\| \leq 1$ for all $\alpha \in A$ by converging ultraweakly to y because of Kaplansky's density theorem and Proposition 2.1. Since ω is ultraweakly continuous, $\omega(y_\alpha) \to \omega(y)$ as well, implying $|\omega(y_\alpha)| \to |\omega(y)|$. Since

$$|\omega(y_\alpha)| \leq \sup\{|\omega(y)| \,|\, y \in \pi(\mathcal{A}),\ \|y\| \leq 1\}$$

for all $\alpha \in A$, it follows that $|\omega(y)| \leq \sup\{|\omega(y)| \,|\, y \in \pi(\mathcal{A}),\ \|y\| \leq 1\}$ as well, proving the other inequality. Finally, as

$$\sup\{|\omega(y)| \,|\, y \in \mathscr{M},\ \|y\| \leq 1\} = \|\omega\|,$$

we have proved that $\|\Omega(\omega)\| = \|\omega\|$ for all $\omega \in \mathscr{M}_*$. Hence Ω is an isometry.

To prove that Ω is surjective, let $\varphi \in \mathcal{A}^*$. By Theorem 2.34, it follows that $\varphi = \sum_{i=1}^4 \lambda_i \varphi_i$ for $\lambda_i \in \mathbb{C}$ and $\varphi_i \in S(\mathcal{A})$ where $i = 1, 2, 3, 4$. Using the GNS representations of $\mathcal{A}$, define elements ξ and η in $\mathcal{H}$ by

$$\xi = \sum_{i=1}^4 \lambda_i \xi_{\varphi_i}, \quad \eta = \sum_{i=1}^4 \xi_{\varphi_i},$$

where each ξ_{φ_i} is considered as an element in $\mathcal{H}$. If $\omega \in \mathscr{M}_*$ is given by $\omega(T) = \langle T\xi, \eta\rangle$ for $T \in \mathscr{M}$, then for all $x \in \mathcal{A}$ we have

$$\Omega(\omega)(x) = \omega(\pi(x)) = \langle \pi(x)\xi, \eta\rangle = \sum_{i=1}^{4} \lambda_i \langle \pi(x)\xi_{\varphi_i}, \xi_{\varphi_i}\rangle = \sum_{i=1}^{4} \lambda_i \langle \pi_{\varphi_i}(x)\xi_{\varphi_i}, \xi_{\varphi_i}\rangle = \sum_{i=1}^{4} \lambda_i \varphi_i(x) = \varphi(x).$$

Hence Ω is surjective, so $\mathscr{M}_*$ is isometrically isomorphic to $\mathcal{A}^*$ under Ω.

For the first statement, note that

$$\|\Omega^*(\varphi)\| = \sup\{|\varphi(\Omega(\omega))| \,|\, \omega \in \mathscr{M}_*,\ \|\omega\| \leq 1\} = \sup\{|\varphi(\psi)| \,|\, \psi \in \mathcal{A}^*,\ \|\psi\| \leq 1\} = \|\varphi\|$$

by Ω being a surjective isometry, so Ω^* is an isometry. Finally, for $\varphi \in (\mathscr{M}_*)^*$, define $\psi\colon \mathcal{A}^* \to \mathbb{C}$ by $\psi(\Omega(\omega)) = \varphi(\omega)$, possible as Ω is surjective. Then ψ is well-defined, linear and bounded above by $\|\varphi\|$ by Ω being an isometry, and $\Omega^*(\psi) = \varphi$, so Ω^* is surjective. Therefore $\tilde{\pi}$ is a surjective isometry, and since $(\Omega^*)^{-1} = (\Omega^{-1})^*$, it follows that Ω^* is a weak*-to-weak* homeomorphism, as dual mappings are always weak*-to-weak* continuous. It therefore follows that $\tilde{\pi}$ is also a weak*-to-ultraweak operator topology homeomorphism.

Finally, π is a universal representation. Indeed, let $\rho\colon \mathcal{A} \to B(\mathcal{K})$ be a nondegenerate representation of $\mathcal{A}$ onto some Hilbert space $\mathcal{K}$. Then by Theorem 2.62, ρ induces a linear map $\tilde{\rho}\colon \mathcal{A}^{**} \to \rho(\mathcal{A})''$. Define $\rho_0 = \tilde{\rho} \circ \tilde{\pi}^{-1}$. Then ρ_0 is an ultraweakly continuous linear map of $\pi(\mathcal{A})''$ onto $\rho(\mathcal{A})''$, and $\rho_0(\pi(x)) = \tilde{\rho}(\iota(x)) = \rho(x)$ for all $x \in \mathcal{A}$. Furthermore, ρ_0 is also a *-homomorphism (see page 59) so π is indeed universal. □

The above theorem is truly a gold mine. The above surjective isometry allows us to identify the universal enveloping von Neumann algebra of $\mathcal{A}$ with $\mathcal{A}^{**}$ and $\mathcal{A}^{**}$ can hence be endowed with a C^*-algebra structure, whereupon the surjective isometry $\tilde{\pi}$ of Theorem 2.63 becomes a *-isomorphism. By Theorem 2.63, the inclusion $\iota\colon \mathcal{A} \to \mathcal{A}^{**}$ becomes a *-homomorphism. Indeed, if π is a universal representation of $\mathcal{A}$, then by the identification under the map $\tilde{\pi}\colon \mathcal{A}^{**} \to \pi(\mathcal{A})''$, then ι is just π. Furthermore, as $\iota(\mathcal{A})$ (or $\mathcal{A}$) is weak*-dense in the dual space $\mathcal{A}^{**}$, it is ultraweakly dense in the von Neumann algebra $\mathcal{A}^{**}$, and $(\mathcal{A}^{**})_* \cong \mathcal{A}^*$ by the isometric isomorphism $\Omega\colon (\mathcal{A}^{**})_* \cong \mathcal{A}^*$ given by

$$\Omega(\omega)(a) = \omega(\iota(a)), \quad \omega \in (\mathcal{A}^{**})_*,\ a \in \mathcal{A}.$$

This is of course easy to remember; one of the asterisks cancels out!

Equally important is that the defining property of universal representations tells us that for any nondegenerate representation $\rho\colon \mathcal{A} \to B(\mathcal{H})$, there exists a surjective normal *-homomorphism $\rho_0\colon \mathcal{A}^{**} \to \rho(\mathcal{A})''$ such that $\rho = \rho_0 \circ \iota$. Any linear map $\varphi\colon \mathcal{A} \to \mathcal{B}$ of C^*-algebras also induces a second adjoint linear map $\varphi^{**}\colon \mathcal{A}^{**} \to \mathcal{B}^{**}$ of von Neumann algebras by composing with appropriate surjective isometries, the aforementioned theorem again helping us out. We will investigate this before treading other waters.

Proposition 2.64. *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras and $\varphi\colon \mathcal{A} \to \mathcal{B}$ is a bounded linear map. Let $\varphi^{**}\colon \mathcal{A}^{**} \to \mathcal{B}^{**}$ denote its second adjoint.*

(i) *φ^{**} has the same norm as φ;*

(ii) *φ^{**} is ultraweakly-to-ultraweakly continuous (i.e. normal) when considered as a map between the enveloping von Neumann algebras;*

(iii) *if $\psi\colon \mathcal{B} \to \mathcal{C}$ is a bounded linear map, then*

$$(\psi \circ \varphi)^{**} = \psi^{**} \circ \varphi^{**}$$

as maps between the enveloping von Neumann algebras;

(iv) *if $\iota_{\mathcal{A}}\colon \mathcal{A} \to \mathcal{A}^{**}$ and $\iota_{\mathcal{B}}\colon \mathcal{B} \to \mathcal{B}^{**}$ denote the natural inclusions, then $\varphi^{**} \circ \iota_{\mathcal{A}} = \iota_{\mathcal{B}} \circ \varphi$ as maps of $\mathcal{A}$ into the enveloping von Neumann algebra of $\mathcal{B}$.*

Proof. (ii) is clear from the outset (see page 60). To prove (iv), note first that as maps of normed spaces, then for $\psi \in \mathcal{B}^*$, we have

$$\varphi^{**}(\iota_{\mathcal{A}}(a))(\psi) = (\iota_{\mathcal{A}}(a) \circ \varphi^*)(\psi) = \varphi^*(\psi)(a) = \psi(\varphi(a)) = \iota_{\mathcal{B}}(\varphi(a))(\psi),$$

so that the equality holds for $\mathcal{A}^{**}$ and $\mathcal{B}^{**}$ as Banach spaces. Let π and ρ be universal representations of $\mathcal{A}$ and $\mathcal{B}$ respectively, inducing surjective isometries $\tilde{\pi}\colon \mathcal{A}^{**} \to \pi(\mathcal{A})''$ and $\tilde{\rho}\colon \mathcal{B}^{**} \to \pi(\mathcal{B})''$ by Theorem 2.63. When considered as maps between the C^*-algebras and their enveloping von Neumann algebras, $\iota_{\mathcal{A}}$ and $\iota_{\mathcal{B}}$ are just the maps π and ρ, i.e. $\iota_{\mathcal{A}} = \tilde{\pi}^{-1} \circ \pi$ and $\iota_{\mathcal{B}} = \tilde{\rho}^{-1} \circ \rho$. Likewise, φ^{**} as a map of von Neumann algebras is just the map $\tilde{\rho} \circ \varphi^{**} \circ \tilde{\pi}^{-1}$, so as

$$\tilde{\rho} \circ \varphi^{**} \circ \tilde{\pi}^{-1} \circ \pi = \tilde{\rho} \circ \varphi^{**} \circ \iota_{\mathcal{A}} = \tilde{\rho} \circ \iota_{\mathcal{B}} \circ \varphi = \rho \circ \varphi.$$

Hence we obtain (iv).

For (i), it is clear that $\|\varphi^{**}\| \leq \|\varphi^*\| \leq \|\varphi\|$. Let $a \in \mathcal{A}$ be arbitrary. By the Hahn-Banach theorem [13, Theorem 5.8], there exists $\psi \in \mathcal{B}^*$ such that $\|\psi\| = 1$ and $\psi(\varphi(a)) = \|\varphi(a)\|$. Since

$$\varphi^{**}(\iota_{\mathcal{A}}(a))(\psi) = \iota_{\mathcal{A}}(a) \circ \varphi^*(\psi) = \varphi^*(\psi)(a) = \psi(\varphi(a)) = \|\varphi(a)\|,$$

it follows that

$$\|\varphi(a)\| = |\varphi^{**}(\iota_{\mathcal{A}}(a))(\psi)| \leq \|\varphi^{**}(\iota(a))\| \leq \|\varphi^{**}\|\|a\|.$$

Hence $\|\varphi\| \leq \|\varphi^{**}\|$, so $\|\varphi\| = \|\varphi^{**}\|$ when considering φ^{**} as a map of dual spaces. For the von Neumann algebra case, then note that because $\tilde{\pi}$ and $\tilde{\rho}$ as defined in Theorem 2.63 are isometries, it follows that $\|\tilde{\rho} \circ \varphi^{**} \circ \tilde{\pi}^{-1}\| \leq \|\varphi\|$. For $a \in (\mathcal{A})_1$, $\iota_{\mathcal{A}}(a) \in (\mathcal{A}^{**})_1$ and hence $T = \tilde{\pi}(\iota_{\mathcal{A}}(a)) \in (\pi(\mathcal{A})'')_1$. Remembering that $\varphi^{**} \circ \iota_{\mathcal{A}} = \iota_{\mathcal{B}} \circ \varphi$, we have

$$\|\tilde{\rho} \circ \varphi^{**} \circ \tilde{\pi}^{-1}\| \geq \|\tilde{\rho} \circ \varphi^{**} \circ \tilde{\pi}^{-1}(T)\| = \|\tilde{\rho}(\iota_{\mathcal{B}}(\varphi(a)))\| = \|\varphi(a)\|,$$

so $\|\varphi\| \leq \|\tilde{\rho} \circ \varphi^{**} \circ \tilde{\pi}^{-1}\|$. Hence φ^{**} has the same norm as φ.

To prove (iii), let ω be a universal representation of $\mathcal{C}$. As maps of dual spaces, we have $(\psi \circ \varphi)^{**} = (\varphi^* \circ \psi^*)^* = \psi^{**} \circ \varphi^{**}$ (see 60). With ψ^{**} seen as the von Neumann algebra map given by $\tilde{\omega} \circ \psi^{**} \circ \tilde{\rho}^{-1}$ and φ^{**} seen as above, we have

$$(\tilde{\omega} \circ \psi^{**} \circ \tilde{\rho}^{-1}) \circ (\tilde{\rho} \circ \varphi^{**} \circ \tilde{\pi}^{-1}) = \tilde{\omega} \circ (\psi \circ \varphi)^{**} \circ \tilde{\pi}^{-1},$$

which is precisely the map $(\psi \circ \varphi)^{**}$ considered as a map over the enveloping von Neumann algebras of $\mathcal{A}$ and $\mathcal{C}$. Hence (iii) follows, and the proof is complete. □

In the proof above, we used the explicit properties of the map connecting second duals to von Neumann algebras; however, as one can see, it is rather tedious notation-wise which is why we will not make use of this connection too often. We will return to other properties of the second adjoint map shortly in the next chapter after introducing the next important concept, with (hopefully) more concise proofs.

We end the chapter with a nice application of the enveloping von Neumann algebra.

Corollary 2.65. *Let $\mathcal{A}$ be a C^*-algebra. If $\varphi \in \mathcal{A}^*$, there exists a representation $\pi\colon \mathcal{A} \to B(\mathcal{H})$ and $\xi, \eta \in \mathcal{H}$ such that*

$$\varphi(x) = \langle \pi(x)\xi, \eta \rangle, \quad x \in \mathcal{A}$$

and $\|\omega\| = \|\xi\|\|\eta\|$.

Proof. Let $\omega \in (\mathcal{A}^{**})_*$ such that $\varphi(x) = \omega(\iota(x))$ for all $x \in \mathcal{A}$. By Proposition 2.44, we can write $\omega = U \cdot \psi$ for some partial isometry $U \in \mathcal{A}^{**}$ and a positive linear functional $\psi \in (\mathcal{A}^{**})_*$ such that $\|\psi\| = \|\omega\|$. Let $\psi' = \|\psi\|^{-1}\psi$, so that ψ' is a state. Letting $(\mathcal{H}, \pi', \xi')$ be the GNS triple associated to ψ', then

$$\varphi(x) = \omega(\iota(x)) = \psi(\iota(x)U) = \|\psi\|\langle \pi'(\iota(x)U)\xi', \xi' \rangle = \langle \pi(x)\xi, \eta \rangle$$

for all $x \in \mathcal{A}$ where $\pi = \pi' \otimes \iota$, $\xi = \pi(U)\xi'$ and $\eta = \|\psi\|\xi'$. Clearly $\|\varphi\| \leq \|\xi\|\|\eta\|$, and as ξ' is a unit vector, we have

$$\|\xi\|\|\eta\| \leq \|\psi\| = \|\omega\| = \|\varphi\|,$$

completing the proof. □

If you, kind reader, have found the structure of the past two chapters too busy, I don't blame you. Focus has not exactly been the word of the day, but the next three chapters will hopefully make up for it.

CHAPTER 3

COMPLETELY POSITIVE MAPS

It might be very easy to realize what it means for a map to be positive. Indeed, there is no way it could be mean anything other than sending positive elements to positive elements. If one were to be told that there existed higher degrees of posivitity, there is at least some possibility that one could derive the notion by transforming the original map into maps over matrix algebras, but more on that later. For our definition to be the most encompassing, we will start out by defining a notion of positivity for dual matrix algebras. The definition will then be given in the following section.

3.1 A matter of dual spaces

One should remember that there exists a notion of positivity for linear functionals and hence we can derive one for maps over dual spaces of C^*-algebras (sending positive functionals to positive functionals). This section will bring along a couple of isomorphisms so that the definition of positivity in the next section needs no explanation.

We will start out by classifying positive matrices with C^*-algebra entries.

Lemma 3.1. *Let $\mathcal{A}$ be a C^*-algebra and $n \geq 1$. An element in $\mathcal{A} \otimes M_n(\mathbb{C})$ is positive if and only if it is a finite sum of elements of the form*

$$a = \sum_{i,j=1}^{n} a_i^* a_j \otimes e_{ij}, \quad a_1, \dots, a_n \in \mathcal{A},$$

where $(e_{ij})_{i,j=1}^n$ denotes the canonical set of matrix units of $M_n(\mathbb{C})$. Hence an element of $M_n(\mathcal{A})$ is positive if and only if it is a finite sum of matrices of the form $(a_i^* a_j)_{i,j=1}^n$ for $a_1, \dots, a_n \in \mathcal{A}$.

Proof. For $a_1, \dots, a_n \in \mathcal{A}$ we have

$$\sum_{i,j=1}^{n} a_i^* a_j \otimes e_{ij} = \left(\sum_{i=1}^{n} a_i^* \otimes e_{i1} \right) \left(\sum_{j=1}^{n} a_j \otimes e_{1j} \right) = \left(\sum_{i=1}^{n} a_i \otimes e_{1i} \right)^* \left(\sum_{j=1}^{n} a_j \otimes e_{1j} \right) \geq 0.$$

Hence elements of the above form are positive, so finite sums are as well (Proposition 0.6). Assuming that $a \in \mathcal{A} \otimes M_n(\mathbb{C})$ is positive, there exists $b \in \mathcal{A} \otimes M_n(\mathbb{C})$ such that $a = b^* b$. Since b is of the form $\sum_{i,j=1}^n b_{ij} \otimes e_{ij}$ for $b_{ij} \in \mathcal{A}$, then we have

$$a = \left(\sum_{i,j=1}^{n} b_{ij} \otimes e_{ij} \right)^* \left(\sum_{k,l=1}^{n} b_{kl} \otimes e_{kl} \right) = \sum_{i,j} \sum_{k,l} b_{ij}^* b_{kl} \otimes e_{ji} e_{kl} = \sum_{i=1}^{n} \sum_{j,l=1}^{n} b_{ij}^* b_{il} \otimes e_{jl},$$

completing the proof. $\square$

Lemma 3.2. *Let $\mathcal{A}$ be a unital C^*-algebra and $n \geq 1$. Then for any $a = (a_{ij})_{i,j=1}^n \in M_n(\mathcal{A})$, the following are equivalent:*

(i) *a is positive.*

(ii) *For all $b \in M_{n,1}(\mathcal{A})$, $b^* a b = \sum_{i,j=1}^n b_i^* a_{ij} b_j$ is positive in $M_{1,1}(\mathcal{A}) = \mathcal{A}$.*

Proof. (i) $\Rightarrow$ (ii) is clear. For the converse implication, assume that a is not positive. Let $\mathcal{B}$ be the separable unital C^*-subalgebra generated by the entries a_{ij} for $i, j = 1, \ldots, n$. Then $a \in M_n(\mathcal{B})$, and a is not positive in $M_n(\mathcal{B})$. Since $\mathcal{B}$ is separable, we can take a faithful state $\omega \in S(\mathcal{B})$ by Proposition A.17. Let $(\mathcal{H}, \pi, \xi)$ be the associated GNS triple, in which π is a faithful representation of $\mathcal{B}$. By the construction of the matrix algebra $M_n(\mathcal{B})$, the induced map $\hat{\pi}\colon M_n(\mathcal{B}) \to B(\mathcal{H}^n)$ is faithful as well, and hence $\hat{\pi}(a)$ is not positive in $B(\mathcal{H}^n)$.

It is now easy to verify that $\{(\pi(b_1)\xi, \ldots, \pi(b_n)\xi) \mid b_1, \ldots, b_n \in \mathcal{B}\}$ is dense in $\mathcal{H}^n$. This implies that there exist $b_1, \ldots, b_n \in \mathcal{B}$ such that $\langle \hat{\pi}(a)\eta, \eta\rangle$ is not a positive number, where $\eta = (\pi(b_1)\xi, \ldots, \pi(b_n)\xi)$. As

$$\langle \hat{\pi}(a)\eta, \eta\rangle = \sum_{i,j=1}^{n} \langle \pi(a_{ij})\pi(b_j)\xi, \pi(b_i)\xi\rangle = \sum_{i,j=1}^{n} \langle \pi(b_i^* a_{ij} b_j)\xi, \xi\rangle = \langle \pi(b^*ab)\xi, \xi\rangle,$$

where $b = (b_1, \ldots, b_n) \in M_{n,1}(\mathcal{A})$, we see that $\pi(b^*ab)$ is not positive, so b^*ab is not positive. Hence the proof is complete. □

It will become useful in the following discussion to identify the dual of a matrix algebra with other vector spaces, establishing a notion of positivity by means of the positive linear functionals in the dual. We will do this by establishing, not just one, but *two* linear isomorphisms on the dual, allowing for a wider view on the identification.

The first isomorphism is created as follows. For a given C^*-algebra $\mathcal{A}$ and $n \geq 1$, we let $M_n(\mathcal{A}^*)$ denote the *vector space* of matrices with entries in $\mathcal{A}^*$. For $\varphi = (\varphi_{ij})_{i,j=1}^n \in M_n(\mathcal{A}^*)$, define

$$\Omega(\varphi)(a) = \sum_{i,j=1}^{n} \varphi_{ij}(a_{ij})$$

for $a = (a_{ij})_{i,j=1}^n \in M_n(\mathcal{A})$. Then we have $\Omega(\varphi) \in M_n(\mathcal{A})^*$ for all $\varphi \in M_n(\mathcal{A}^*)$, as it is linear and

$$|\Omega(\varphi)(a)| \leq n^2 \max_{i,j} \|\varphi_{ij}\| \|a\|,$$

for all $a \in M_n(\mathcal{A})$ by Lemma 1.25, and Ω is linear as well. If $\Omega(\varphi) = 0$ for some $\varphi = (\varphi_{ij})_{i,j=1}^n \in M_n(\mathcal{A}^*)$, then it is easy to see that $\varphi_{ij} = 0$ for all $i, j = 1, \ldots, n$, so φ is the zero matrix. Moreover, if $\psi \in M_n(\mathcal{A})^*$ then by letting $\rho_{ij}\colon \mathcal{A} \to M_n(\mathcal{A})$ be the linear isometry that inserts a at place (i, j) in a $n \times n$ matrix and puts 0 everywhere else, we can define a bounded linear functional $\varphi_{ij} = \psi \circ \rho_{ij}\colon \mathcal{A} \to \mathbb{C}$. If $\varphi = (\varphi_{ij})_{i,j=1}^n \in M_n(\mathcal{A}^*)$ we then have

$$\Omega(\varphi)(a) = \psi\left(\sum_{i,j=1}^{n} \rho_{ij}(a_{ij})\right) = \psi(a)$$

for all $a = (a_{ij})_{i,j=1}^n \in M_n(\mathcal{A})$. Hence we can identify $M_n(\mathcal{A})^*$ with $M_n(\mathcal{A}^*)$ by the isomorphism Ω.

Also, by Corollary 1.12, any element of the vector space $\mathcal{A}^* \odot M_n(\mathbb{C})$ can be written uniquely as an element of the form

$$\sum_{i,j} \varphi_{ij} \otimes e_{ij},$$

where $(e_{ij})_{i,j=1}^n$ denotes the canonical set of matrix units of $M_n(\mathbb{C})$. Defining a map $M_n(\mathcal{A}^*) \to \mathcal{A}^* \odot M_n(\mathbb{C})$ by

$$(\varphi_{ij})_{i,j=1}^n \mapsto \sum_{i,j} \varphi_{ij} \otimes e_{ij},$$

it is clear that it is a linear isomorphism. This is the second isomorphism that we seek. We will say that an element of $\mathcal{A}^* \odot M_n(\mathbb{C})$ or $M_n(\mathcal{A}^*)$ is positive if it is identifiable with a positive linear functional on $M_n(\mathcal{A})^*$ by either of these two isomorphisms. It will be useful to know when such elements are positive, and the next lemma will clarify this matter.

Lemma 3.3. *Let $\mathcal{A}$ be a C^*-algebra and let $\varphi = (\varphi_{ij})_{i,j=1}^n \in M_n(\mathcal{A}^*)$. Then φ is positive if and only if*

$$\sum_{i,j=1}^n \varphi_{ij}(a_i^* a_j) \geq 0$$

for all $a_1, \ldots, a_n \in \mathcal{A}$.

Proof. Since

$$\varphi\left((a_i^* a_j)_{i,j=1}^n\right) = \sum_{i,j=1}^n \varphi_{ij}(a_i^* a_j)$$

for all $a_1, \ldots, a_n \in \mathcal{A}$, the result follows from Lemma 3.1. □

If E is a subspace of the dual $\mathcal{A}^*$ of a C^*-algebra $\mathcal{A}$, then we say that $\varphi = (\varphi_{ij})_{i,j=1}^n \in M_n(E)$ if $\varphi_{ij} \in E$ for all $i,j = 1, \ldots, n$, and that $\varphi \in M_n(E)$ is positive if φ is positive as an element in $M_n(\mathcal{A}^*)$.

In the case where $\mathscr{M}$ is a von Neumann algebra, it will be useful to know that the above isomorphisms preserve the notion of ultraweak continuity. Here is a proof.

Proposition 3.4. *For any von Neumann algebra $\mathscr{M} \subseteq B(\mathcal{H})$ and $n \geq 1$, then $\phi\colon M_n(\mathscr{M}_*) \to M_n(\mathscr{M})_*$ given by*

$$\phi((\omega_{ij})_{i,j=1}^n)((T_{ij})_{i,j=1}^n) = \sum_{i,j=1}^n \omega_{ij}(T_{ij})$$

is a linear isomorphism. Moreover,

$$\max_{i,j=1,\ldots,n} \|\omega_{ij}\| \leq \|\phi(\omega)\| \leq \sum_{i,j=1}^n \|\omega_{ij}\|$$

for all $\omega = (\omega_{ij})_{i,j=1}^n \in M_n(\mathscr{M}_*)$.

Proof. To see that ϕ is well-defined, let $\omega = (\omega_{ij})_{i,j=1}^n \in M_n(\mathscr{M}_*)$ and for all $i,j = 1, \ldots, n$ write

$$\omega_{ij} = \sum_{m=1}^\infty \omega_{\xi_{ij}^m, \eta_{ij}^m}$$

for suitable sequences in $\mathcal{H}$. For any $m \geq 1$, let

$$\xi_{(i,j)}^m = \iota_j(\xi_{ij}^m), \quad \eta_{(i,j)}^m = \iota_i(\eta_{ij}^m), \quad i,j = 1, \ldots, n,$$

where ι_k denotes the inclusion of $\mathcal{H}$ into the k'th copy of $\mathcal{H}$ in $\mathcal{H}^n$. Then for all $T = (T_{ij})_{i,j=1}^n \in M_n(\mathscr{M})$, we have

$$\sum_{i,j=1}^n \sum_{m=1}^\infty \langle T\xi_{(i,j)}^m, \eta_{(i,j)}^m\rangle = \sum_{i,j=1}^n \sum_{m=1}^\infty \langle T_{ij}\xi_{ij}^m, \eta_{ij}^m\rangle = \sum_{i,j=1}^n \omega_{ij}(T_{ij}) = \omega(T).$$

Hence $\phi(T) \in M_n(\mathscr{M})_*$, so ϕ is well-defined. Not surprisingly, ϕ is linear as well. For any $\omega \in M_n(\mathscr{M})_*$, define $\omega_{ij}\colon \mathscr{M} \to \mathbb{C}$ for $i,j = 1, \ldots, n$ by

$$\omega_{ij}(T) = \omega(\rho_{ij}(T))$$

where $\rho_{ij}(T)$ is the element of $M_n(\mathscr{M})$ with T in position (i,j) and 0 everywhere else. To prove that $\omega_{ij} \in \mathscr{M}_*$, write $\omega = \sum_{m=1}^\infty \omega_{\xi_m, \eta_m}$ for suitable sequences in $\mathcal{H}^n$ and write $\xi_m = (\xi_m^1, \ldots, \xi_m^n)$ and $\eta_m = (\eta_m^1, \ldots, \eta_m^n)$ for all $m \geq 1$. Then

$$\omega_{ij}(T) = \sum_{m=1}^\infty \langle \rho_{ij}(T)\xi_m, \eta_m\rangle = \sum_{m=1}^\infty \langle T\xi_m^j, T\eta_m^i\rangle$$

for all $T \in \mathscr{M}$, proving that $\omega_{ij} \in \mathscr{M}_*$. It is then easily seen that $\phi((\omega_{ij})_{i,j=1}^n) = \omega$, proving that ϕ is a linear isomorphism, since it is injective by the remark after Lemma 3.2. To prove the inequalities, we do the following: note that for all $i, j = 1, \ldots, n$, then $\|\rho_{ij}(T)\| = \|T\|$ by Lemma 1.25 for any $T \in (\mathscr{M})_1$. Hence for $\omega = (\omega_{ij})_{i,j=1}^n \in M_n(\mathscr{M}_*)$, we have

$$|\omega_{ij}(T)| = |\phi(\omega)(\rho_{ij}(T))| \leq \|\phi(\omega)\|,$$

proving the first inequality. Moreover, for any $T = (T_{ij})_{i,j=1}^n \in (M_n(\mathscr{M}))_1$ we have

$$|\phi(\omega)(T)| \leq \sum_{i,j=1}^n \|\omega_{ij}\|\|T_{ij}\| \leq \sum_{i,j=1}^n \|\omega_{ij}\|\|T\|,$$

proving the second. $\square$

3.2 Positive and completely positive maps

The next two sections should now be completely understandable.

Definition 3.1. Let $\mathcal{A}$ and $\mathcal{B}$ be either of a C^*-algebra or a subspace of the dual of a C^*-algebra. A linear map $\varphi\colon \mathcal{A} \to \mathcal{B}$ is called *positive* if it maps positive elements to positive elements.

Definition 3.2. Once again, let $\mathcal{A}$ and $\mathcal{B}$ be either of a C^*-algebra or a subspace of the dual of a C^*-algebra. For $n \geq 1$, a linear map $\varphi\colon \mathcal{A} \to \mathcal{B}$ is called *n-positive* if the tensor product map

$$\varphi \odot \mathrm{id}_n\colon \mathcal{A} \odot M_n(\mathbb{C}) \to \mathcal{B} \odot M_n(\mathbb{C})$$

is positive, where $\mathrm{id}_n\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is the identity mapping. If φ is n-positive for all $n \geq 1$, we say that φ is *completely positive.*

We shall often write $\varphi \odot \mathrm{id}_n = \varphi^{(n)}$. By identifying $\mathcal{A} \odot M_n(\mathbb{C})$ and $\mathcal{B} \odot M_n(\mathbb{C})$ with $M_n(\mathcal{A})$ and $M_n(\mathcal{B})$ respectively, we see that $\varphi^{(n)}$ is also a map $M_n(\mathcal{A}) \to M_n(\mathcal{B})$ given by

$$\varphi^{(n)}((a_{ij})_{i,j=1}^n) = (\varphi(a_{ij}))_{i,j=1}^n.$$

As compositions of positive maps are again positive, it follows that compositions of completely positive maps are again completely positive.

The discussion of duals of C^*-algebras will be set aside for the moment, and we will now concentrate on positivity and complete positivity for C^*-algebras only.

Proposition 3.5. *Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras. Then the set of completely positive maps $\mathcal{A} \to \mathcal{B}$ is a cone, i.e. if φ and ψ are completely positive maps $\mathcal{A} \to \mathcal{B}$ and $\lambda \geq 0$, then $\varphi+\psi$ and $\lambda\varphi$ are completely positive.*

Proof. Let $n \geq 1$ and let $a \in M_n(\mathcal{A})$ be positive. Then $(\varphi + \psi)^{(n)}(a) = \varphi^{(n)}(a) + \psi^{(n)}(a) \geq 0$ by Proposition 0.6 and $(\lambda\varphi^{(n)})(a) = \lambda\varphi^{(n)}(a) \geq 0$, so the result follows. $\square$

As positive linear functionals are Hermitian, it might be useful to know whether a similar property holds for positive maps. We will deal with this straight away.

Definition 3.3. A linear map $\varphi\colon \mathcal{A} \to \mathcal{B}$ of C^*-algebras is called *Hermitian* if $\varphi(a^*) = \varphi(a)^*$ for all $a \in \mathcal{A}$.

Proposition 3.6. *A linear map $\varphi\colon \mathcal{A} \to \mathcal{B}$ of C^*-algebras is Hermitian if and only if $\varphi(a) \in \mathcal{B}_{\mathrm{sa}}$ for all $a \in \mathcal{A}_{\mathrm{sa}}$.*

Proof. If φ is Hermitian, then it clearly satisfies the other condition as well. If $\varphi(a) \in \mathcal{B}_{\mathrm{sa}}$ for all $a \in \mathcal{A}_{\mathrm{sa}}$, let $a \in \mathcal{A}$ and write $a = a_1 + ia_2$ with self-adjoint elements $a_1, a_2 \in \mathcal{A}_{\mathrm{sa}}$. Then

$$\varphi(a^*) = \varphi(a_1) - i\varphi(a_2) = (\varphi(a_1) + i\varphi(a_2))^* = \varphi(a)^*$$

for all $a \in \mathcal{A}$, so φ is Hermitian. $\square$

Proposition 3.7. *Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras. Then all positive linear maps $\mathcal{A} \to \mathcal{B}$ are Hermitian.*

Proof. Let $\varphi\colon \mathcal{A} \to \mathcal{B}$ be a positive linear map and assume $a \in \mathcal{A}_{\mathrm{sa}}$. Because $|a|+a \geq 0$ and $|a|-a \geq 0$ by the continuous functional calculus (note that $x \mapsto |x|$ maps 0 to 0, so that $|a|$ can actually be obtained by approximating with polynomials sending 0 to 0 – in short, it does not matter if $\mathcal{A}$ is non-unital), it follows that $\varphi(a) = \frac{1}{2}\left(\varphi(|a|+a) - \varphi(|a|-a)\right)$ is self-adjoint, so that φ is Hermitian by Proposition 3.6. □

In Chapter 2, we depended upon the notion of positivity to define normal maps. As we now have the concept and notation fully laid out, we will prove one last helpful thing about normal maps.

Lemma 3.8. *If $\varphi\colon \mathscr{M} \to \mathscr{N}$ is a normal linear map on von Neumann algebras $\mathscr{M}$ and $\mathscr{N}$ and $n \geq 1$, then $\varphi^{(n)}$ is normal as well.*

Proof. Letting $\omega \in M_n(\mathscr{N})_*$, then Proposition 3.4 yields $\omega_{ij} \in \mathscr{N}_*$, $i,j = 1,\ldots,n$ in a way such that for all $T = (T_{ij})_{i,j=1}^n \in M_n(\mathscr{M})$ we have

$$\omega(\varphi^{(n)}(T)) = \sum_{i,j=1}^{n} \omega_{ij}(\varphi(T_{ij})).$$

Proposition 2.45 tells us that $\omega_{ij} \circ \varphi \in \mathscr{M}_*$ for all $i,j = 1,\ldots,n$. Hence $(\omega_{ij} \circ \varphi)_{i,j=1}^n \in M_n(\mathscr{M}_*)$ defines an element ψ of $M_n(\mathscr{M})_*$ in the manner of Proposition 3.4, so that we have

$$\omega(\varphi^{(n)}(T)) = \psi(T).$$

Hence $\omega \circ \varphi^{(n)} = \psi \in M_n(\mathscr{M})_*$ for all $\omega \in M_n(\mathscr{N})_*$, so by Proposition 2.45, $\varphi^{(n)}$ is normal. □

We will now derive some additional properties of positive maps as well as a property implying positivity.

Proposition 3.9. *Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras and let $\varphi\colon \mathcal{A} \to \mathcal{B}$ be a linear map. Then:*

(i) *If φ is positive, then φ is bounded.*
(ii) *If $\mathcal{A}$ is unital and φ is positive and 2-positive, then $\|\varphi\| = \|\varphi(1_{\mathcal{A}})\|$.*

Proof. (i) Let $f \in S(\mathcal{B})$. Then $f \circ \varphi$ is a positive linear functional on $\mathcal{A}$, so it is bounded as well. As

$$|(f \circ \varphi)(a)| \leq \|f\|\|\varphi(a)\| = \|\varphi(a)\|$$

for all $a \in \mathcal{A}$, then the Uniform Boundedness Principle [13, Theorem 5.13] yields that the set of bounded linear functionals $\{f \circ \varphi \mid f \in S(\mathcal{B})\}$ is uniformly bounded. Hence there exists $K \in \mathbb{R}_+$ such that

$$|(f \circ \varphi)(a)| \leq K\|a\|$$

for all $f \in S(\mathcal{B})$ and $a \in \mathcal{A}$. For $a \in \mathcal{A}_{\mathrm{sa}}$, there exists a state $\psi \in S(\mathcal{B})$ such that $|\psi(\varphi(a))| = \|\varphi(a)\|$ by Theorem 2.49, and hence $\|\varphi(a)\| \leq K\|a\|$. For $a \in \mathcal{A}$, then by decomposing a into the sum $a = a_1 + ia_2$ for $a_1, a_2 \in \mathcal{A}_{\mathrm{sa}}$, we obtain

$$\|\varphi(a)\| \leq \|\varphi(a_1)\| + \|\varphi(a_2)\| \leq K(\|a_1\| + \|a_2\|) = 2K\|a\|.$$

Hence φ is bounded.

(ii) We clearly have $\|\varphi(1_{\mathcal{A}})\| \leq \|\varphi\|$. For the other inequality, we will pass to the matrix algebra $M_2(\mathcal{A})$ for useful information. Since $-\|a\|1_{\mathcal{A}} \leq a \leq \|a\|1_{\mathcal{A}}$ for $a \in \mathcal{A}_{\mathrm{sa}}$, then by positivity we obtain

$$-\|a\|\varphi(1_{\mathcal{A}}) \leq \varphi(a) \leq \|a\|\varphi(1_{\mathcal{A}})$$

and hence $\|\varphi(a)\| \leq \|a\|\|\varphi(1_{\mathcal{A}})\|$. Given any $a \in \mathcal{A}$, put

$$\tilde{a} = \begin{pmatrix} 0 & a^* \\ a & 0 \end{pmatrix} \in M_2(\mathcal{A}).$$

Clearly $a = a^*$. To calculate the norm of $\tilde{a}$, pick a faithful unital *-representation π of $\mathcal{A}$ onto some Hilbert space $\mathcal{H}$, and let $\pi^{(2)}$ denote the induced faithful unital *-representation of $M_2(\mathcal{A})$ onto $\mathcal{H}^2$ defined as in Proposition 1.23. For $\xi = (\xi_1, \xi_2) \in \mathcal{H}^2$, we have

$$\|\pi^{(2)}(\tilde{a})\xi\|^2 = \left\| \begin{pmatrix} \pi(a^*)\xi_2 \\ \pi(a)\xi_1 \end{pmatrix} \right\|^2 = \|\pi(a)^*\xi_2\|^2 + \|\pi(a)\xi_1\|^2 \leq \|a\|^2\|\xi\|^2.$$

Hence $\|\tilde{a}\| = \|\pi^{(2)}(\tilde{a})\| \leq \|a\|$. For $\varepsilon > 0$, then if η is a unit vector of $\mathcal{H}$ such that $\|\pi(a)\eta\| + \varepsilon \geq \|\pi(a)\|$, then

$$\left\| \pi^{(2)}(\tilde{a}) \begin{pmatrix} 0 \\ \eta \end{pmatrix} \right\| = \left\| \begin{pmatrix} 0 \\ \pi(a)\eta \end{pmatrix} \right\| = \|\pi(a)\eta\| \geq \|\pi(a)\| - \varepsilon = \|a\| - \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we conclude $\|a\| \leq \|\tilde{a}\|$, thereby proving $\|a\| = \|\tilde{a}\|$. One proves in the same manner that

$$\varphi^{(2)}(\tilde{a}) = \begin{pmatrix} 0 & \varphi(a)^* \\ \varphi(a) & 0 \end{pmatrix}$$

has norm $\|\varphi(a)\|$ and that $\|\varphi^{(2)}(1_{M_2(\mathcal{A})})\| = \|\varphi(1_{\mathcal{A}})\|$. Since φ is 2-positive, then by what we proved above, we have

$$\|\varphi(a)\| = \|\varphi^{(2)}(\tilde{a})\| \leq \|\tilde{a}\| \|\varphi^{(2)}(1_{M_2(\mathcal{A})})\| = \|a\| \|\varphi(1_{\mathcal{A}})\|$$

for all $a \in \mathcal{A}$, and thus we conclude $\|\varphi\| \leq \|\varphi(1_{\mathcal{A}})\|$ and hence equality. $\square$

Proposition 3.10 (Russo-Dye, 1966). *Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras and let $\varphi\colon \mathcal{A} \to \mathcal{B}$ be a unital linear contraction. Then φ is positive.*

Proof. Suppose φ is contractive and let π be a faithful unital representation of $\mathcal{B}$ on some Hilbert space $\mathcal{H}$ (see e.g. page viii). For any vector $\xi \in \mathcal{H}$, define $\omega\colon \mathcal{A} \to \mathbb{C}$ by

$$\omega(a) = \langle \pi(\varphi(a))\xi, \xi \rangle.$$

Then ω is a linear functional on $\mathcal{A}$, $\|\omega\| \leq \|\xi\|^2$ and $\omega(1_{\mathcal{A}}) = \|\xi\|^2$, so by [31, Theorem 13.5], ω is positive. Hence if $a \geq 0$, then

$$0 \leq \omega(a) = \langle \pi(\varphi(a))\xi, \xi \rangle.$$

Since ξ was arbitrary, it follows that $\langle \pi(\varphi(a))\xi, \xi \rangle \geq 0$ for all positive $a \geq 0$ and $\xi \in \mathcal{H}$, so $\pi(\varphi(a)) \geq 0$ and hence $\varphi(a) \geq 0$ for all $a \geq 0$, since π was faithful. $\square$

We finally look into some examples of completely positive maps.

Proposition 3.11. *Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras. Then every *-homomorphism $\varphi\colon \mathcal{A} \to \mathcal{B}$ is completely positive.*

Proof. Since any *-homomorphism is positive, and $\varphi^{(n)}$ is a *-homomorphism for all $n \geq 1$, the result follows. $\square$

Proposition 3.12. *Let $\mathcal{A}$ be a C^*-algebra. Then any positive linear functional $\varphi\colon \mathcal{A} \to \mathbb{C}$ is completely positive.*

Proof. Let $\varphi \in \mathcal{A}^*$ be positive on $\mathcal{A}$. We have to prove for any $n \geq 1$ that $\varphi^{(n)}\colon M_n(\mathcal{A}) \to M_n(\mathbb{C})$ is positive. By identifying $M_n(\mathbb{C})$ with $B(\mathbb{C}^n)$, then for $a_1, \ldots, a_n \in \mathcal{A}$ and $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{C}^n$, we have

$$\begin{aligned} \langle \varphi^{(n)}((a_i^* a_j)_{i,j=1}^n)\xi, \xi \rangle &= \langle (\varphi(a_i^* a_j)_{i,j=1}^n)\xi, \xi \rangle \\ &= \sum_{i,j=1}^n \langle \varphi(a_i^* a_j)\xi_j, \xi_i \rangle \\ &= \sum_{i,j=1}^n \varphi(a_i^* a_j)\xi_j \overline{\xi_i} = \varphi\left(\left(\sum_{i=1}^n \xi_i a_i \right)^* \left(\sum_{j=1}^n \xi_j a_j \right) \right) \geq 0. \end{aligned}$$

Hence $\varphi^{(n)}((a_i^* a_j)_{i,j=1}^n) \geq 0$. From Lemma 3.1 it follows that $\varphi^{(n)}$ is positive for all $n \geq 1$. $\square$

3.3 Properties of second adjoint maps

In Chapter 2, we took our time to define the enveloping von Neumann algebras and how we could obtain maps of these from maps over the original C^*-algebras. Some properties were deduced in Proposition 2.64. We now return immediately to the properties of second adjoint maps between double duals (or enveloping von Neumann algebras) of C^*-algebras derived from linear maps, in a manner that sums up everything that is needed for now.

Proposition 3.13. *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras and $\varphi\colon \mathcal{A} \to \mathcal{B}$ is a bounded linear map. Let $\varphi^{**}\colon \mathcal{A}^{**} \to \mathcal{B}^{**}$ denote its second adjoint.*

(i) *φ^{**} has the same norm as φ;*
(ii) *φ^{**} is normal;*
(iii) *if φ is Hermitian, then so is φ^{**};*
(iv) *if φ is positive, then so is φ^{**};*
(v) *if φ is a *-homomorphism, then so is φ^{**};*
(vi) *if φ is completely positive, then so is φ^{**};*

Proof. Let $\iota_{\mathcal{A}}\colon \mathcal{A} \to \mathcal{A}^{**}$ and $\iota_{\mathcal{B}}\colon \mathcal{B} \to \mathcal{B}^{**}$ denote the canonical inclusions with ultraweakly dense images. (i) and (ii) was the content of Proposition 2.64, (i) and (ii).

For (iii), let $T \in \mathcal{A}^{**}$ be self-adjoint. By Kaplansky's density theorem (Theorem 2.33), there is a bounded net $(\iota_{\mathcal{A}}(a_\alpha))_{\alpha\in A}$ of self-adjoint operators in $\iota_{\mathcal{A}}(\mathcal{A})$ converging weakly to T and hence ultraweakly by Proposition 2.1. Since $\iota_{\mathcal{A}}$ is faithful, a_α is self-adjoint for all $\alpha \in A$. Since the adjoint operation is ultraweakly continuous, it follows from (ii) that

$$\begin{aligned}\varphi^{**}(T) &= \lim_{\alpha\in A} \varphi^{**}(\iota(a_\alpha)) = \lim_{\alpha\in A} \iota_{\mathcal{B}}(\varphi(a_\alpha))\\ &= \lim_{\alpha\in A} \iota_{\mathcal{B}}(\varphi(a_\alpha))^* = \left(\lim_{\alpha\in A} \iota_{\mathcal{B}}(\varphi(a_\alpha))\right)^* = \left(\lim_{\alpha\in A} \varphi^{**}(\iota(a_\alpha))\right)^*\\ &= \varphi^{**}(T)^*.\end{aligned}$$

Hence $\varphi^{**}(T)$ is self-adjoint, so φ^{**} is Hermitian by Proposition 3.6.

(iv) Let $T \in \mathcal{A}^{**}$ be positive. By Kaplansky's density theorem, there is a bounded net of positive operators in $\iota_{\mathcal{A}}(\mathcal{A})$ converging strongly and hence weakly to T. As φ^{**} is normal, it is weakly-to-weakly continuous on bounded sets by Proposition 2.1, so $\iota_{\mathcal{B}}(\varphi(x_\alpha)) = \varphi^{**}(\iota_{\mathcal{A}}(x_\alpha)) \to \varphi^{**}(T)$ weakly; as $\iota_{\mathcal{B}}(\varphi(x_\alpha))$ is positive for all $\alpha \in A$ and $(\mathcal{B}^{**})_+$ is weakly closed (indeed, $\mathscr{M}_+$ is weakly closed for any von Neumann algebra $\mathscr{M}$), it follows that $\varphi^{**}(T)$ is positive.

(v) We only need to prove that φ is multiplicative by (iii). Let $S, T \in \mathcal{A}^{**}$ and take nets $(\iota_{\mathcal{A}}(x_\alpha))_{\alpha\in A}$ and $(\iota_{\mathcal{A}}(y_\beta))_{\beta\in B}$ in $\mathcal{A}$ converging ultraweakly to S and T respectively. Because the product is ultraweakly continuous in each variable and φ^{**} is ultraweakly continuous, it follows that

$$\begin{aligned}\varphi^{**}(ST) &= \lim_{\alpha\in A}\lim_{\beta\in B} \varphi^{**}(\iota_{\mathcal{A}}(x_\alpha y_\beta))\\ &= \lim_{\alpha\in A}\lim_{\beta\in B} \iota_{\mathcal{B}}(\varphi(x_\alpha))\iota_{\mathcal{B}}(\varphi(y_\beta)) = \lim_{\alpha\in A} \iota_{\mathcal{B}}(\varphi(x_\alpha))\varphi^{**}(T) = \varphi^{**}(S)\varphi^{**}(T).\end{aligned}$$

(vi) Let $n \geq 1$ and let $T \in M_n(\mathcal{A}^{**})$ be positive; we will show that $(\varphi^{**})^{(n)}(T)$ is positive. Since $\iota(\mathcal{A})$ is ultraweakly and hence weakly dense in $\mathcal{A}^{**}$, it follows from Proposition 1.36 that $M_n(\iota_{\mathcal{A}}(\mathcal{A}))$ is weakly dense in $M_n(\mathcal{A}^{**})$. By Kaplansky's density theorem, there exists a bounded net of positive operators $(T_\alpha)_{\alpha\in A}$ in $M_n(\iota_{\mathcal{A}}(\mathcal{A}))$ such that $T_\alpha \to T$ strongly. Each T_α is of the form $\iota_{\mathcal{A}}^{(n)}(x_\alpha^{ij})_{i,j=1}^n$ where $x_\alpha^{ij} \in \mathcal{A}$ for all $\alpha \in A$ and $i, j = 1, \ldots, n$. Since $\iota_{\mathcal{A}}^{(n)}$ is an injective *-homomorphism, then $M_n(\iota_{\mathcal{A}}(\mathcal{A}))$ is a C^*-algebra and hence each $x_\alpha = (x_\alpha^{ij})_{i,j=1}^n \in M_n(\mathcal{A})$ is a positive matrix itself. Note that for all $\alpha \in A$ we have

$$(\varphi^{**})^{(n)}(T_\alpha) = (\varphi^{**}(\iota_{\mathcal{A}}(x_\alpha^{ij})))_{i,j=1}^n = (\iota_{\mathcal{B}}(\varphi(x_\alpha^{ij})))_{i,j=1}^n = \iota_{\mathcal{B}}^{(n)}\varphi^{(n)}(x_\alpha) \geq 0$$

since φ is completely positive. As φ^{**} is normal by (ii) then $(\varphi^{**})^{(n)}$ is normal by Lemma 3.8, so Proposition 2.1 yields that $(\varphi^{**})^{(n)}(T_\alpha) \to (\varphi^{**})^{(n)}(T)$ ultraweakly. Since $M_n(\mathcal{B}^{**})_+$ is weakly closed and $(\varphi^{**})^{(n)}(T_\alpha) \in M_n(\mathcal{B}^{**})_+$ for all α as found above, we conclude that $(\varphi^{**})^{(n)}(T)$ is positive. $\square$

A wonderful consequence of the properties of second adjoint maps is the following: it can be used every day when taking a shower and twice on New Year's Eve if there is no soup and cigarettes left.

Proposition 3.14. *Let $\mathcal{A}$ be a C^*-algebra with a closed two-sided ideal $\mathfrak{J}$. Then*

$$\mathcal{A}^{**} \cong \mathfrak{J}^{**} \oplus (\mathcal{A}/\mathfrak{J})^{**}.$$

Proof. Note that $\mathfrak{J}$ and $\mathcal{A}/\mathfrak{J}$ are C^*-algebras (see e.g. [24, Theorem 8.1]). Let $j\colon \mathfrak{J} \to \mathcal{A}$ denote the inclusion map and let $\pi\colon \mathcal{A} \to \mathcal{A}/\mathfrak{J}$ denote the canonical quotient map. Then $j^{**}\colon \mathfrak{J}^{**} \to \mathcal{A}^{**}$ and $\pi^{**}\colon \mathcal{A}^{**} \to (\mathcal{A}/\mathfrak{J})^{**}$ are normal homomorphisms by Proposition 3.13, and j^{**} is injective by Lemma A.18. Furtheremore, $(j^{**}(\mathfrak{J}^{**}))_r = j^{**}((\mathfrak{J}^{**})_r)$ and $(\pi^{**}(\mathfrak{X}^{**}))_r = \pi^{**}((\mathfrak{X}^{**})_r)$ are ultraweakly compact and hence ultraweakly closed for all $r > 0$ by normality of j^{**} and π^{**} as well as Corollary 2.9. Now it follows from Theorem 2.11 that $j^{**}(\mathfrak{J}^{**})$ and $\pi^{**}(\mathfrak{X}^{**})$ are ultraweakly closed. Since Lemma A.18 yields that $\ker \pi^{**}$ equals the ultraweak closure of $j^{**}(\mathfrak{J}^{**})$ and that $(\mathcal{A}/\mathfrak{J})^{**}$ equals the ultraweak closure of $\pi^{**}(\mathfrak{X}^{**})$, it then follows that $\ker \pi^{**} = j^{**}(\mathfrak{J}^{**})$ and that π^{**} is surjective. Finally noting that $\mathfrak{J}^{**} \cong j^{**}(\mathfrak{J}^{**})$, the result follows from Proposition 2.53. □

Corollary 3.15. *Let $\mathcal{A}$ be a non-unital C^*-algebra, and let $\tilde{\mathcal{A}}$ denote its unitization. Then*

$$(\tilde{\mathcal{A}})^{**} \cong \mathcal{A}^{**} \oplus \mathbb{C}.$$

Proof. As $\mathbb{C}^{**} = \mathbb{C}$, the result clearly follows from Proposition 3.14. □

3.4 Stinespring's representation theorem

Let $\mathcal{A}$ be a C^*-algebra. The *-isomorphism $\psi\colon B(\mathbb{C}) \to \mathbb{C}$ given by $V \mapsto V(1)$ is of course a positive map, so if $\varphi\colon \mathcal{A} \to B(\mathbb{C})$ is a positive linear map, then the corresponding linear functional $\psi\circ\varphi\colon \mathcal{A} \to \mathbb{C}$ is positive. The GNS representation now yields a GNS triple $(\mathcal{K}, \pi, \xi)$ such that $\psi(\varphi(x)) = \langle \pi(x)\xi, \xi\rangle$ for all $x \in \mathcal{A}$. Defining $V\colon \mathbb{C} \to \mathcal{H}$ by $V(\lambda) = \lambda\xi$, then as

$$\langle V(\lambda), \eta\rangle = \lambda\langle \xi, \eta\rangle = \lambda\overline{\langle \eta, \xi\rangle}, \quad \lambda \in \mathbb{C},\ \eta \in \mathcal{H},$$

we see that $V^*\colon \mathcal{H} \to \mathbb{C}$ is given by $V^*\eta = \langle \eta, \xi\rangle$ for $\eta \in \mathcal{H}$. Hence

$$\varphi(x)(1) = \psi(\varphi(x)) = \langle \pi(x)\xi, \xi\rangle = V^*(\pi(x)\xi) = V^*(\pi(x)V(1)) = (V^*\pi(x)V)(1)$$

for all $x \in \mathcal{A}$, so $\varphi(x) = V^*\pi(x)V$. To summarize, if $\mathcal{H} = \mathbb{C}$, then for a given positive linear map $\varphi\colon \mathcal{A} \to B(\mathcal{H})$ we have found a Hilbert space $\mathcal{K}$, a representation $\pi\colon \mathcal{A} \to B(\mathcal{K})$ and a bounded linear operator $V\colon \mathcal{H} \to \mathcal{K}$ such that $\varphi(x) = V^*\pi(x)V$ for all $x \in \mathcal{A}$. As it turns out, we can in fact generalize this to any Hilbert space $\mathcal{H}$ if we put a notable restriction on φ – it has to be completely positive. The proof is somewhat long, but it has a wide range of applications we cannot afford to miss out on.

Theorem 3.16 (Stinespring's representation theorem, 1955)**.** *Let $\mathcal{A}$ be a C^*-algebra and $\mathcal{H}$ a Hilbert space. Moreover, let $\varphi\colon \mathcal{A} \to B(\mathcal{H})$ be a linear map. Then the following are equivalent:*

(i) *φ is completely positive.*

(ii) *There is a Hilbert space $\mathcal{K}$, a representation $\pi\colon \mathcal{A} \to B(\mathcal{K})$ and a bounded linear operator $V\colon \mathcal{H} \to \mathcal{K}$ such that*

$$\varphi(a) = V^*\pi(a)V, \quad a \in \mathcal{A}.$$

If $\mathcal{A}$ is unital, π can be chosen to be unital.

If $\mathcal{A}$ is a von Neumann algebra and φ is normal, then π in (ii) can be chosen to be normal.

Proof. The easy part is proving that the second condition implies the first one. Assume that (ii) is satisfied, let $n \geq 1$ and let $a = (a_{ij})_{i,j=1}^n \in M_n(\mathcal{A})$ be positive. We will identify $M_n(B(\mathcal{H}))$ with $B(\mathcal{H}^n)$. For any $\xi = (\xi_1, \ldots, \xi_n) \in \mathcal{H}^n$, note that

$$\langle \varphi^{(n)}(a)\xi, \xi\rangle_{\mathcal{H}^n} = \sum_{i,j=1}^n \langle \varphi(a_{ij})\xi_j, \xi_i\rangle_{\mathcal{H}} = \sum_{i,j=1}^n \langle \pi(a_{ij})V\xi_j, V\xi_i\rangle_{\mathcal{K}} = \langle \pi^{(n)}(a)V\xi, V\xi\rangle_{\mathcal{K}^n},$$

where $V\xi = (V\xi_1, \dots, V\xi_n) \in \mathcal{K}^n$. Since π is a *-homomorphism, then $\pi^{(n)}$ is positive by Proposition 3.11, and therefore $\langle \varphi^{(n)}(a)\xi, \xi\rangle = \langle \pi^{(n)}(a)V\xi, V\xi\rangle \geq 0$. Therefore $\varphi^{(n)}(a)$ is positive, so $\varphi^{(n)}$ is positive for $n \geq 1$. Hence φ is completely positive.

Now assume that (i) is satisfied. The aim is first to define a sesquilinear form $\langle \cdot, \cdot \rangle_\varphi : \mathcal{A}\odot\mathcal{H}\times\mathcal{A}\odot\mathcal{H} \to \mathbb{C}$ on the algebraic tensor product $\mathcal{A} \odot \mathcal{H}$ such that

$$\left\langle \sum_{i=1}^n a_i \otimes \xi_i, \sum_{j=1}^m b_j \otimes \eta_j \right\rangle_\varphi = \sum_{i,j} \langle \varphi(b_j^* a_i)\xi_i, \eta_j\rangle_{\mathcal{H}}. \tag{3.1}$$

The question is whether we can obtain a well-defined sesquilinear form satisfying the above equation. To address this, we will show that there exists a sesquilinear form on $\mathcal{A} \odot \mathcal{H}$ that satisfies (3.1). For $a \in \mathcal{A}$ and $\xi \in \mathcal{H}$, the map $(b, \eta) \mapsto \overline{\langle \varphi(b^*a)\xi, \eta\rangle_{\mathcal{H}}}$ is bilinear and hence induces a unique linear map $f_{(a,\xi)} \colon \mathcal{A} \odot \mathcal{H} \to \mathbb{C}$ such that

$$f_{(a,\xi)}(b \otimes \eta) = \langle \eta, \varphi(b^*a)\xi\rangle_{\mathcal{H}}, \quad b \in \mathcal{A},\ \eta \in \mathcal{H},$$

by universality of the algebraic tensor product. Letting $(\mathcal{A} \odot \mathcal{H})^*$ denote the vector space of (not necessarily bounded) linear functionals $\mathcal{A} \odot \mathcal{H} \to \mathbb{C}$, the map $\sigma \colon \mathcal{A} \times \mathcal{H} \to (\mathcal{A} \odot \mathcal{H})^*$ given by

$$\sigma(a, \xi)(v) = \overline{f_{(a,\xi)}(v)}, \quad v \in \mathcal{A} \odot \mathcal{H},$$

is bilinear, so universality again yields a linear map $\tilde{\sigma} \colon \mathcal{A} \odot \mathcal{H} \to (\mathcal{A} \odot \mathcal{H})^*$ satisfying

$$\tilde{\sigma}(a \otimes \xi)(b \otimes \eta) = \overline{f_{(a,\xi)}(b \otimes \eta)} = \langle \varphi(b^*a)\xi, \eta\rangle_{\mathcal{H}}.$$

By defining $\langle v, w\rangle_\varphi = \tilde{\sigma}(v)(w)$ for $v, w \in \mathcal{A} \odot \mathcal{H}$, $\langle \cdot, \cdot \rangle_\varphi$ is sesquilinear and satisfies (3.1).

Observe that for $a_1, \dots, a_n \in \mathcal{A}$ and $\xi = (\xi_1, \dots, \xi_n) \in \mathcal{H}^n$, then if we define $x \in M_n(\mathcal{A})$ by

$$x = \begin{pmatrix} a_1^*a_1 & \dots & a_1^*a_n \\ \vdots & & \vdots \\ a_n^*a_1 & \dots & a_n^*a_n \end{pmatrix}, \tag{3.2}$$

Lemma 3.1 yields that A is positive and hence

$$\left\langle \sum_{i=1}^n a_i \otimes \xi_i, \sum_{i=1}^n a_i \otimes \xi_i \right\rangle_\varphi = \sum_{i,j=1}^n \langle \varphi(a_j^*a_i)\xi_i, \xi_j\rangle_{\mathcal{H}} = \langle \varphi^{(n)}(x)\xi, \xi\rangle_{\mathcal{H}^n} \geq 0,$$

since $\varphi^{(n)}$ was positive by assumption. Hence $\langle \cdot, \cdot \rangle_\varphi$ is a positive semi-definite sesquilinear form, but it is not necessarily an inner product. In order to turn it into an inner product, we need to pass to an appropriate quotient space. Let $\mathscr{N} \subseteq \mathcal{A} \odot \mathcal{H}$ be defined by $\mathscr{N} = \{v \in \mathcal{A} \odot \mathcal{H} \,|\, \langle v, v\rangle_\varphi = 0\}$. By the Cauchy-Schwarz inequality (Proposition 0.1) we have

$$|\langle v, w\rangle_\varphi| \leq \langle v, v\rangle_\varphi \langle w, w\rangle_\varphi$$

for $x, y \in \mathcal{A} \odot \mathcal{H}$, so

$$\mathscr{N} = \{v \in \mathcal{A} \odot \mathcal{H} \,|\, \langle v, w\rangle_\varphi = 0 \text{ for all } w \in \mathcal{A} \odot \mathcal{H}\}.$$

This makes it clear that $\mathscr{N}$ is a subspace of $\mathcal{A} \odot \mathcal{H}$. We now define an inner product on the quotient vector space $(\mathcal{A} \odot \mathcal{H})/\mathscr{N}$ by

$$\langle [x], [y]\rangle_\varphi = \langle x, y\rangle_\varphi,$$

and we let $\mathcal{K}$ denote the Hilbert space completion of the $(\mathcal{A}\odot\mathcal{H})/\mathscr{N}$ with respect to this inner product.

We now assume that $\mathcal{A}$ has a unit $1_{\mathcal{A}}$. Given $a \in \mathcal{A}$, then since the map $\mathcal{A} \times \mathcal{H} \to \mathcal{A} \odot \mathcal{H}$ given by $(b, \xi) \mapsto ab \otimes \xi$ is bilinear, it induces a unique linear map $\pi'(a) \colon \mathcal{A} \odot \mathcal{H} \to \mathcal{A} \odot \mathcal{H}$ given by

$$\pi'(a)\left(\sum_{i=1}^n a_i \otimes \xi_i\right) = \sum_{i=1}^n aa_i \otimes \xi_i.$$

If $v = \sum_{i=1}^n a_i \otimes \xi_i \in \mathcal{A} \odot \mathcal{H}$, note that by defining

$$\tilde{a} = \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & a & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a \end{pmatrix}, \quad y = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix},$$

then $y^*\tilde{a}^*\tilde{a}y \leq \|\tilde{a}\|^2 y^*y = \|a\|^2 y^*y$. Hence if we put $\xi = (\xi_1, \dots, \xi_n)$, we obtain

$$\begin{aligned} \langle \pi'(a)v, \pi'(a)v \rangle_\varphi &= \left\langle \sum_{i=1}^n aa_i \otimes \xi_i, \sum_{i=1}^n aa_i \otimes \xi_i \right\rangle_\varphi \\ &= \sum_{i,j=1}^n \langle \varphi(a_j^* a^* a a_i)\xi_i, \xi_j \rangle_{\mathcal{H}} \\ &= \langle \varphi^{(n)}(y^*\tilde{a}^*\tilde{a}y)\xi, \xi \rangle_{\mathcal{H}^n} \leq \|a\|^2 \langle \varphi^{(n)}(y^*y)\xi, \xi \rangle_{\mathcal{H}^n} = \|a\|^2 \langle v, v \rangle_\varphi. \end{aligned}$$

Hence it is possible to define a bounded linear map $\tilde{\pi}(a) \colon (\mathcal{A} \odot \mathcal{H})/\mathscr{N} \to (\mathcal{A} \odot \mathcal{H})/\mathscr{N}$ by $\tilde{\pi}(a)([v]) = [\pi'(a)(v)]$ for $v \in \mathcal{A} \odot \mathcal{H}$ and by passing to completions (using Proposition A.1) we obtain a unique bounded linear operator $\pi(a) \in B(\mathcal{K})$ such that $\pi(a)w = \tilde{\pi}(a)(w)$ for $w \in (\mathcal{A} \odot \mathcal{H})/\mathscr{N}$. This gives us a map $\pi \colon \mathcal{A} \to B(\mathcal{K})$. We claim that π is actually a unital *-homomorphism. Let $a, b \in \mathcal{A}$ and $w = [\sum_{i=1}^n a_i \otimes \xi_i] \in (\mathcal{A} \odot \mathcal{H})/\mathscr{N}$. Clearly $\pi(1_\mathcal{A})w = w$ and $\pi(a+b)w = \pi(a)w + \pi(b)w$; moreover,

$$\pi(ab)\left[\sum_{i=1}^n a_i \otimes \xi_i\right] = \left[\sum_{i=1}^n aba_i \otimes \xi_i\right] = \pi(a)\left[\sum_{i=1}^n ba_i \otimes \xi_i\right] = \pi(a)\pi(b)\left[\sum_{i=1}^n a_i \otimes \xi_i\right],$$

so $\pi(ab)w = \pi(a)\pi(b)w$. As

$$\begin{aligned} \left\langle \pi(a^*)\left[\sum_{i=1}^n a_i \otimes \xi_i\right], \left[\sum_{j=1}^m b_j \otimes \eta_j\right] \right\rangle_{\mathcal{K}} &= \left\langle \left[\sum_{i=1}^n a^*a_i \otimes \xi_i\right], \left[\sum_{j=1}^m b_j \otimes \eta_j\right] \right\rangle_\varphi \\ &= \sum_{i,j} \langle \varphi(b_j^* a^* a_i)\xi_i, \eta_j \rangle_{\mathcal{H}} \\ &= \sum_{i,j} \langle \varphi((ab_j)^* a_i)\xi_i, \eta_j \rangle_{\mathcal{H}} \\ &= \left\langle \left[\sum_{i=1}^n a_i \otimes \xi_i\right], \left[\sum_{j=1}^m ab_j \otimes \eta_j\right] \right\rangle_\varphi \\ &= \left\langle \left[\sum_{i=1}^n a_i \otimes \xi_i\right], \pi(a)\left[\sum_{j=1}^m b_j \otimes \eta_j\right] \right\rangle_{\mathcal{K}} \\ &= \left\langle \pi(a)^*\left[\sum_{i=1}^n a_i \otimes \xi_i\right], \left[\sum_{j=1}^m b_j \otimes \eta_j\right] \right\rangle_{\mathcal{K}} \end{aligned}$$

we also have $\pi(a^*)w = \pi(a)^*w$. Using continuity of $\pi(1_\mathcal{A})$, $\pi(a)$ and $\pi(b)$, we conclude that π is indeed a unital *-homomorphism.

Define $V \colon \mathcal{H} \to \mathcal{K}$ by $V(\xi) = [1_\mathcal{A} \otimes \xi]$ for $\xi \in \mathcal{H}$. Since V is clearly linear and

$$\|V(\xi)\|_{\mathcal{K}}^2 = \langle [1_\mathcal{A} \otimes \xi], [1_\mathcal{A} \otimes \xi] \rangle_\varphi = \langle 1_\mathcal{A} \otimes \xi, 1_\mathcal{A} \otimes \xi \rangle_\varphi = \langle \varphi(1_\mathcal{A})\xi, \xi \rangle_{\mathcal{H}} \leq \|\varphi\| \|\xi\|_{\mathcal{H}}^2$$

for all $\xi \in \mathcal{H}$, V is a bounded linear operator. Finally, for $a \in \mathcal{A}$, then we have for all $\xi, \eta \in \mathcal{H}$ that

$$\begin{aligned} \langle V^*\pi(a)V\xi, \eta \rangle_{\mathcal{H}} &= \langle \pi(a)V\xi, V\eta \rangle_{\mathcal{K}} \\ &= \langle \pi(a)[1_\mathcal{A} \otimes \xi], [1_\mathcal{A} \otimes \eta] \rangle_{\mathcal{K}} \\ &= \langle [a \otimes \xi], [1_\mathcal{A} \otimes \eta] \rangle_\varphi \\ &= \langle a \otimes \xi, 1_\mathcal{A} \otimes \eta \rangle_\varphi \\ &= \langle \varphi(1_\mathcal{A}^* a)\xi, \eta \rangle_{\mathcal{H}} \\ &= \langle \varphi(a)\xi, \eta \rangle_{\mathcal{H}}. \end{aligned}$$

Hence $V^*\pi(a)V = \varphi(a)$, completing the proof of the unital case.

If $\mathcal{A}$ is non-unital, we consider instead the enveloping von Neumann algebra $\mathcal{A}^{**}$. It follows from Proposition 3.13 that $\varphi^{**}\colon \mathcal{A}^{**} \to B(\mathcal{H})^{**}$ is completely positive. Since $B(\mathcal{H})$ is a von Neumann algebra, it follows from the discussion after Theorem 2.63 that there exists a representation $\theta\colon B(\mathcal{H})^{**} \to B(\mathcal{H})$ such that $\theta(\iota_{B(\mathcal{H})}(T)) = T$ for all $T \in B(\mathcal{H})$. As θ is completely positive, $\theta \circ \varphi^{**}$ is completely positive. Because we know that (i) $\Rightarrow$ (ii) holds for unital C^*-algebras, there exists a Hilbert space $\mathcal{K}$, a representation $\pi\colon \mathcal{A}^{**} \to B(\mathcal{K})$ and a bounded linear operator $V\colon \mathcal{H} \to \mathcal{K}$ such that

$$\theta \circ \varphi^{**}(a) = V^*\pi(a)V, \quad a \in \mathcal{A}^{**}.$$

Since $\varphi^{**} \circ \iota_{\mathcal{A}} = \iota_{B(\mathcal{H})} \circ \varphi$ then for any $a \in \mathcal{A}$,

$$V^*\pi(\iota_{\mathcal{A}}(a))V = \theta \circ \varphi^{**}(\iota_{\mathcal{A}}(a)) = \varphi(a),$$

so $\pi \circ \iota_{\mathcal{A}}\colon \mathcal{A} \to B(\mathcal{K})$ is the sought-after representation.

Finally, assume that $\mathcal{A}$ is a von Neumann algebra and that φ is normal and completely positive. Let $(T_\alpha)_{\alpha\in A}$ be a bounded, increasing net of self-adjoint operators in $\mathcal{A}$ with $T \in \mathcal{A}$ being its strong operator limit and least upper bound $T \in \mathcal{A}$. For any R and S in $\mathcal{A}$, then because $T_\alpha \to T$ ultraweakly by Proposition 2.1 we see that $R^*T_\alpha S \to R^*TS$ ultraweakly. By normality of φ, $\varphi(R^*T_\alpha S) \to \varphi(R^*TS)$ ultraweakly. For $w = [\sum_{i=1}^n S_i \otimes \xi_i] \in (\mathcal{A} \odot \mathcal{H})/\mathscr{N}$ we then have

$$\begin{aligned}
\langle \pi(T_\alpha)w, w\rangle_\varphi &= \sum_{i,j=1}^n \langle \pi(T_\alpha)[S_i \otimes \xi_i], [S_j \otimes \xi_j]\rangle_\varphi \\
&= \sum_{i,j=1}^n \langle T_\alpha S_i \otimes \xi_i, S_j \otimes \xi_j\rangle_\varphi \\
&= \sum_{i,j=1}^n \langle \varphi(S_j^* T_\alpha S_i)\xi_i, \xi_j\rangle_{\mathcal{H}} \\
&\to \sum_{i,j=1}^n \langle \varphi(S_j^* T S_i)\xi_i, \xi_j\rangle_{\mathcal{H}} = \langle \pi(T)w, w\rangle_\varphi.
\end{aligned}$$

As π is a *-homomorphism, $(\pi(T_\alpha))_{\alpha\in A}$ is a bounded, increasing net of self-adjoint operators in $B(\mathcal{H})$ and hence has a least upper bound S that is also its strong operator limit. As this implies $\langle \pi(T_\alpha)w, w\rangle \to \langle Sw, w\rangle$ for all $w \in (\mathcal{A} \odot \mathcal{H})/\mathscr{N}$, it follows that $\langle \pi(T)w, w\rangle = \langle Sw, w\rangle$ for all $w \in (\mathcal{A}\odot\mathcal{H})/\mathscr{N}$. Because $(\mathcal{A}\odot\mathcal{H})/\mathscr{N}$ is dense in $\mathcal{K}$ by construction, it follows that $\langle \pi(T)\xi, \xi\rangle = \langle S\xi, \xi\rangle$ for all $\xi \in \mathcal{K}$, so $\pi(T) = S$. Therefore π is normal. $\square$

Since the main concern of the project is von Neumann algebras, we are now interested in looking at what consequences it has for completely positive maps on these. The two next results will reduce our future work greatly.

Corollary 3.17. *Let $(\mathscr{M}_i)_{i\in I}$ and $(\mathscr{N}_i)_{i\in I}$ be families of von Neumann algebras with $\mathscr{M}_i \subseteq B(\mathcal{H}_i)$ and $\mathscr{N}_i \subseteq B(\mathcal{K}_i)$ for families of Hilbert spaces $(\mathcal{H}_i)_{i\in I}$ and $(\mathcal{K}_i)_{i\in I}$. Define $\mathscr{M} = \bigoplus_{i\in I} \mathscr{M}_i$ and $\mathscr{N} = \bigoplus_{i\in I} \mathscr{N}_i$ and let $(\varphi_i)_{i\in I}$ be a family of completely positive maps $\mathscr{M}_i \to \mathscr{N}_i$ such that $\varphi_i(1_{\mathscr{M}_i}) = 1_{\mathscr{N}_i}$ for all $i \in I$. Then $\varphi = \bigoplus_{i\in I} \varphi_i\colon \mathscr{M} \to \mathscr{N}$ given by*

$$\varphi((T_i)_{i\in I}) = (\varphi_i(T_i))_{i\in I}$$

is a completely positive map with $\varphi(1_{\mathscr{M}}) = 1_{\mathscr{N}}$. If all φ_i are normal, then φ is normal as well.

Proof. By Proposition 3.9, $\|\varphi_i\| = 1$ for all $i \in I$, so φ is well-defined, and clearly linear and bounded as well with $\varphi(1) = 1$. By Stinespring's representation theorem then for all $i \in I$ we have a Hilbert space $\mathcal{L}_i$, a representation $\pi_i\colon \mathscr{M}_i \to B(\mathcal{L}_i)$ and a bounded linear operator $V_i\colon \mathcal{K}_i \to \mathcal{L}_i$ such that

$$\varphi_i(T_i) = V_i^*\pi_i(T_i)V_i, \quad T_i \in \mathscr{M}_i.$$

Note that $1 = \|\varphi_i(1)\| = \|V_i^*\pi(1)V_i\| = \|V_i^*V_i\| = \|V_i\|^2$, so $\|V_i\| = 1$ for all $i \in I$. It is now possible to define a representation $\pi\colon \mathscr{M} \to B(\bigoplus_{i\in I}\mathcal{L}_i)$ by

$$\pi((T_i)_{i\in I})(\xi_i)_{i\in I} = (\pi_i(T_i)\xi_i)_{i\in I}, \quad T_i \in \mathscr{M}_i,\ \xi_i \in \mathcal{L}_i,$$

and a bounded linear operator $V\colon \bigoplus_{i\in I}\mathcal{K}_i \to \bigoplus_{i\in I}\mathcal{L}_i$ by $V((\xi_i)_{i\in I}) = (V_i\xi_i)_{i\in I}$. In this case we see that $\varphi((T_i)_{i\in I}) = V^*\pi((T_i)_{i\in I})V$ for all $(T_i)_{i\in I} \in \mathscr{M}$, so Stinespring's representation theorem yields that φ is completely positive.

Assume that all φ_i are normal and let $\omega \in \mathscr{N}_*$. Then there is a family $(\omega_i)_{i\in I} \in \bigoplus_{i\in I}(\mathscr{N}_i)_*$ corresponding to ω in the manner of Proposition 2.57, so $\omega_i \circ \varphi_i \in (\mathscr{M}_i)_*$ for all $i \in I$ by Proposition 2.45. As $\sum_{i\in I}\|\omega_i\circ\varphi_i\| \le \sum_{i\in I}\|\omega_i\| < \infty$, so there is a $\psi \in \mathscr{M}_*$ corresponding to $(\omega_i\circ\varphi_i)_{i\in I} \in \bigoplus_{i\in I}(\mathscr{M}_i)_*$, again in the manner of Proposition 2.57. Since

$$\omega(\varphi((T_i)_{i\in I})) = \omega((\varphi_i(T_i))_{i\in I}) = \sum_{i\in I}\omega_i\circ\varphi_i(T_i) = \psi((T_i)_{i\in I}),$$

we have that $\omega\circ\varphi \in \mathscr{M}_*$ for all $\omega \in \mathscr{N}_*$. Hence by Proposition 2.45, φ is normal. □

Corollary 3.18. *Let $\mathscr{M}_1$, $\mathscr{M}_2$, $\mathscr{N}_1$ and $\mathscr{N}_2$ be von Neumann algebras. Let $\varphi_i\colon \mathscr{M}_i \to \mathscr{N}_i$, $i = 1,2$ be completely positive normal maps. Then there is exactly one completely positive normal map $\varphi\colon \mathscr{M}_1 \mathbin{\overline{\otimes}} \mathscr{M}_2 \to \mathscr{N}_1 \mathbin{\overline{\otimes}} \mathscr{N}_2$ such that*

$$\varphi(T_1\otimes T_2) = \varphi_1(T_1)\otimes\varphi_2(T_2), \quad T_1 \in \mathscr{M}_1,\ T_2 \in \mathscr{M}_2.$$

Moreover, $\|\varphi\| = \|\varphi_1\|\|\varphi_2\|$.

Proof. Assume that $\mathscr{N}_i$ acts on the Hilbert space $\mathcal{H}_i$ for $i = 1,2$ so that φ_i maps into $B(\mathcal{H}_i)$. Stinespring's representation theorem yields the existence of Hilbert spaces $\mathcal{K}_i$, unital normal representations $\pi_i\colon \mathscr{M}_i \to B(\mathcal{K}_i)$ and bounded linear maps $V_i\colon \mathcal{H}_i \to \mathcal{K}_i$ such that

$$\varphi_i(T_i) = V_i^*\pi_i(T_i)V_i, \quad T_i \in \mathscr{M}_i$$

for $i = 1,2$. Proposition 2.56 in turn yields the existence of a unital representation

$$\pi\colon \mathscr{M}_1 \mathbin{\overline{\otimes}} \mathscr{M}_2 \to B(\mathcal{K}_1\otimes\mathcal{K}_2)$$

satisfying $\pi(T_1\otimes T_2) = \pi_1(T_1)\otimes\pi_2(T_2)$ for $T_i \in \mathscr{M}_i$, $i = 1,2$. From [14, Proposition 2.6.12] we obtain a bounded linear operator $V\colon \mathcal{H}_1\otimes\mathcal{H}_2 \to \mathcal{K}_1\otimes\mathcal{K}_2$ from V_1 and V_2 that uniquely satisfies

$$V(\xi_1\otimes\xi_2) = V_1\xi\otimes V_2\xi_2, \quad \xi_i \in \mathcal{H}_i,\ i = 1,2.$$

Moreover, $\|V\| \le \|V_1\|\|V_2\|$ and $V^* = V_1^*\otimes V_2^*$. Define

$$\varphi(T) = V^*\pi(T)V, \quad T \in \mathscr{M}_1 \mathbin{\overline{\otimes}} \mathscr{M}_2.$$

φ is clearly linear, and it follows from Stinespring's representation theorem that φ is completely positive. Moreover, φ is normal, since the map $T \mapsto V^*TV$ is ultraweakly-to-ultraweakly continuous and π is normal. For $T_1 \in \mathscr{M}_1$ and $T_2 \in \mathscr{M}_2$, we have

$$\varphi(T_1\otimes T_2) = (V_1^*\pi_1(T_1)V_1)\otimes(V_2^*\pi_2(T_2)V_2) = \varphi_1(T_1)\otimes\varphi_2(T_2) \in \mathscr{N}_1\odot\mathscr{N}_2,$$

so φ maps $\mathscr{M}_1\odot\mathscr{M}_2$ into $\mathscr{N}_1\odot\mathscr{N}_2$. Since $\mathscr{N}_1 \mathbin{\overline{\otimes}} \mathscr{N}_2$ is the ultraweak closure of $\mathscr{N}_1\odot\mathscr{N}_2$ by von Neumann's density theorem and φ is normal, it follows that φ maps into $\mathscr{N}_1 \mathbin{\overline{\otimes}} \mathscr{N}_2$ and satisfies the wanted elementary tensor property. Moreover, it is uniquely determined by this property since it is normal. Finally, Proposition 3.9 yields

$$\|\varphi\| = \|\varphi(1_{\mathscr{M}_1\overline{\otimes}\mathscr{M}_2})\| = \|\varphi_1(1_{\mathscr{M}_1})\otimes\varphi_2(1_{\mathscr{M}_2})\| = \|\varphi_1(1_{\mathscr{M}_1})\|\|\varphi_2(1_{\mathscr{M}_2})\| = \|\varphi_1\|\|\varphi_2\|,$$

completing the proof. □

Let $\mathscr{M} \subseteq B(\mathcal{H})$ and $\mathscr{N} \subseteq B(\mathcal{K})$ be von Neumann algebras. We will make an attempt to describe the predual of $\mathscr{M} \mathbin{\overline{\otimes}} \mathscr{N}$. For any positive $\omega \in \mathscr{M}_*$ and $\varphi \in \mathscr{N}_*$, then by Proposition 3.12, ω and φ are completely positive. As they are also ultraweakly continuous and hence normal, it follows from Corollary 3.18 that there is a unique completely positive normal map $\omega \otimes \varphi \colon \mathscr{M} \mathbin{\overline{\otimes}} \mathscr{N} \to \mathbb{C}$ such that

$$(\omega \otimes \varphi)(S \otimes T) = \omega(S) \otimes \varphi(T) = \omega(S)\varphi(T), \quad S \in \mathscr{M},\ T \in \mathscr{N},$$

that also satisfies $\|\omega \otimes \varphi\| = \|\omega\|\|\varphi\|$. In the above expression, we have identified the von Neumann algebras $\mathbb{C} \mathbin{\overline{\otimes}} \mathbb{C} = \mathbb{C} \odot \mathbb{C}$ (Lemma 1.34) and $\mathbb{C}$ by means of the *-isomorphism given by $\lambda \otimes \mu \mapsto \lambda\mu$ – this map is normal by Proposition 2.48. Hence $\omega \otimes \varphi \in (\mathscr{M} \mathbin{\overline{\otimes}} \mathscr{N})_*$. We define $\mathscr{M}_* \odot \mathscr{N}_*$ to be the linear span of all $\omega \otimes \varphi$ in $(\mathscr{M} \mathbin{\overline{\otimes}} \mathscr{N})_*$ constructed this way for positive $\omega \in \mathscr{M}_*$ and $\varphi \in \mathscr{N}_*$.

For arbitrary $\omega \in \mathscr{M}_*$ and $\varphi \in \mathscr{N}_*$, Theorem 2.40 yields that each of these is a linear combination of positive ultraweakly continuous linear functionals on their respective von Neumann algebras, i.e.

$$\omega = \sum_{i=1}^{n} \lambda_i \omega_i, \quad \varphi = \sum_{j=1}^{m} \mu_j \varphi_j,$$

where each of the summands is a positive ultraweakly continuous functional. By defining

$$\omega \otimes \varphi := \sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_i \mu_j \omega_i \otimes \varphi_j \in (\mathscr{M} \mathbin{\overline{\otimes}} \mathscr{N})_*,$$

then for all $S \in \mathscr{M}$ and $T \in \mathscr{N}$, we have

$$(\omega \otimes \varphi)(S \otimes T) = \sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_i \mu_j \omega_i(S) \varphi_j(T) = \omega(S)\varphi(T).$$

Hence any $\omega \in \mathscr{M}_*$ and $\varphi \in \mathscr{N}_*$ induce an element $\omega \otimes \varphi \in \mathscr{M}_* \odot \mathscr{N}_* \subseteq (\mathscr{M} \mathbin{\overline{\otimes}} \mathscr{N})_*$ satisfying the above equality for elementary tensors, and moreover, it is the *only* linear functional in $(\mathscr{M} \mathbin{\overline{\otimes}} \mathscr{N})_*$ to satisfy the above equality; if any other functional $\psi \in (\mathscr{M} \mathbin{\overline{\otimes}} \mathscr{N})_*$ satisfies $\psi(S \otimes T) = \omega(S)\varphi(T)$, then ψ and $\omega \otimes \varphi$ are equal on $\mathscr{M} \odot \mathscr{N}$, and since $\mathscr{M} \mathbin{\overline{\otimes}} \mathscr{N}$ is the ultraweak closure of $\mathscr{M} \odot \mathscr{N}$ by von Neumann's density theorem, it follows from ultraweak continuity that $\psi = \omega \otimes \varphi$. By this uniqueness, some relevant calculus for these tensor functionals follows, namely

(i) $(\omega_1 + \omega_2) \otimes \varphi = \omega_1 \otimes \varphi + \omega_2 \otimes \varphi$ for $\omega_1, \omega_2 \in \mathscr{M}_*$ and $\varphi \in \mathscr{N}_*$;
(ii) $\omega \otimes (\varphi_1 + \varphi_2) = \omega \otimes \varphi_1 + \omega \otimes \varphi_1$ for $\omega \in \mathscr{M}_*$ and $\varphi_1, \varphi_2 \in \mathscr{N}_*$;
(iii) $(\lambda\omega) \otimes \varphi = \omega \otimes (\lambda\varphi) = \lambda(\omega \otimes \varphi)$ for $\omega \in \mathscr{M}_*$, $\varphi \in \mathscr{N}_*$ and $\lambda \in \mathbb{C}$.

Finally, for any $\xi \in \mathcal{H}$ and $\eta \in \mathcal{K}$, then $\omega_\xi \in \mathscr{M}_*$ and $\omega_\eta \in \mathscr{N}_*$ are positive, hence inducing a unique ultraweakly continuous linear functional $\omega_\xi \otimes \omega_\eta \in \mathscr{M}_* \odot \mathscr{N}_*$ by Proposition 3.18, satisfying

$$\Omega(S \otimes T) = \langle S\xi, \xi\rangle\langle T\eta, \eta\rangle = \langle (S \otimes T)\xi \otimes \eta, \xi \otimes \eta\rangle, \quad S \in \mathscr{M},\ T \in \mathscr{N}.$$

Hence

$$\omega_{\xi \otimes \eta} = \omega_\xi \otimes \omega_\eta \in \mathscr{M}_* \odot \mathscr{N}_*$$

by uniqueness, so $\omega_\xi \in \mathscr{M}_* \odot \mathscr{N}_*$ for all $\xi \in \mathcal{H} \odot \mathcal{K}$. Therefore ω_ξ is contained in the norm closure of $\mathscr{M}_* \odot \mathscr{N}_*$ for all $\xi \in \mathcal{H} \otimes \mathcal{K}$; as this norm closure is also a subspace of $(\mathscr{M} \mathbin{\overline{\otimes}} \mathscr{N})^*$, Theorem 2.40 finally yields the following elegant solution to our problem:

Proposition 3.19. *For any two von Neumann algebras $\mathscr{M}$ and $\mathscr{N}$, $\mathscr{M}_* \odot \mathscr{N}_*$ is norm-dense in $(\mathscr{M} \mathbin{\overline{\otimes}} \mathscr{N})_*$.*

However, this statement, although helpful, does not provide the entire story about the product functionals $\omega \otimes \varphi$ for $\omega \in \mathscr{M}_*$ and $\varphi \in \mathscr{N}_*$. What are their norms? $\|\omega \otimes \varphi\| = \|\omega\|\|\varphi\|$ is what one would expect and this statement is indeed true; the proof requires the fact that any ultraweakly continuous linear functional has a *polar decomposition*, proved in Proposition 2.44.

Proposition 3.20. *If $\omega \in \mathscr{M}_*$ and $\varphi \in \mathscr{N}_*$, then $\|\omega \otimes \varphi\| = \|\omega\|\|\varphi\|$.*

Proof. For any $\omega \in \mathscr{M}_*$ and $\varphi \in \mathscr{N}_*$, Proposition 2.44 yields the existence of partial isometries $U \in \mathscr{M}$ and $V \in \mathscr{N}$ as well as positive linear functionals $\eta \in \mathscr{M}_*$ and $\psi \in \mathscr{N}_*$ such that

$$\omega = U \cdot \eta, \quad \eta = U^* \cdot \omega, \quad \varphi = V \cdot \psi, \quad \psi = V^* \cdot \varphi,$$

and moreover, $\|\omega\| = \|\eta\|$ and $\|\varphi\| = \|\psi\|$. Consider the positive linear functional $\eta \otimes \psi \in (\mathscr{M} \mathbin{\overline{\otimes}} \mathscr{N})_*$ and note that for all $S \in \mathscr{M}$ and $T \in \mathscr{N}$ we have

$$(\omega \otimes \varphi)(S \otimes T) = \omega(S)\varphi(T) = \eta(SU)\psi(TV) = (\eta \otimes \psi)((S \otimes T)(U \otimes V)) = (U \otimes V) \cdot (\eta \otimes \psi)(S \otimes T)$$

and similarly

$$(\eta \otimes \psi)(S \otimes T) = \omega(SU^*)\varphi(TV^*) = (U^* \otimes V^*) \cdot (\omega \otimes \varphi)(S \otimes T).$$

Since $(U \otimes V) \cdot (\eta \otimes \psi)$ and $(U^* \otimes V^*) \cdot (\omega \otimes \varphi)$ are contained in $(\mathscr{M} \mathbin{\overline{\otimes}} \mathscr{N})_*$ by Lemma 2.36 and Theorem 2.40, it follows by uniqueness that

$$\omega \otimes \varphi = (U \otimes V) \cdot (\eta \otimes \psi), \quad \eta \otimes \psi = (U^* \otimes V^*) \cdot (\omega \otimes \varphi).$$

In particular, this implies $\|\omega \otimes \varphi\| = \|\eta \otimes \psi\|$. Hence

$$\|\omega \otimes \varphi\| = \|\eta \otimes \psi\| = \|\eta\|\|\psi\| = \|\omega\|\|\varphi\|,$$

completing the proof. □

3.5 Completely positive maps over dual spaces

It may seem peculiar to the reader that we have defined the notion of positive linear maps for not just C^*-algebras, but also duals of these, and yet have not even been close to considering the lastly mentioned case. This is where the dual spaces return, for only a short time but with a vengeance, paving the way for some of the later and greater results of this thesis.

The first question one could ask is whether dual maps preserve positivity, and the answer is affirmative.

Proposition 3.21. *Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras. If $\varphi\colon \mathcal{A} \to \mathcal{B}$ is a linear positive map, then $\varphi^*\colon \mathcal{B}^* \to \mathcal{A}^*$ is positive. If φ is completely positive, then φ^* is completely positive.*

Proof. The second statement follows immediately from the first, since for all $n \geq 1$, $\psi = (\psi_{ij})_{i,j=1}^n \in M_n(\mathcal{B}^*)$ and $a = (a_{ij})_{i,j=1}^n \in M_n(\mathcal{A})$, then

$$(\varphi^{(n)})^*(\psi)(a) = (\psi \circ \varphi^{(n)})(a) = \sum_{i,j=1}^n \psi_{ij}(\varphi(a_{ij})) = \sum_{i,j=1}^n \varphi^*(\psi_{ij})(a_{ij}) = (\varphi^* \circ \psi_{ij})_{i,j=1}^n(a) = (\varphi^*)^{(n)}(\psi)(a),$$

so $(\varphi^{(n)})^* = (\varphi^*)^{(n)}$. Assuming that φ is positive, then if $\psi \in \mathcal{B}^*$ and $a \in \mathcal{A}$ are positive, then we have

$$\varphi^*(\psi)(a) = \psi(\varphi(a)) \geq 0,$$

implying that φ^* is positive. □

We will soon need to know when a bounded linear map $\mathcal{A} \to \mathcal{B}^*$ for C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ is completely positive. It turns out that there is a straightforward criterion for this to be true.

Proposition 3.22. *Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras and $\varphi\colon \mathcal{A} \to \mathcal{B}^*$ bounded and linear. Then the following are equivalent:*

(i) *φ is completely positive.*
(ii) *$\sum_{i,j=1}^n \varphi(a_i^* a_j)(b_i^* b_j) \geq 0$ for all $n \geq 1$, $a_1, \ldots, a_n \in \mathcal{A}$ and $b_1, \ldots, b_n \in \mathcal{B}$.*

Proof. If φ is completely positive, then for $n \geq 1$ and any $a_1, \ldots, a_n \in \mathcal{A}$, $\varphi^{(n)}((a_i^* a_j)_{i,j=1}^n) = (\varphi(a_i^* a_j))_{i,j=1}^n$ is a positive element of $M_n(\mathcal{B}^*)$, so (ii) holds by Lemma 3.3. Assuming instead that (ii) holds, then for $n \geq 1$ and $a_1, \ldots, a_n \in \mathcal{A}$, then Lemma 3.3 allows us to go backwards and say that $\varphi^{(n)}((a_i^* a_j)_{i,j=1}^n) = (\varphi(a_i^* a_j))_{i,j=1}^n$ is positive in $M_n(\mathcal{B}^*)$, so Lemma 3.1 tells us that $\varphi^{(n)}(a)$ is positive for all positive $a \in M_n(\mathcal{A})$, as sums of positive linear functionals are positive. □

The next proposition concerns functionals that we have met before – they are siblings of the functionals described in Proposition 2.38.

Proposition 3.23. *Let $\mathcal{A}$ be a C^*-algebra and let $\pi\colon \mathcal{A} \to B(\mathcal{H})$ be a representation such that $\mathscr{M} = \pi(\mathcal{A})$ has a cyclic unit vector $\xi \in \mathcal{H}$. Let ω_ξ be the corresponding vector state and let E denote the complex linear span of*

$$C_\xi = \{\varphi \in \mathcal{A}^* \mid 0 \leq \varphi \leq \omega_\xi \circ \pi\}.$$

Finally, let $\theta\colon \mathscr{M}' \to \mathcal{A}^*$ be given by

$$\theta(T)(a) = \langle \pi(a)T\xi, \xi\rangle, \quad T \in \mathscr{M}', \quad a \in \mathcal{A}.$$

Then θ is a completely positive linear isomorphism of $\mathscr{M}'$ onto E with a completely positive inverse.

Proof. θ is clearly well-defined, linear, bounded and maps into $\mathcal{A}^*$. For any $\varphi \in E$, write $\varphi = \sum_{i=1}^n \lambda_i\varphi_i$ where $\lambda_i \in \mathbb{C}$ and $\varphi_i \in C_\xi$ for $i = 1, \dots, n$. By Proposition 2.38, there exists a positive operator $T_i \in \mathscr{M}'$ such that $\theta_{T_i} \circ \pi = \varphi_i$, in the notation of the aforementioned proposition. Since

$$\varphi_i(a) = \theta_{T_i}(\pi(a)) = \langle \pi(a)T_i\xi, T_i\xi\rangle = \langle \pi(a)(T_i)^2\xi, \xi\rangle = \theta((T_i)^2)(a),$$

we have $\theta(\sum_{i=1}^n \lambda_i(T_i')^2) = \varphi$; hence φ is surjective. Moreover, for any non-zero positive operator $T \in \mathscr{M}'$ with $\lambda = \|T\|$ we have $0 \leq \lambda^{-1/2}T^{1/2} \leq 1_{\mathcal{H}}$, so by Proposition 2.38 we find that the positive linear functional functional $\theta(\lambda^{-1}T) = \theta_{\lambda^{-1/2}T^{1/2}}$ is mapped into C_ξ, whereupon $\theta(T) \in E$. Since any operator in the unital C^*-algebra $\mathscr{M}'$ is a finite linear combination of positive operators [31, Theorem 11.2], it follows that θ maps $\mathscr{M}'$ onto E.

Assuming that $\theta(T) = 0$ for some $T \in \mathscr{M}'$, then $0 = \theta(T)(b^*a) = \langle T\pi(a)\xi, \pi(b)\xi\rangle$ for all $a, b \in \mathcal{A}$. Since ξ is cyclic for $\mathscr{M}$, it follows that $T = 0$. Hence $\theta\colon \mathscr{M}' \to E$ is a linear isomorphism.

In order to prove that θ is completely positive, it is sufficient by Proposition 3.22 to prove that

$$\sum_{i,j=1}^n \theta(T_i^*T_j)(a_i^*a_j) \geq 0, \quad n \geq 1,\ T_1, \dots, T_n \in \mathscr{M}',\ a_1, \dots, a_n \in \mathcal{A}.$$

By straightforward calculation, we indeed see that

$$\begin{aligned}\sum_{i,j=1}^n \theta(T_i^*T_j)(a_i^*a_j) &= \sum_{i,j=1}^n \langle (T_i^*T_j)\pi(a_i^*a_j)\xi, \xi\rangle \\ &= \sum_{i,j=1}^n \langle T_j\pi(a_j)\xi, T_i\pi(a_i)\xi\rangle \\ &= \left\|\sum_{i=1}^n T_i\pi(a_i)\xi\right\|^2 \geq 0.\end{aligned}$$

Hence θ is completely positive.

To prove that $\theta^{-1}\colon E \to \mathscr{M}'$ is completely positive, let $n \geq 1$ and let $\varphi = (\varphi_{ij})_{i,j=1}^n$ be a positive element in $M_n(E)$; we must prove that $(\theta^{-1})^{(n)}(\varphi)$ is a positive element of $M_n(\mathscr{M}')$. We will identify $M_n(B(\mathcal{H}))$ with $B(\mathcal{H}^n)$ in the following, and prove that $(\theta^{-1})^{(n)}(\varphi)$ is positive as an operator in $B(\mathcal{H}^n)$. Let $a_1, \dots, a_n \in \mathcal{A}$ and define

$$\tilde{\xi} = (\pi(a_1)\xi, \dots, \pi(a_n)\xi) \in \mathcal{H}^n.$$

Noting that $\theta^{-1}(\varphi_{ij})$ is in $\mathscr{M}'$ and hence commutes with all $\pi(a_i)$ for all $i, j = 1, \ldots, n$, we find that

$$\begin{aligned}\left\langle (\theta^{-1})^{(n)}(\varphi)\tilde{\xi}, \tilde{\xi} \right\rangle &= \sum_{i,j=1}^{n} \langle \theta^{-1}(\varphi_{ij})\pi(a_j)\xi, \pi(a_i)\xi\rangle \\ &= \sum_{i,j=1}^{n} \langle \theta^{-1}(\varphi_{ij})\pi(a_i^* a_j)\xi, \xi\rangle \\ &= \sum_{i,j=1}^{n} (\theta \circ \theta^{-1})(\varphi_{ij})(T_i^* T_j) \\ &= \sum_{i,j=1}^{n} \varphi_{ij}(T_i^* T_j) = \varphi((T_i^* T_j)_{i,j=1}^n) \geq 0,\end{aligned}$$

since $(a_i^* a_j)_{i,j=1}^n$ is positive by Lemma 3.1. Because ξ is cyclic for $\mathscr{M}$, it is easy to see that elements of the form $(\pi(a_1)\xi, \ldots, \pi(a_n)\xi)$ for $a_1, \ldots, a_n \in \mathcal{A}$ are dense in $\mathcal{H}^n$. Hence $(\theta^{-1})^{(n)}(\varphi) \in M_n(\mathscr{M}')$ is positive, so θ^{-1} is completely positive. $\square$

Our last result will be concerning the isometric isomorphism connecting duals to preduals of double duals.

Proposition 3.24. *Let $\mathcal{A}$ be a C^*-algebra. Then $\Omega\colon (\mathcal{A}^{**})_* \to \mathcal{A}^*$ given by*

$$\Omega(\omega)(a) = \omega(\iota(a)), \quad a \in \mathcal{A}$$

is an isometric isomorphism, where $\iota\colon \mathcal{A} \to \mathcal{A}^{**}$ is the inclusion homomorphism. Moreover, Ω and Ω^{-1} are completely positive.

Proof. Define $\mathscr{M} = \mathcal{A}^{**}$. We already know that Ω is an isometric isomorphism from the remarks after Theorem 2.63. To see that Ω is completely positive, let $n \geq 1$ and $\omega = (\omega_{ij})_{i,j=1}^n \in M_n(\mathscr{M}_*)$ be positive. For any positive $a \in M_n(\mathcal{A})$, we then have that $\iota^{(n)}(a)$ is positive by Proposition 3.11, so $\omega(\iota^{(n)}(a))$ is positive. Therefore Ω is completely positive. For the case of Ω^{-1}, assume instead that $\varphi = (\varphi_{ij})_{i,j=1}^n \in M_n(\mathcal{A}^*)$ is positive and let $T_1, \ldots, T_n \in \mathscr{M}$. For each $\varphi_{ij} \in \mathcal{A}^*$, there is an $\omega_{ij} \in \mathscr{M}_*$ such that $\Omega(\omega_{ij}) = \varphi_{ij}$. By Kaplansky's density theorem (Theorem 2.33), we can find bounded nets $(\iota(x_\alpha^i))_{\alpha \in A_i}$ of operators in $\iota(\mathcal{A})$ such that $\iota(x_\alpha^i) \to T_i$ strongly for all $i = 1, \ldots, n$. Note that all the index sets of these nets may be different; nonetheless, the fact that the nets are bounded gives us a clear advantage, namely that

$$\iota(x_\alpha^i)^* \iota(x_\beta^j) \to T_i^* T_j$$

strongly for all $i, j = 1, \ldots, n$. Using Lemma 3.3 and Corollary 2.12, we have for all $(\alpha_1, \ldots, \alpha_n) \in \prod_{i=1}^n A_i$ that

$$0 \leq \sum_{i,j=1}^{n} \varphi_{ij}(x_{\alpha_i}^{i*} x_{\alpha_j}^j) = \sum_{i,j=1}^{n} \omega_{ij}(\iota(x_{\alpha_i}^{i*})\iota(x_{\alpha_j}^j)) \to \sum_{i,j=1}^{n} \omega_{ij}(T_i^* T_j),$$

so by Lemma 3.3, $(\omega_{ij})_{i,j=1}^n$ is positive in $M_n(\mathscr{M}_*)$. Hence Ω^{-1} is completely positive. $\square$

Chapter 4

Injective von Neumann algebras

We now, after more than 79 pages, hit upon the first of the two big concepts that the project is supposed to be about.

Definition 4.1. A C^*-algebra $\mathcal{A}$ is *injective* if the following holds: Given C^*-algebras $\mathcal{B}$ and $\mathcal{B}_1$ with $\mathcal{B} \subseteq \mathcal{B}_1$ and a completely positive map $\varphi\colon \mathcal{B} \to \mathcal{A}$, then there exists a completely positive map $\varphi_1 : \mathcal{B}_1 \to \mathcal{A}$ such that $\varphi_1|_{\mathcal{B}} = \varphi$, i.e. the following diagram commutes:

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "\iota"] \arrow[d, "\varphi"'] & \mathcal{B}_1 \arrow[dl, dotted, "\varphi_1"] \\
\mathcal{A} &
\end{tikzcd}$$

where ι denotes the inclusion.

Note that if one considers the category of C^*-algebras with completely positive maps as the morphisms, the above notion of injectivity is exactly the property that defines the homological algebra version of injectivity. We will not be using concepts from homological algebra to develop this particular concept, though.

Whenever one introduces a property for an object, it is a natural question to ask whether isomorphic objects has the property. Asking whether injectivity is not preserved by isomorphisms is essentially the same as asking whether Elvis is still alive. (I'm sorry, but he isn't.)

Proposition 4.1. *Let $\mathcal{A}$ and $\mathcal{C}$ be C^*-algebras for which there exists a *-isomorphism $\rho\colon \mathcal{A} \to \mathcal{C}$. If $\mathcal{A}$ is injective, then $\mathcal{C}$ is injective.*

Proof. Let $\mathcal{B}$ and $\mathcal{B}_1$ be C^*-algebras with $\mathcal{B} \subseteq \mathcal{B}_1$ and let $\varphi\colon \mathcal{B} \to \mathcal{C}$ be completely positive. Then $\rho^{-1} \circ \varphi\colon \mathcal{B} \to \mathcal{A}$ is completely positive by Proposition 3.11. Hence there exists a completely positive map $\pi\colon \mathcal{B}_1 \to \mathcal{A}$ such that $\pi|_{\mathcal{B}} = \rho^{-1} \circ \varphi$, since $\mathcal{A}$ is injective. Then $\rho \circ \pi$ is a completely positive map by Proposition 3.11, such that $\rho \circ \pi|_{\mathcal{B}} = \varphi$. Therefore $\mathcal{C}$ is injective. □

4.1 Injectivity and projections of norm one

A von Neumann algebra is injective if it is injective as a C^*-algebra – so no different notion exists for von Neumann algebras and no confusion can occur. The main aim of this section is to find a criterion equivalent to injectivity for von Neumann algebras, but to get there, we will need to swing by a downright shocking result, namely Tomiyama's theorem (Theorem 4.5) that does something so big with so little to the degree that I couldn't believe it at first.

We will rely on theory not covered in the project to prove this next theorem. It should be noted that there are other ways to prove it; [4] does it by means of completely positive maps and the so-called *point-ultraweak topology.*

Proposition 4.2. *The C^*-algebra $B(\mathcal{H})$ is injective for any Hilbert space $\mathcal{H}$.*

Proof. Let $\mathcal{B}$ and $\mathcal{B}_1$ be C^*-algebras such that $\mathcal{B} \subseteq \mathcal{B}_1$ and let $\varphi\colon \mathcal{B} \to B(\mathcal{H})$ be a completely positive map. From Stinespring's representation theorem (Theorem 3.16), we obtain a Hilbert space $\mathcal{K}_0$ along with a *-representation $\pi_0 : \mathcal{B} \to B(\mathcal{K}_0)$ and a bounded linear operator $V : \mathcal{H} \to \mathcal{K}_0$ such that

$$\varphi(b) = V^*\pi_0(b)V, \quad b \in \mathcal{B}.$$

[9, Proposition 2.10.2] then yields the existence of a Hilbert space $\mathcal{K}$ together with an isometric imbedding $W\colon \mathcal{K}_0 \to \mathcal{K}$ and a *-representation $\pi\colon \mathcal{B}_1 \to B(\mathcal{K})$ such that

$$\pi_0(b) = W^*\pi(b)W, \quad b \in \mathcal{B}.$$

Now define $\varphi_1\colon \mathcal{B}_1 \to B(\mathcal{H})$ by

$$\varphi_1(b) = V^*W^*\pi(b)WV, \quad b \in \mathcal{B}_1.$$

We claim that φ_1 is the wanted completely positive extension of φ to $\mathcal{B}_1$. Clearly φ_1 is linear and $\varphi_1(b) = V^*(W^*\pi(b)W)V = V^*\pi_0(b)V = \varphi(b)$ for all $b \in \mathcal{B}$. Additionally, for $n \geq 1$, a positive matrix $x = (x_{ij}) \in M_n(\mathcal{B}_1)$ and $\xi = (\xi_1, \ldots, \xi_n) \in \mathcal{H}^n$, we have

$$\langle \varphi_1^{(n)}(x)\xi, \xi\rangle_{\mathcal{H}^n} = \sum_{i,j=1}^{n} \langle \varphi_1(x_{ij})\xi_j, \xi_i\rangle_{\mathcal{H}} = \sum_{i,j=1}^{n} \langle \pi(x_{ij})WV\xi_j, WV\xi_i\rangle_{\mathcal{K}} = \langle \pi^{(n)}(x)y, y\rangle_{\mathcal{K}^n},$$

where $y = (WV\xi_1, \ldots, WV\xi_n) \in \mathcal{K}^n$. Since π is a *-homomorphism, it follows that $\varphi_1^{(n)}(x)$ is positive, completing the proof. □

The next definition will make life a lot easier from now on.

Definition 4.2. Let $\mathcal{A}$ and $\mathcal{B}$ be vector spaces with $\mathcal{B} \subseteq \mathcal{A}$ and $E\colon \mathcal{A} \to \mathcal{B}$ a surjective linear map. E is called a *projection* if $E(b) = b$ for all $b \in \mathcal{B}$, or equivalently $E \circ E = E$.

Recall that the image of a unital *-homomorphism is closed [31, Theorem 11.1].

Corollary 4.3. *Let $\mathcal{A}$ be a unital C^*-algebra, and let $\pi\colon \mathcal{A} \to B(\mathcal{H})$ be a unital representation of $\mathcal{A}$ on some Hilbert space $\mathcal{H}$. Then the following are equivalent:*

(i) *$\pi(\mathcal{A})$ is injective.*
(ii) *There is a completely positive projection $E\colon B(\mathcal{H}) \to \pi(\mathcal{A})$.*

Proof. For (i) ⇒ (ii), assume that $\pi(\mathcal{A})$ is injective. The identity map on $\pi(\mathcal{A})$ is a *-homomorphism, so it is completely positive and can therefore be extended by injectivity to a completely positive map $E\colon B(\mathcal{H}) \to \pi(\mathcal{A})$ which must also be a projection.

To prove (ii) ⇒ (i), assume that $E\colon B(\mathcal{H}) \to \pi(\mathcal{A})$ is a completely positive projection onto $\pi(\mathcal{A})$. Let $\mathcal{B}$ and $\mathcal{B}_1$ be C^*-algebras with $\mathcal{B} \subseteq \mathcal{B}_1$ and let $\varphi\colon \mathcal{B} \to \pi(\mathcal{A})$ be a completely positive map. Since $\pi(\mathcal{A}) \subseteq B(\mathcal{H})$, φ is also a completely positive map $\mathcal{B} \to B(\mathcal{H})$. Because $B(\mathcal{H})$ is injective we now obtain a completely positive map $\varphi_1\colon \mathcal{B}_1 \to B(\mathcal{H})$ such that $\varphi_1|_{\mathcal{B}} = \varphi$. The map $E \circ \varphi_1\colon \mathcal{B}_1 \to \pi(\mathcal{A})$ is now linear and completely positive. Moreover, for $b \in \mathcal{B}$ we have $E(\varphi_1(b)) = E(\varphi(b)) = \varphi(b)$ since φ maps into $\pi(\mathcal{A})$, so $E \circ \varphi_1$ also extends φ, and hence $\pi(\mathcal{A})$ is injective. □

Next up before the theorem of the day is the notion of a conditional expectation. I am not sure whether it has something to do with probability theory, as the name suggests something of the kind.

Definition 4.3. Let $\mathcal{B}$ and $\mathcal{A}$ be C^*-algebras such that $\mathcal{B} \subseteq \mathcal{A}$. A *conditional expectation* is a contractive and completely positive projection $E\colon \mathcal{A} \to \mathcal{B}$ satisfying

$$E(bxb') = bE(x)b', \quad x \in \mathcal{A},\ b, b' \in \mathcal{B},$$

i.e. E is a $\mathcal{B}$-bimodule map.

Lemma 4.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras with $\mathcal{B} \subseteq \mathcal{A}$. Moreover, let $j\colon \mathcal{B} \to \mathcal{A}$ be the inclusion and let $E\colon \mathcal{A} \to \mathcal{B}$ be a projection, so that $E \circ j = \mathrm{id}_{\mathcal{B}}$ where $\mathrm{id}_{\mathcal{B}}$ denotes the identity map on $\mathcal{B}$. Then $j^{**}\colon \mathcal{B}^{**} \to \mathcal{A}^{**}$ is an injective *-homomorphism, $j^{**}(\mathcal{B}^{**})$ is a unital and ultraweakly closed *-subalgebra of $\mathcal{A}^{**}$, and $j^{**} \circ E^{**}$ is a projection of $\mathcal{A}^{**}$ onto $j^{**}(\mathcal{B}^{**})$.*

Proof. The map j is a homomorphism, so j^{**} is a *-homomorphism by Proposition 3.13 and $j^{**}(\mathcal{B}^{**})$ is a unital C^*-subalgebra of $\mathcal{A}^{**}$. Note that $\mathcal{A}^{**}$ and $j^{**}(\mathcal{B}^{**})$ may not share the same unit. Moreover, since $(\mathcal{B}^{**})_r$ is ultraweakly compact for all $r > 0$ by Corollary 2.9, it follows from Corollary 2.61 that $(j^{**}(\mathcal{B}^{**}))_r$ is ultraweakly compact and hence ultraweakly closed for all $r > 0$, so $j^{**}(\mathcal{B}^{**})$ is ultraweakly closed by Theorem 2.11. Since $E^{**} \circ j^{**} = \mathrm{id}_{\mathcal{B}^{**}}$ by Proposition 2.64(iii), j^{**} is injective and $j^{**} \circ E^{**}$ is a projection of $\mathcal{A}^{**}$ onto $j^{**}(\mathcal{B}^{**}) \subseteq \mathcal{A}^{**}$. □

As so it comes: the godsend.

Theorem 4.5 (Tomiyama, 1957). *Let $\mathcal{B}$ and $\mathcal{A}$ be C^*-algebras such that $\mathcal{B} \subseteq \mathcal{A}$, and let $E\colon \mathcal{A} \to \mathcal{B}$ be a projection of $\mathcal{A}$ onto $\mathcal{B}$. Then the following are equivalent:*

(i) *E is a conditional expectation.*
(ii) *E is contractive and completely positive.*
(iii) *E is contractive.*

Proof. It is obvious that (i) ⇒ (ii) ⇒ (iii). Therefore we "only" have to prove that (iii) ⇒ (i). Assume that E is contractive. By Proposition 3.13, the second adjoint map $E^{**}\colon \mathcal{A}^{**} \to \mathcal{B}^{**}$ is contractive, and as the inclusion $j\colon \mathcal{B} \to \mathcal{A}$ is contractive, it follows that $j^{**} \circ E^{**}$ is contractive as well. By Lemma 4.4, $j^{**} \circ E^{**}$ is a projection of $\mathscr{M} = \mathcal{A}^{**}$ onto the unital ultraweakly closed *-subalgebra $\mathscr{N} = j^{**}(\mathcal{B}^{**})$. Assuming that we have proved that $j^{**} \circ E^{**}$ is a conditional expectation, then if $\iota_{\mathcal{A}}\colon \mathcal{A} \to \mathcal{A}^{**}$ and $\iota_{\mathcal{B}}\colon \mathcal{B} \to \mathcal{B}^{**}$ denotes the canonical inclusions into the double duals, we have $\iota_{\mathcal{A}} \circ j = j^{**} \circ \iota_{\mathcal{B}}$ and $\iota_{\mathcal{B}} \circ E = E^{**} \circ \iota_{\mathcal{A}}$, and therefore for all $x \in \mathcal{A}$ and $b, b' \in \mathcal{B}$

$$\begin{aligned} j^{**} \circ \iota_{\mathcal{B}}(E(bxb')) &= j^{**} \circ E^{**}(\iota_{\mathcal{A}}(j(b)xj(b'))) \\ &= j^{**} \circ E^{**}(\iota_{\mathcal{A}}(j(b))\iota_{\mathcal{A}}(x)\iota_{\mathcal{A}}(j(b'))) \\ &= \iota_{\mathcal{A}}(j(b))\,[j^{**} \circ E^{**}(\iota_{\mathcal{A}}(x))]\,\iota_{\mathcal{A}}(j(b')) \\ &= j^{**}\,[\iota_{\mathcal{B}}(b)\iota_{\mathcal{B}}(E(x))\iota_{\mathcal{B}}(b')] \\ &= j^{**}(\iota_{\mathcal{B}}(bE(x)b')), \end{aligned}$$

so since j^{**} and $\iota_{\mathcal{B}}$ are injective, it follows that $E(bxb') = bE(x)b'$. Moreover, as $\iota_{\mathcal{A}}$ and the inverses of $j^{**}\colon \mathcal{B}^{**} \to \mathscr{N}$ and $\iota_{\mathcal{B}}\colon \mathcal{B} \to \iota_{\mathcal{B}}(\mathcal{B})$ are *-homomorphisms, it follows that

$$E = \iota_{\mathcal{B}}^{-1} \circ (j^{**})^{-1} \circ j^{**} \circ (\iota_{\mathcal{B}} \circ E) = (\iota_{\mathcal{B}}^{-1} \circ j^{**})^{-1} \circ (j^{**} \circ E^{**}) \circ \iota_{\mathcal{A}}$$

is completely positive. Hence to prove (iii) ⇒ (i), it suffices to prove the implication for a projection $E\colon \mathscr{M} \to \mathscr{N}$ where $\mathscr{M}$ is a von Neumann algebra and $\mathscr{N}$ is an ultraweakly closed unital *-subalgebra of $\mathscr{N}$ (not necessarily sharing the same unit). Note that in the above case, $\mathcal{B}^{**}$ is the norm closure of the linear span of its projections, so since j^{**} is a *-homomorphism and hence also contractive, the same holds for $\mathscr{N}$. We can therefore also assume that the linear span of the projections in $\mathscr{N}$ is norm-dense in $\mathscr{N}$.

Assume now that $E\colon \mathscr{M} \to \mathscr{N}$ is a contractive projection. To prove that E is an $\mathscr{N}$-bimodule map, it suffices to check that $E(pxp') = pE(x)p'$ for $x \in \mathscr{M}$ and projections $p, p' \in \mathscr{N}$, by the aforementioned assumption that the linear span of the projections of $\mathscr{N}$ is norm-dense in $\mathscr{N}$ and the fact that E is contractive. Fix $x \in \mathscr{M}$. For any projection $p \in \mathcal{B}$, let $p^\perp = 1_{\mathscr{M}} - p$. Then we must have

$$pE(p^\perp x) = E(pE(p^\perp x)).$$

For any $t \in \mathbb{R}$, then if $y = p^\perp x + tpE(p^\perp x)$ we see that

$$y^*y = (x^*p^\perp x + tE(p^\perp x)^*p)(p^\perp x + tpE(p^\perp x)) = x^*p^\perp x + t^2E(p^\perp x)^*pE(p^\perp x).$$

Hence

$$\begin{aligned} \|y\|^2 = \|y^*y\| &= \|x^*p^\perp x + t^2E(p^\perp x)^*pE(p^\perp x)\| \\ &\le \|x^*p^\perp x\| + \|t^2E(p^\perp x)^*pE(p^\perp x)\| \\ &= \|p^\perp x\|^2 + t^2\|pE(p^\perp x)\|^2, \end{aligned}$$

so

$$\begin{aligned} (1+t)^2\|pE(p^\perp x)\|^2 &= \|pE(p^\perp x) + tp(pE(p^\perp x))\|^2 \\ &= \|pE(p^\perp x + tpE(p^\perp x))\|^2 \\ &\le \|y\|^2 \\ &\le \|p^\perp x\|^2 + t^2\|pE(p^\perp x)\|^2. \end{aligned}$$

Hence $(2t+1)\|pE(p^\perp x)\|^2 \leq \|p^\perp x\|^2$ for all $t \in \mathbb{R}$, implying $pE(p^\perp x) = 0$. Letting $1_{\mathscr{M}}$ and $1_{\mathscr{N}}$ denote the units of $\mathscr{M}$ and $\mathscr{N}$ respectively, then for $p = 1_{\mathscr{N}}$, we see that $E(1_{\mathscr{N}}^\perp x) = 1_{\mathscr{N}} E(1_{\mathscr{N}}^\perp x) = 0$. For any projection $p \in \mathscr{N}$, then because

$$(1_{\mathscr{N}} - p)^\perp = 1_{\mathscr{M}} - 1_{\mathscr{N}} + p = 1_{\mathscr{N}}^\perp + p,$$

we also see that

$$0 = (1_{\mathscr{N}} - p)E((1_{\mathscr{N}} - p)^\perp x) = (1_{\mathscr{N}} - p)E((1_{\mathscr{N}}^\perp + p)x) = (1_{\mathscr{N}} - p)E(px),$$

so $E(px) = pE(px)$. This finally implies

$$E(px) = pE(px) = pE(x - p^\perp x) = pE(x) - pE(p^\perp x) = pE(x).$$

Similarly one shows for any projection $p \in \mathscr{N}$ and $x \in \mathscr{M}$ that $E(xp^\perp)p = 0$, implying $E(x1_{\mathscr{N}}^\perp) = 0$, $E(xp) = E(xp)p$ and finally $E(xp) = E(x)p$. Hence

$$E(pxp') = pE(xp') = pE(x)p'$$

for all $x \in \mathscr{M}$ and projections $p, p' \in \mathscr{N}$, so E is an $\mathscr{N}$-bimodule map.

It only remains to show that E is completely positive. Since $yE(1_{\mathcal{A}}) = E(1_{\mathcal{A}})y = E(y) = y$ for all $y \in \mathscr{N}$, it follows that $E(1_{\mathcal{A}}) = 1_{\mathcal{B}}$. Therefore E is a unital contraction and hence positive by Proposition 3.10. Let $n \geq 1$. If $x \in M_n(\mathscr{M})$ is positive and $b = (b_1, \ldots, b_n) \in M_{n,1}(\mathscr{N})$, then

$$b^* E^{(n)}(x) b = \sum_{i,j=1}^n b_i^* E(x_{ij}) b_j = \sum_{i,j=1}^n E(b_i^* x_{ij} b_j) = E\left(\sum_{i,j=1}^n b_i^* x_{ij} b_j\right) \geq 0$$

by Lemma 3.2. Hence by the same lemma, $E^{(n)}(x)$ is positive, so E is completely positive. This concludes the proof. $\square$

I might note that the above proof of Tomiyama's theorem is the most precise and thorough one that I can think of; it combines the proof contained in [4] with the humility that the theorem deserves. (In other words, if we want it to be true we'd better make sure that the proof is correct.) Nonetheless, it yields the following important corollary.

Corollary 4.6. *Let $\mathcal{H}$ be a Hilbert space and let $\mathscr{M} \subseteq B(\mathcal{H})$ be a von Neumann algebra. Then the following are equivalent:*

(i) *$\mathscr{M}$ is injective.*

(ii) *There is a projection $E\colon B(\mathcal{H}) \to \mathscr{M}$ of $B(\mathcal{H})$ onto $\mathscr{M}$ with $\|E\| = 1$.*

Proof. Let $\pi\colon \mathscr{M} \to B(\mathcal{H})$ denote the inclusion. Assuming (i), then Corollary 4.3 used with π yields that there is a completely positive projection E of $B(\mathcal{H})$ onto $\pi(\mathscr{M}) = \mathscr{M}$. By Proposition 3.9, $\|E\| = \|E(1_{\mathcal{H}})\| = \|1_{\mathcal{H}}\| = 1$. If (ii) holds, then E is completely positive by Tomiyama's theorem (Theorem 4.5), so by Corollary 4.3 yields that $\pi(\mathscr{M}) = \mathscr{M}$ is injective and hence we obtain (i). $\square$

The above result does not rely of $\mathscr{M}$ being a von Neumann algebra – any C^*-subalgebra of $B(\mathcal{H})$ will do just fine. As we shall almost exclusively be working with von Neumann algebras from now on, there is really no reason to deal with this generality and *much* less with the original definition of injectivity. From now on *whenever* we state or assume that a von Neumann algebra $\mathscr{M} \subseteq B(\mathcal{H})$ is injective, we assume that

$\mathscr{M}$ satisfies condition (ii) of Corollary 4.6.

Having done all necessary preliminary work, let us construct some injective von Neumann algebras!

4.2 The construction of injective von Neumann algebras

There is no point in introducing all the upcoming results one at a time, so suffice to say that we will investigate whether injectivity is preserved for some of the most typical von Neumann algebra constructions.

Proposition 4.7. *Let $(\mathscr{M}_i)_{i\in I}$ be a family of von Neumann algebras with $\mathscr{M}_i \subseteq B(\mathcal{H}_i)$ for Hilbert spaces $(\mathcal{H}_i)_{i\in I}$. Then $\mathscr{M} = \bigoplus_{i\in I} \mathscr{M}_i$ is injective if and only if $\mathscr{M}_i$ is injective for all $i \in I$.*

Proof. Putting $\mathcal{H} = \bigoplus_{i\in I} \mathcal{H}_i$, and using the natural injections $\iota_i \colon \mathcal{H}_i \to \mathcal{H}$ and projections $\pi_i \colon \mathcal{H} \to \mathcal{H}_i$ defined after Proposition 0.4, then $P_i = \iota_i\iota_i^*$ is the projection onto the closed subspace $\iota_i(\mathcal{H}_i)$ of $\mathcal{H}$ for all $i \in I$. Clearly $\iota_i B(\mathcal{H}_i)\pi_i \subseteq B(\mathcal{H})$ and $\iota_i\mathscr{M}_i\pi_i \subseteq \mathscr{M}$ and we shall identify $\mathscr{M}_i$ with $\pi_i\mathscr{M}\iota_i$ (see page xi).

Assume first that $\mathscr{M}$ is injective; then there exists a projection $E \colon B(\mathcal{H}) \to \mathscr{M}$ of norm 1. For any $i \in I$, define $E_i \colon B(\mathcal{H}_i) \to \mathscr{M}_i$ by $E_i(T) = \pi_i E(\iota_i T\pi_i)\iota_i$ for $T \in B(\mathcal{H}_i)$. Then $E_i(T) \in \mathscr{M}$ for all $T \in B(\mathcal{H})$, $\|E_i(T)\| \le \|E(\iota_i T\pi_i)\| \le \|T\|$ and for $T \in \mathscr{M}$, then $\iota_i T\pi_i \in \mathscr{M}$, so $E_i(T) = \pi_i\iota_i T\pi_i\iota_i = T$. Therefore each E_i is a projection of norm 1 of $B(\mathcal{H}_i)$ onto $\mathscr{M}_i$.

If $\mathscr{M}_i$ is injective for all $i \in I$, we have projections $E_i \colon B(\mathcal{H}_i) \to \mathscr{M}_i$ of norm 1 for all $i \in I$. Define $E \colon B(\mathcal{H}) \to \mathscr{M}$ by

$$E(T) = (E_i(\pi_i T\iota_i))_{i\in I}, \quad T \in B(\mathcal{H}).$$

To see that E is well-defined, note that for all $i \in I$ and all $T \in B(\mathcal{H})$, then $\pi_i T\iota_i \in B(\mathcal{H}_i)$, so $E_i(\pi_i T\iota_i) \in \mathscr{M}_i$. Moreover, as $E_i(\pi_i T\iota_i)$ is bounded by $\|T\|$, it follows that $E(T)$ is a well-defined bounded operator in $\mathscr{M}$. It follows immediately that $\|E\| \le 1$ and if $T \in \mathscr{M}$, then $T = (T_i)_{i\in I}$ where $T_i \in \mathscr{M}_i$ for all $i \in I$; therefore since $\pi_i T\iota_i = T_i$ it follows that $E_i(\pi_i T\iota_i) = E_i(T_i) = T_i$, so $E(T) = T$. Therefore E is a projection of norm 1. $\square$

Proposition 4.8. *Let $\mathscr{M} \subseteq B(\mathcal{H})$ be an injective von Neumann algebra and let $P \in \mathscr{M}$ be a projection. Then $\mathscr{M}_P$ is injective.*

Proof. Recall that $\mathscr{M}_P$ consists of all operators in $B(P(\mathcal{H}))$ of the form $PT|_{P(\mathcal{H})}$ for $T \in \mathscr{M}$. Let $E \colon B(\mathcal{H}) \to \mathscr{M}$ be a projection of norm 1 and define $E' \colon B(P(\mathcal{H})) \to \mathscr{M}_P$ by

$$E'(T) = PE(PTP)|_{P(\mathcal{H})},$$

where PTP is seen as an operator on $\mathcal{H}$. First of all, for all $T \in B(P(\mathcal{H}))$, we have $E'(T) \in \mathscr{M}_P$ by the definition of E and $\mathscr{M}_P$. Clearly $\|E'\| \le 1$, as we have $\|PTP\xi\| \le \|T\|\|P\xi\|$ and thus

$$\|E(PTP)\| \le \|PTP\| \le \|T\|, \quad T \in B(P(\mathcal{H})).$$

Finally, if $T \in \mathscr{M}_P$, then $T = PS|_{P(\mathcal{H})}$ for some $S \in \mathscr{M}$, so $PTP\xi = PSP\xi$ for all $\xi \in \mathcal{H}$. Hence $PTP = PSP \in \mathscr{M}$, so

$$E'(T) = PE(PTP)|_{P(\mathcal{H})} = PE(PSP)|_{P(\mathcal{H})} = PS|_{P(\mathcal{H})} = T$$

since E is a projection onto $\mathscr{M}$. Therefore E' is a projection of norm 1, so $\mathscr{M}_P$ is injective. $\square$

Corollary 4.9. *Let $\mathscr{M}$ and $\mathscr{N}$ be von Neumann algebras and let $\varphi \colon \mathscr{M} \to \mathscr{N}$ be a normal surjective *-homomorphism. Then $\mathscr{M}$ is injective if and only if* $\ker\varphi$ *and $\mathscr{N}$ are injective.*

Proof. This follows from Propositions 2.53, 4.7 and 4.1. $\square$

Corollary 4.10. *Let $\mathscr{M}$ be a von Neumann algebra. Then*

(i) *If $\pi \colon \mathscr{M} \to B(\mathcal{H})$ is a normal representation and $\mathscr{M}$ is injective, then $\pi(\mathscr{M})$ is injective.*

(ii) *If $(\pi_\alpha)_{\alpha\in A}$ is a separating family of normal representations of $\mathscr{M}$ such that $\pi_\alpha(\mathscr{M})$ is injective for all $\alpha \in A$, then $\mathscr{M}$ is injective.*

Proof. (i) is a consequence of Corollary 4.9. For (ii), let $(P_\beta)_{\beta\in B}$ be a maximal family of orthogonal central projections in $\mathscr{M}$ such that there for all $\beta\in B$ exists an $\alpha(\beta)\in A$ such that $\pi_{\alpha(\beta)}$ is an injective map on $\mathscr{M}P_\beta$. Define $P=\sum_{\beta\in B}P_\beta$. Assuming for contradiction that $P\neq 1_{\mathscr{M}}$, there exists a non-zero $T\in\mathscr{M}$ such that $TP=0$. Because $(\pi_\alpha)_{\alpha\in A}$ was separating, there exists $\alpha\in A$ such that $\pi_\alpha(T)\neq 0$. Note now that there exists a central projection $P'\in\mathscr{M}$ such that

$$\ker\pi_\alpha=\mathscr{M}(1_{\mathscr{M}}-P')$$

by Proposition 2.32. Then $T(1_{\mathscr{M}}-P)P'=TP'$. Assuming that $TP'=0$ yields

$$\pi_\alpha(T)=\pi_\alpha(TP')+\pi_\alpha(T(1_{\mathscr{M}}-P'))=0,$$

a contradiction. Hence $T(1_{\mathscr{M}}-P)P'\neq 0$ so $P''=(1_{\mathscr{M}}-P)P'$ is a non-zero central projection. Moreover, π_α is injective on $\mathscr{M}P''$; indeed, if $S\in\mathscr{M}P''$ satisfies $\pi_\alpha(S)=0$, then $S=T(1_{\mathscr{M}}-P')$ for some $T\in\mathscr{M}$, but then

$$S=S(1_{\mathscr{M}}-P)P'=T(1_{\mathscr{M}}-P')(1_{\mathscr{M}}-P)P'=0.$$

Lastly, as $P''P_\beta=0$ for all $\beta\in B$, we obtain a contradiction of the maximality of the family $(P_\beta)_{\beta\in B}$, so $P=1_{\mathscr{M}}$, and hence

$$\mathscr{M}\cong\bigoplus_{\beta\in B}\mathscr{M}_{P_\beta}$$

by Proposition 2.19. Since $\mathscr{M}P_\beta\cong\pi_{\alpha(\beta)}(\mathscr{M}P_\beta)=\pi_{\alpha(\beta)}(\mathscr{M})\pi_{\alpha(\beta)}(P_\beta)$ by injectivity of each $\pi_{\alpha(\beta)}$ and $\pi_{\alpha(\beta)}(e_\beta)$ is central in the injective von Neumann algebra $\pi_{\alpha(\beta)}(\mathscr{M})$, it follows that all $\mathscr{M}_{P_\beta}$ are injective from Propositions 2.17 and 4.8. Hence $\mathscr{M}$ is injective by Proposition 4.7. □

For our next result we will prepare ourselves by defining some helpful maps. Let $\mathscr{M}$ and $\mathscr{N}$ be von Neumann algebras. For any $\omega\in\mathscr{M}_*$ and $\varphi\in\mathscr{N}_*$, $\omega\otimes\varphi$ denotes the ultraweakly continous linear functional on $\mathscr{M}\mathbin{\overline{\otimes}}\mathscr{N}$ defined on page 76 that uniquely satisfies

$$(\omega\otimes\varphi)(S\otimes S')=\omega(S)\varphi(S'),\quad S\in\mathscr{M},S'\in\mathscr{N}.$$

For fixed $\omega\in\mathscr{M}_*$, $T\in\mathscr{M}\mathbin{\overline{\otimes}}\mathscr{N}$, let $f_{\omega,T}\colon\mathscr{N}_*\to\mathbb{C}$ be given by

$$f_{\omega,T}(\varphi)=(\omega\otimes\varphi)(T).$$

As $\|f_{\omega,T}(\varphi)\|\leq\|\omega\otimes\varphi\|\|T\|=\|\omega\|\|\varphi\|\|T\|$, each $f_{\omega,T}$ is a bounded linear functional on $\mathscr{N}_*$ and by Theorem 2.7 hence corresponds to a unique element $R_\omega(T)$ of $\mathscr{N}$. Likewise, one obtains a map $\mathscr{M}\mathbin{\overline{\otimes}}\mathscr{N}\to\mathscr{M}$, $T\mapsto L_\varphi(T)$ for each $\varphi\in\mathscr{M}_*$.

Lemma 4.11. *$R_\omega\colon\mathscr{M}\mathbin{\overline{\otimes}}\mathscr{N}\to\mathscr{N}$ and $L_\varphi\colon\mathscr{M}\mathbin{\overline{\otimes}}\mathscr{N}\to\mathscr{M}$ as defined above are bounded linear mappings with $\|R_\omega\|=\|\omega\|$ and $\|L_\varphi\|=\|\varphi\|$. Moreover, they satisfy*

$$R_\omega\left(\sum_{i=1}^n S_i\otimes T_i\right)=\sum_{i=1}^n\omega(S_i)T_i,\quad L_\varphi\left(\sum_{i=1}^n S_i\otimes T_i\right)=\sum_{i=1}^n\varphi(T_i)S_i,\quad S_i\in\mathscr{M},\ T_i\in\mathscr{N}.$$

Proof. R_ω is clearly a linear map; indeed, if $\lambda,\mu\in\mathbb{C}$ and $S,T\in\mathscr{M}\mathbin{\overline{\otimes}}\mathscr{N}$, then

$$f_{\omega,\lambda S+\mu T}(\varphi)=(\omega\otimes\varphi)(\lambda S+\mu T)=\lambda(\omega\otimes\varphi)(S)+\mu(\omega\otimes\varphi)(T)=(\lambda f_{\omega,S}+\mu f_{\omega,T})(\varphi)$$

for all $\varphi\in\mathscr{N}_*$, so by uniqueness and linearity of the canonical identification $\Lambda\colon\mathscr{N}\to(\mathscr{N}_*)^*$ (again Theorem 2.7) we have

$$R_\omega(\lambda S+\mu T)=\Lambda^{-1}(\lambda f_{\omega,S}+\mu f_{\omega,T})=\lambda R_\omega(S)+\mu R_\omega(T).$$

For all $\varphi\in\mathscr{N}_*$ and $T\in\mathscr{M}\mathbin{\overline{\otimes}}\mathscr{N}$, we have

$$\varphi(R_\omega(T))=\varphi(\Lambda^{-1}(f_{\omega,T}))=\Lambda(\Lambda^{-1}(f_{\omega,T}))(\varphi)=f_{\omega,T}(\varphi)=(\omega\otimes\varphi)(T),$$

so for $\varphi\in\mathscr{N}_*$ we have

$$\varphi\left(R_\omega\left(\sum_{i=1}^n S_i\otimes T_i\right)\right)=(\omega\times\varphi)\left(\sum_{i=1}^n S_i\otimes T_i\right)=\sum_{i=1}^n\omega(S_i)\varphi(T_i)=\varphi\left(\sum_{i=1}^n\omega(S_i)T_i\right).$$

Since all ultraweakly continuous linear functionals on $\mathscr{N}$ agree on the two vectors, they must be equal; indeed, if $\mathscr{N} \subseteq B(\mathcal{K})$ and it holds for two operators $S, T \in \mathscr{N}$ that $\varphi(S) = \varphi(T)$ for all $\varphi \in \mathscr{N}_*$, then in particular Proposition 2.2 yields that for all vectors $\xi, \eta \in \mathcal{K}$ we have $\langle S\xi, \eta\rangle = \langle T\xi, \eta\rangle$ and hence $S = T$.

Using Lemma 1.24, one easily sees that

$$\|S\| = \sup\{|\varphi(S)| \mid \varphi \in \mathscr{N}_*,\ \|\varphi\| \leq 1\}, \quad S \in \mathscr{N}.$$

Therefore for all $T \in \mathscr{M} \overline{\otimes} \mathscr{N}$ and $\varphi \in \mathscr{N}_*$ we see that $|\varphi(R_\omega(T))| = |(\omega \otimes \varphi)(T)| \leq \|\omega\|\|\varphi\|\|T\|$, implying $\|R_\omega(T)\| \leq \|\omega\|\|T\|$ and thus $\|R_\omega\| \leq \|\omega\|$. For the converse inequality, note that for all $S \in \mathscr{M}$ we have

$$|\omega(S)| = \|\omega(S)1_{\mathscr{N}}\| = \|R_\omega(S \otimes 1_{\mathscr{N}})\| \leq \|R_\omega\|\|S \otimes 1_{\mathscr{N}}\| = \|R_\omega\|\|S\|,$$

implying $\|\omega\| \leq \|R_\omega\|$ and hence equality. The results for L_φ follow similarly. □

As one might expect, we will now investigate a statement about injectivity for the von Neumann algebra tensor product.

Proposition 4.12. *Let $\mathscr{M} \subseteq B(\mathcal{H})$ and $\mathscr{N} \subseteq B(\mathcal{K})$ be von Neumann algebras. Then the von Neumann algebra $\mathscr{M} \overline{\otimes} \mathscr{N} \subseteq B(\mathcal{H} \otimes \mathcal{K})$ is injective if and only if $\mathscr{M}$ and $\mathscr{N}$ are injective.*

Proof. Assume first that $\mathscr{M} \overline{\otimes} \mathscr{N}$ is injective and let $E \colon B(\mathcal{H} \otimes \mathcal{K}) \to \mathscr{M} \overline{\otimes} \mathscr{N}$ be a projection of norm 1. Let $\omega \in \mathscr{M}_*$ be an ultraweakly continuous state, and let $\theta \colon B(\mathcal{K}) \to \mathbb{C}1_{\mathcal{H}} \otimes B(\mathcal{K})$ be the amplification. Define $E' \colon B(\mathcal{K}) \to \mathscr{N}$ by $E' = R_\omega \circ E \circ \theta$. Then for $T \in \mathscr{N}$, we have

$$E'(T) = R_\varphi(E(1_{\mathcal{H}} \otimes T)) = R_\varphi(1_{\mathcal{H}} \otimes T) = \varphi(1_{\mathcal{H}})T = T.$$

As R_ω, E and θ have norm 1, it follows that $\|E'\| \leq 1$, and as E' is isometric on $\mathscr{N}$, it follows that E' has norm 1. Therefore $\mathscr{N}$ is injective. A similar reasoning with an ultraweakly continuous state $\varphi \in \mathscr{N}_*$ proves that $\mathscr{M}$ is injective.

The converse statement is not as easy. Assume that $\mathscr{M}$ and $\mathscr{N}$ are injective and let $E_1 \colon B(\mathcal{H}) \to \mathscr{M}$ and $E_2 \colon B(\mathcal{K}) \to \mathscr{N}$ be projections of norm 1. Let $(P_i)_{i \in I}$ be minimal orthogonal projections in $B(\mathcal{K})$ corresponding to an orthonormal basis for $\mathcal{K}$ and for finite subsets J of I, define $P_J \in B(\mathcal{K})$ by $P_J = \sum_{i \in J} P_i$ and define $\tilde{P}_J \in B(\mathcal{H} \otimes \mathcal{K})$ by $\tilde{P}_J = 1_{\mathcal{H}} \otimes P_J$. As P_J is a projection, it follows that $\tilde{P}_J$ is a projection in $B(\mathcal{H} \otimes \mathcal{K})$ for all finite subsets J of I. The set $\mathfrak{J}$ of finite subsets of I is of course a directed set, ordered by inclusion, so $(P_J)_{J \in \mathfrak{J}}$ and $(\tilde{P}_J)_{J \in \mathfrak{J}}$ are nets. Since $P_J \to 1_{\mathcal{H}}$ strongly and thus ultraweakly by Proposition 2.1, it follows from Proposition 2.50 that $\tilde{P}_J \to 1_{\mathcal{H} \otimes \mathcal{K}}$ ultraweakly. Hence for all $S \in B(\mathcal{H}) \overline{\otimes} B(\mathcal{K})$, we have

$$\tilde{P}_J S \tilde{P}_J \to S$$

ultraweakly on $B(\mathcal{H}) \overline{\otimes} B(\mathcal{K})$. For any $J \in \mathfrak{J}$, Propositions 2.20, 2.17 and 1.34 yield that

$$(B(\mathcal{H}) \overline{\otimes} B(\mathcal{K}))_{\tilde{P}_J} = B(\mathcal{H}) \overline{\otimes} B(\mathcal{K})_{P_J} = B(\mathcal{H}) \overline{\otimes} B(P_J(\mathcal{K})) = B(\mathcal{H}) \odot B(P_J(\mathcal{K}))$$

because $P_J(\mathcal{K})$ is finite-dimensional.

For $J \in \mathfrak{J}$ define $E_J \colon B(\mathcal{H}) \odot B(P_J(\mathcal{K})) \to \mathscr{M} \odot B(P_J(\mathcal{K}))$ by $E_J = E_1 \odot 1_J$ where 1_J denotes the identity map $B(P_J(\mathcal{K})) \to B(P_J(\mathcal{K}))$. E_J is then clearly a projection of norm 1, and helps define a map $\tilde{E}_J \colon B(\mathcal{H}) \overline{\otimes} B(\mathcal{K}) \to \mathscr{M} \odot B(P_J(\mathcal{K}))$ by

$$\tilde{E}_J(T) = E_J(\tilde{P}_J T \tilde{P}_J), \quad T \in B(\mathcal{H}) \overline{\otimes} B(\mathcal{K}),$$

where $\tilde{P}_J T \tilde{P}_J$ is the operator $\tilde{P}_J T|_{\mathcal{H} \otimes P_J(\mathcal{K})}$. $\tilde{E}_J$ is then also a projection of norm 1.

For any $J \in \mathfrak{J}$, any operator $S \in \mathscr{M} \odot B(P_J(\mathcal{K}))$ is of the form $S = \sum_{i=1}^n S_i \otimes T_i$ for operators $S_1, \ldots, S_n \in \mathscr{M}$ and $T_1, \ldots, T_n \in B(P_J(\mathcal{K}))$. Because any T_i can be extended naturally to an operator on $\mathcal{K}$, just by defining it to be 0 on the orthogonal complement of $P_J(\mathcal{K})$, it follows that S can be seen as contained in $\mathscr{M} \odot B(\mathcal{K}) \subseteq \mathscr{M} \overline{\otimes} B(\mathcal{K})$. Since this operation is clearly contractive, then for any $T \in B(\mathcal{H}) \overline{\otimes} B(\mathcal{K})$ we will have that the net $(\tilde{E}_J(T))_{J \in \mathfrak{J}}$ is contained in the closed ball of $\mathscr{M} \overline{\otimes} B(\mathcal{K})$ of radius $\|T\|$. Since this ball is ultraweakly compact by Corollary 2.9, the net $(\tilde{E}_J(T))_{J\in\mathfrak{J}}$ has a cluster point which we will denote by $\tilde{E}_1(T)$. Note that the net can only have one cluster point: indeed, for $J, J' \in \mathfrak{J}$ with $J \subseteq J'$, we can naturally view $\tilde{P}_J$ as an operator in $\mathscr{M} \odot B(P_{J'}(\mathcal{K}))$, so

$$\begin{aligned}\tilde{P}_J\tilde{E}_{J'}(T)\tilde{P}_J &= \tilde{P}_J E_{J'}(\tilde{P}_{J'}T\tilde{P}_{J'})\tilde{P}_J\\ &= E_{J'}(\tilde{P}_J T\tilde{P}_J)\\ &= E_J(\tilde{P}_J T\tilde{P}_J)\\ &= \tilde{E}_J(T)\end{aligned}$$

where we used Tomiyama's theorem at the second equality. As there is a subnet $(\tilde{E}_S(T))_{S\in\mathscr{S}}$ of $(\tilde{E}_J(T))_{J\in\mathfrak{J}}$ converging ultraweakly to $\tilde{E}_1(T)$ by compactness of the aforementioned closed ball, we can derive another subnet $(\tilde{E}_S(T))_{S'\in\mathscr{S}'}$ $\mathscr{S}' \subseteq \mathscr{S}$ by defining $\mathscr{S}' = \{S \in \mathscr{S} \mid J \subseteq S\}$. This subnet also converges ultraweakly to $\tilde{E}_1(T)$. By considering the above equalities for $J' \in \mathscr{S}'$ we find that $\tilde{E}_J(T) = \tilde{P}_J\tilde{E}_1(T)\tilde{P}_J$, but we have also found that $\tilde{E}_J(T) = \tilde{P}_J\tilde{E}_1(T)\tilde{P}_J \to \tilde{E}_1(T)$ ultraweakly. Therefore $\tilde{E}_1(T)$ is uniquely determined by the elements $\tilde{E}_J(T)$ as the ultraweak topology is Hausdorff, and we thus obtain a well-defined map

$$\tilde{E}_1 \colon B(\mathcal{H}) \mathbin{\overline{\otimes}} B(\mathcal{K}) \to \mathscr{M} \mathbin{\overline{\otimes}} B(\mathcal{K}).$$

$\tilde{E}_1$ is in fact linear, since for $\lambda_1, \lambda_2 \in \mathbb{C}$ and $T_1, T_2 \in B(\mathcal{H}) \mathbin{\overline{\otimes}} B(\mathcal{K})$ we have

$$\tilde{E}_1(\lambda_1 T_1 + \lambda_2 T_2) = \lim_{J\in\mathfrak{J}} \tilde{E}_J(\lambda_1 T_1 + \lambda_2 T_2) = \lambda_1 \lim_{J\in\mathfrak{J}} \tilde{E}_J(T_1) + \lambda_2 \lim_{J\in\mathfrak{J}} \tilde{E}_J(T_2) = \lambda_1\tilde{E}_1(T_1) + \lambda_2\tilde{E}_1(T_2),$$

where the limits are in the ultraweak topology. Since $\|\tilde{E}_1(T)\| \le \|T\|$ by construction, it follows that $\|\tilde{E}_1\| \le 1$. Moreover, for $T \in \mathscr{M} \mathbin{\overline{\otimes}} B(\mathcal{K})$, we have

$$\tilde{E}_J(T) = E_J(\tilde{P}_J T\tilde{P}_J) = \tilde{P}_J T\tilde{P}_J \to T$$

ultraweakly, so $\tilde{E}_1(T) = T$ and hence $\tilde{E}_1$ is a projection of norm 1 onto $\mathscr{M} \mathbin{\overline{\otimes}} B(\mathcal{K})$. Moreover,

$$\tilde{E}_J(S \otimes T) = E_J(\tilde{P}_J(S \otimes T)\tilde{P}_J) = E_J(S \otimes P_J T P_J) = E_1(S) \otimes P_J T P_J,$$

so $\tilde{E}_1(S \otimes T) = E_1(S) \otimes T$ for all $S \in \mathscr{M}$, $T \in \mathscr{N}$. Similarly we can define a projection

$$\tilde{E}_2 \colon B(\mathcal{H}) \mathbin{\overline{\otimes}} B(\mathcal{K}) \to B(\mathcal{H}) \mathbin{\overline{\otimes}} \mathscr{N}$$

of norm 1, satisfying $\tilde{E}_2(S \otimes T) = S \otimes E_2(T)$ for all $S \in \mathscr{M}$, $T \in \mathscr{N}$.

Now let $S \in \mathscr{M} \mathbin{\overline{\otimes}} B(\mathcal{K})$ and $T \in (\mathscr{M} \mathbin{\overline{\otimes}} B(\mathcal{K}))' = \mathscr{M}' \odot \mathbb{C}1_{\mathcal{K}} \subseteq B(\mathcal{H}) \mathbin{\overline{\otimes}} \mathscr{N}$. Then

$$\tilde{E}_2(S)T = \tilde{E}_2(ST) = \tilde{E}_2(TS) = T\tilde{E}_2(S)$$

by Tomiyama's theorem, so $\tilde{E}_2(S) \in (\mathscr{M} \mathbin{\overline{\otimes}} B(\mathcal{K}))'' = \mathscr{M} \mathbin{\overline{\otimes}} B(\mathcal{K})$. Therefore, for $S \in \mathscr{M} \mathbin{\overline{\otimes}} B(\mathcal{K})$, we have

$$\tilde{E}_2(S) \in \mathscr{M} \mathbin{\overline{\otimes}} B(\mathcal{K}) \cap B(\mathcal{H}) \mathbin{\overline{\otimes}} \mathscr{N} = ((\mathscr{M}' \otimes \mathbb{C}1_{\mathcal{K}}) \cup (\mathbb{C}1_{\mathcal{K}} \otimes \mathscr{M}'))' = (\mathscr{M}' \mathbin{\overline{\otimes}} \mathscr{N}')' = \mathscr{M} \mathbin{\overline{\otimes}} \mathscr{N},$$

using the so-called *commutation theorem for von Neumann algebra tensor products* [15, Theorem 11.2.16]. We can therefore define $E \colon B(\mathcal{H} \otimes \mathcal{K}) \to \mathscr{M} \mathbin{\overline{\otimes}} \mathscr{N}$ by $E = \tilde{E}_2 \circ \tilde{E}_1$. E is linear, and since $\tilde{E}_1$ and $\tilde{E}_2$ both have norm 1, $\|E\| \le 1$. Moreover, for $T \in \mathscr{M} \mathbin{\overline{\otimes}} \mathscr{N}$, note that $\tilde{E}_1(T) = T$ and $\tilde{E}_2(T) = T$, so $E(T) = T$, and therfore E is a projection of $B(\mathcal{H} \otimes \mathcal{K})$ onto $\mathscr{M} \mathbin{\overline{\otimes}} \mathscr{N}$ of norm 1. This proves that $\mathscr{M} \mathbin{\overline{\otimes}} \mathscr{N}$ is injective. □

What we have done now is made sure that certain types of "new" von Neumann algebras inherit injectivity from "old" ones, and we will in the next section make a digression that seems very much out of place at the moment. One reason that we will even consider looking into the next couple of von Neumann algebra concepts is that we can extract injectivity results somewhat similar to the ones proved in this section, but the main reason is because of the next chapter: some of the next two sections become absolutely indispensable when we will attempt to prove the main theorem. For the curious, Theorem 5.18 is the place to look...

4.3 Continuous crossed products

However, the fact that the results of this and the next section depend on group theory (of all things) is a bit surprising. This section will contain only one proof, a self-made one; the rest can be found in [8] (a book I can only recommend, as the writing style is very clear and the material is self-contained). For the rest of the project we will only need the definitions given herein, along with a minimal amount of the theorems.

The first two definitions are probably familiar to any C^*-algebraist; nonetheless we include them here for completion.

Definition 4.4. Let $\mathcal{A}$ be a unital C^*-algebra. An **-automorphism* of $\mathcal{A}$ is a unital *-isomorphism $\varphi\colon \mathcal{A} \to \mathcal{A}$. The group of *-automorphisms of $\mathcal{A}$ is denoted by $\mathrm{Aut}(\mathcal{A})$.

Definition 4.5. Let Γ be a group and let $\mathcal{H}$ be a Hilbert space. A *unitary representation* of Γ is a group homomorphism $\Gamma \to \mathcal{U}(\mathcal{H})$.

The main point of this section is to create a new von Neumann algebra from a given von Neumann algebra $\mathscr{M}$, a group Γ and a homomorphism $\Gamma \to \mathrm{Aut}(\mathscr{M})$. The following definition is where it all starts.

Definition 4.6. Let $\mathscr{M}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$ and let Γ be a locally compact group. A *continuous action of* Γ *on* $\mathscr{M}$ is a group homomorphism $\alpha\colon \Gamma \to \mathrm{Aut}(\mathscr{M})$ such that for any $T \in \mathscr{M}$, the map $s \mapsto \alpha_s(T) := \alpha(s)(T)$ is continuous if $\mathscr{M}$ is considered with the strong operator topology. The fact that α is a homomorphism is reflected in the equality

$$\alpha_s \circ \alpha_t = \alpha_{st}, \quad s, t \in \Gamma.$$

In this case $(\mathscr{M}, \Gamma, \alpha)$ is called a *covariant system*.

In the following, we will let Γ be a locally compact group, $\mathscr{M} \subseteq B(\mathcal{H})$ a von Neumann algebra and α a continuous action of Γ on $\mathscr{M}$. We will denote the space of continuous functions on Γ with compact support and values in $\mathcal{H}$ by $C_c(\Gamma, \mathcal{H})$. Hence a $f\colon \Gamma \to \mathcal{H}$ is contained in $C_c(\Gamma, \mathcal{H})$ if and only if it satisfies the following criteria:

(i) If $s_i \to s$ in Γ, then $f(s_i) \to f(s)$ in $\mathcal{H}$.
(ii) The set $\{s \in \Gamma \mid f(s) \neq 0\}$ has compact closure.

Letting $\mathrm{d}s$ be a fixed Haar measure on Γ, we equip $C_c(\Gamma, \mathcal{H})$ with an inner product given by

$$\langle f, g\rangle = \int_\Gamma \langle f(s), g(s)\rangle \mathrm{d}s, \quad f, g \in C_c(\Gamma, \mathcal{H})$$

and we let $L^2(\Gamma, \mathcal{H})$ denote the completion with respect to this inner product. One important fact about $L^2(\Gamma, \mathcal{H})$ is the following.

Proposition 4.13. *There is an isomorphism of Hilbert spaces $U\colon \mathcal{H} \otimes L^2(\Gamma) \to L^2(\Gamma, \mathcal{H})$ satisfying*

$$U(\xi \otimes f)(s) = f(s)\xi, \quad \xi \in \mathcal{H},\ f \in C_c(\Gamma), s \in \Gamma,$$

allowing us to identify $\mathcal{H} \otimes L^2(\Gamma)$ with $L^2(\Gamma, \mathcal{H})$.

Proof. See [8, Proposition I.2.2]. ☐

We now define two types of maps essential for constructing the continuous crossed product:

- For $T \in \mathscr{M}$ we define a bounded linear operator $\pi(T)$ on the Hilbert space $L^2(\Gamma, \mathcal{H})$ by

$$(\pi(T)f)(s) = \alpha_{s^{-1}}(T)f(s), \quad f \in C_c(\Gamma, \mathcal{H}),\ s \in \Gamma.$$

  $\pi\colon \mathscr{M} \to B(L^2(\Gamma, \mathcal{H}))$ is a faithful, normal representation of $\mathscr{M}$ [8, Proposition I.2.5].

❖ For $t \in \Gamma$, we can define a bounded linear operator $\lambda(s)$ on $L^2(\Gamma, \mathcal{H})$ by

$$(\lambda(t)f)(s) = f(t^{-1}s), \quad f \in C_c(\Gamma, \mathcal{H}), s \in \Gamma.$$

$\lambda\colon \Gamma \to B(L^2(\Gamma, \mathcal{H}))$ is then a strongly continuous unitary representation of Γ. [8, Proposition I.2.8]

An important property of these two map types is that they complement each other quite well, as seen in the next lemma.

Lemma 4.14. *For all $T \in \mathscr{M}$ and $t \in \Gamma$, we have*

$$\lambda(t)\pi(T)\lambda(t)^* = \pi(\alpha_t(T)).$$

This implies that the set of finite linear combinations of operators of the form $\pi(T)\lambda(t)$ with $T \in \mathscr{M}$ and $t \in \Gamma$ form a unital *-algebra.

Proof. See [8, Lemma I.2.9]. □

It follows that the double commutant of the above mentioned *-algebra is a von Neumann algebra – the one we want, in fact.

Definition 4.7. Using the above notation, the *(continuous) crossed product* $\mathrm{R}(\mathscr{M}, \alpha_t)$ of $\mathscr{M}$ by the continuous action α of Γ is the von Neumann algebra in $B(L^2(\Gamma, \mathcal{H}))$ generated by the set

$$\{\pi(T),\ \lambda(t) \,|\, T \in \mathscr{M},\ t \in \Gamma\},$$

i.e. $\mathrm{R}(\mathscr{M}, \alpha_t) = \{\pi(T),\ \lambda(t) \,|\, T \in \mathscr{M},\ t \in \Gamma\}''$.

Because of Lemma 4.14 and von Neumann's density theorem, $\mathrm{R}(\mathscr{M}, \alpha_t)$ is also the strong (or weak, ultraweak or ultrastrong) closure of the *-algebra of linear combinations of the operators $\pi(T)\lambda(t)$ for $T \in \mathscr{M}$ and $t \in \Gamma$.

Lemma 4.15. *Let $\mathscr{M} \subseteq B(\mathcal{H})$ be a von Neumann algebra and let Γ be an locally compact group acting continuously on $\mathscr{M}$ by the homomorphism $\theta\colon \Gamma \to \mathrm{Aut}(\mathscr{M})$. Then the set*

$$\mathscr{M}_\theta^\Gamma = \{T \in \mathscr{M} \,|\, \theta_s(T) = T \text{ for all } s \in \Gamma\}$$

is a von Neumann subalgebra of $\mathscr{M}$, called the fixpoint algebra for the action of Γ on $\mathscr{M}$.

Proof. As $\theta_s(I) = I$ by assumption, $\mathscr{M}_\theta^\Gamma$ is non-empty and contains the identity operator. It is now easily seen that $\mathscr{M}_\theta^\Gamma$ is a *-subalgebra of $\mathscr{M}$. If $T \in \mathscr{M}$ and $T_\alpha \to T$ strongly for a net $(T_\alpha)_{\alpha\in A}$ in $\mathscr{M}_\theta^\Gamma$, then

$$T_\alpha = \theta_s(T_\alpha) \to \theta_s(T)$$

for all $g \in \Gamma$, so $T = \theta_s(T)$ for all $g \in \Gamma$ and hence $\mathscr{M}_\theta^\Gamma$ is strongly closed. □

Proposition 4.16. *Let $a\colon \Gamma \to \mathcal{U}(\mathcal{H})$ be a strongly continuous unitary representation with $a_s := a(s)$ for all $s \in \Gamma$, satisfying*

$$\alpha_s(T) = a_s T a_s^*, \quad T \in \mathscr{M},\ s \in \Gamma.$$

Defining a unitary operator $W \in \mathcal{U}(L^2(\Gamma, \mathcal{H}))$ by

$$(Wf)(s) = a_s f(s), \quad f \in C_c(\Gamma, \mathcal{H}),\ s \in \Gamma,$$

we obtain

$$\pi(T) = W^*(T \otimes 1_{\mathcal{H}})W, \quad \lambda(s) = W^*(a_s \otimes \lambda(s))W, \quad T \in \mathscr{M},\ s \in \Gamma.$$

In particular $\mathrm{R}(\mathscr{M}, \alpha_t)$ is spatially isomorphic to the von Neumann algebra acting on $\mathcal{H} \otimes L^2(\Gamma)$ generated by the operators $\{T \otimes 1,\ a_s \otimes \lambda_s \,|\, T \in \mathscr{M},\ s \in \Gamma\}$.

Proof. See [8, Proposition I.2.12]. □

Corollary 4.17. *$\mathrm{R}(\mathscr{M}, \alpha_t)$ embeds into $\mathscr{M} \mathbin{\overline{\otimes}} B(L^2(\Gamma)) \subseteq B(\mathcal{H} \otimes L^2(\Gamma))$.*

Proof. See [8, Lemma I.3.1]. □

Letting Δ denote the modular function of Γ, we define the right translation $\rho_t\colon L^2(\Gamma) \to L^2(\Gamma)$ by

$$\rho_t(f)(s) = \Delta(t)^{1/2} f(st), \quad f \in L^2(\Gamma), \quad s,t \in \Gamma.$$

We then define $\mathrm{ad}\rho_t\colon B(L^2(\Gamma)) \to B(L^2(\Gamma))$ by

$$\mathrm{ad}\rho_t(T) = \rho_t T \rho_t^*, \quad T \in B(L^2(\Gamma)).$$

For $t \in \Gamma$, define θ_t on $\mathscr{M} \mathbin{\overline{\otimes}} B(L^2(\Gamma))$ by $\theta_t = \alpha_t \otimes \mathrm{ad}\rho_t$. Then $t \mapsto \theta_t$ is a continuous action of Γ on $\mathscr{M} \mathbin{\overline{\otimes}} B(L^2(\Gamma))$ [8, Proposition I.3.3]. Furthermore, the following theorem holds:

Theorem 4.18. *With θ_t as above, we have*

$$\mathrm{R}(\mathscr{M}, \alpha_t) = \{T \in \mathscr{M} \mathbin{\overline{\otimes}} B(L^2(\Gamma)) \,|\, \theta_t(T) = T \text{ for all } t \in \Gamma\}.$$

Hence $\mathrm{R}(\mathscr{M}, \alpha_t)$ is the fixpoint algebra of a continuous action of Γ on $\mathscr{M} \mathbin{\overline{\otimes}} B(L^2(\Gamma))$.

Proof. See [8, Theorem I.3.11]. □

In order for the next theorem to make sense, we take an opportunity to remind the reader of the notions of finiteness and semifiniteness for von Neumann algebras.

Definition 4.8. Let $\mathscr{M}$ be a von Neumann algebra. For two projections $P, Q \in \mathscr{M}$ we say that $P \sim Q$ if there exists a partial isometry $V \in \mathscr{M}$ such that $P = V^*V$ and $Q = VV^*$.

(i) A projection $P \in \mathscr{M}$ is *finite* if $P \sim Q$ and $Q \leq P$ imply $Q = P$.
(ii) A projection $P \in \mathscr{M}$ is *semifinite* if there for any non-zero projection $Q \in \mathscr{M}$ with $Q \leq P$ exists a finite non-zero projection $R \in \mathscr{M}$ with $R \leq Q$.
(iii) $\mathscr{M}$ is *finite* if $1_{\mathscr{M}}$ is finite.
(iv) $\mathscr{M}$ is *semifinite* if $1_{\mathscr{M}}$ is semifinite.

One is advised to consult [10] or [1] for results on semifiniteness and finiteness, as we will do in some of the proofs contained in Chapter 5.

Assume now that $\mathscr{M}$ is a σ-finite von Neumann algebra and let $\omega \in \mathscr{M}_*$ be a faithful normal state by Proposition 2.58. By Tomita-Takesaki theory, there exists a *strongly continuous one-parameter group* $(\sigma_t^\omega)_{t\in\mathbb{R}}$ of *-automorphisms of $\mathscr{M}$ – that is, $t \mapsto \sigma_t^\omega$ is a homomorphism $\mathbb{R} \to \mathrm{Aut}(\mathscr{M})$ and if $t_i \to t$ in $\mathbb{R}$ implies $\sigma_{t_i}^\omega(T) \to \sigma_t^\omega(T)$ for all $T \in \mathscr{M}$ – that is uniquely characterized by satisfying the *K. M. S. condition*: namely that for any given $S, T \in \mathscr{M}$, there exists a complex-valued bounded continuous function F defined on $\{z \in \mathbb{C} \,|\, 0 \leq \mathrm{Im} z \leq 1\}$ that is analytic in the interior and satisfies

$$F(t) = \omega(\sigma_t^\omega(S)T), \quad F(t+i) = \omega(T\sigma_t^\omega(S)), \quad t \in \mathbb{R}.$$

The map $t \mapsto \sigma_t^\omega$ is a continuous action of $\mathbb{R}$ on $\mathscr{M}$. $(\sigma_t^\omega)_{t\in\mathbb{R}}$ is called the *modular automorphism group associated to* ω, and the action $t \mapsto \sigma_t^\omega$ is called the *modular action associated to* ω, yielding the continuous crossed product $\mathrm{R}(\mathscr{M}, \sigma_t^\omega)$.

Because of Connes' cocycle Radon-Nikodym theorem and a theorem by Takesaki [8, Theorems II.2.2 and II.2.3], $\mathrm{R}(\mathscr{M}, \sigma_t^\omega)$ is in fact up to isomorphism independent of the faithful normal state that the construction started with. For further comments, see [8, pp. 34-36]. Hence it is possible to write $\mathrm{R}(\mathscr{M}, \sigma_t)$ instead of specifying a state ω, but we will not do this.

The reason we introduce this specific crossed product is the following theorem that we will state without proof.

Theorem 4.19. *Let $\mathscr{M}$ be a σ-finite von Neumann algebra with a faithful normal state $\omega \in \mathscr{M}_*$. Then $\mathrm{R}(\mathscr{M}, \sigma_t^\omega)$ is a semifinite von Neumann algebra acting on $L^2(\mathbb{R}, \mathcal{H}) \cong \mathcal{H} \otimes L^2(\mathbb{R})$.*

Proof. See [8, Theorem II.3.5]. □

For us, the most important fact about $\mathrm{R}(\mathscr{M}, \sigma_t)$ is that it provides a connection between a σ-finite von Neumann algebra $\mathscr{M}$ and a semifinite one; this is in fact almost enough to go where Chapter 5 will take us.

4.4 Amenable locally compact groups

However, before we go on to Chapter 5, we need to make a short stop in the beautiful world of amenability. Amenability is a property for locally compact groups that generalizes finite groups and abelian groups by introducing an function that makes it possible to take "averages" on bounded functions and stays invariant under translation by group elements. For instance, if Γ is a finite group it is easy to take the average of a bounded function $f\colon \Gamma \to \mathbb{C}$, namely

$$\mathfrak{m}(f) = \frac{1}{|\Gamma|}\sum_{s\in\Gamma} f(s).$$

As $\sum_{s\in\Gamma} f(s) = \sum_{s\in\Gamma} f(t^{-1}s)$ for all $t \in \Gamma$ in this case, one obtains a function $\mathfrak{m}\colon \ell^\infty(\Gamma) \to \mathbb{C}$ whose values is not changed by translation by any element of Γ. Our definition of amenability will generalize this.

Definition 4.9. Let Γ be a locally compact group with Haar measure μ. A measurable function $f\colon \Gamma \to \mathbb{C}$ is essentially bounded if there exists a non-negative real number M such that the set

$$\{g \in \Gamma \,|\, |f(g)| > M\}$$

has measure zero under μ. $L^\infty(\Gamma)$ denotes the Banach space of complex measurable essentially bounded functions $\Gamma \to \mathbb{C}$ with the norm

$$\|f\|_\infty = \inf\{M \geq 0 \,|\, \mu(\{s \in \Gamma \,|\, |f(s)| > M\}) = 0\}, \quad f \in L^\infty(\Gamma).$$

A *mean on* $L^\infty(\Gamma)$ is a state on $L^\infty(\Gamma)$, i.e $\mathfrak{m}(1) = 1$ and $\mathfrak{m}(f) \geq 0$ for any non-negative function $f \in L^\infty(\Gamma)$. For any $s \in \Gamma$, the *left translation operator with respect to* s is the map $\tau_s\colon L^\infty(\Gamma) \to L^\infty(\Gamma)$ given by

$$\tau_s(f)(t) = f(s^{-1}t), \quad f \in L^\infty(\Gamma),\ t \in \Gamma.$$

Note that the left translation operator is well-defined by left invariance of the Haar measure. A mean $\mathfrak{m}$ on $L^\infty(\Gamma)$ is said to be *left invariant* if it satisfies the equality $\mathfrak{m} \circ \tau_s = \mathfrak{m}$ for all $s \in \Gamma$. If $L^\infty(\Gamma)$ has a left invariant mean, Γ is said to be *amenable*.

If Γ is a locally compact group with a mean $\mathfrak{m}$ on $L^\infty(\Gamma)$, then it is customary for any $f \in L^\infty(\Gamma)$ to write

$$\mathfrak{m}(f) = \int_\Gamma f(s)\mathrm{d}\mathfrak{m}(s).$$

Hence

$$\int_\Gamma \lambda\, \mathrm{d}\mathfrak{m}(s) = \lambda \text{ and } \int_\Gamma f(s)\mathrm{d}\mathfrak{m}(s) \geq 0, \quad \lambda \in \mathbb{C},\ f \in L^\infty(\Gamma)_+.$$

The above integral is also linear, as

$$\lambda_1 \int_\Gamma f_1(s)\mathrm{d}\mathfrak{m}(s) + \lambda_2 \int_\Gamma f_2(s)\mathrm{d}\mathfrak{m}(s) = \mathfrak{m}(\lambda_1 f_1 + \lambda_2 f_2) = \int_\Gamma (\lambda_1 f_1(s) + \lambda_2 f_2(s))\mathrm{d}\mathfrak{m}(s)$$

for all $f_1, f_2 \in L^\infty(\Gamma)$ and $\lambda_1, \lambda_2 \in \mathbb{C}$. The condition that $\mathfrak{m}$ is left invariant is reflected in the equality

$$\int_\Gamma f(s_0^{-1}s)\, \mathrm{d}\mathfrak{m}(s) = \int_\Gamma f(s)\, \mathrm{d}\mathfrak{m}(s), \quad f \in L^\infty(\Gamma),\ s_0 \in \Gamma.$$

Since $\mathfrak{m}$ is a state, it is also contractive and hence we have

$$\left|\int_\Gamma f(s)\mathrm{d}\mathfrak{m}(s)\right| \leq \|f\|_\infty, \quad f \in L^\infty(\Gamma).$$

If Γ is a discrete group, then a function $f\colon \Gamma \to \mathbb{C}$ is essentially bounded if and only if it is bounded, and in this case it is customary to denote $L^\infty(\Gamma)$ by $\ell^\infty(\Gamma)$.

For our purposes, we will need to know that a very well-known group is amenable.

Proposition 4.20. *The locally compact group* $(\mathbb{R}, +)$ *with Lebesgue measure is an amenable locally compact group.*

Proof. The Lebesgue measure on $\mathbb{R}$ is a Haar measure. For $n \geq 1$, define $\mathfrak{m}_n\colon L^\infty(\mathbb{R}) \to \mathbb{C}$ by

$$\mathfrak{m}_n(f) = \frac{1}{2n}\int_{-n}^{n} f(t)\mathrm{d}t, \quad f \in L^\infty(\mathbb{R}).$$

It is clear that $\mathfrak{m}_n$ is a state on $L^\infty(\mathbb{R})$ for all $n \geq 1$. Since $L^\infty(\mathbb{R})$ is a unital C^*-algebra, the state space of $L^\infty(\mathbb{R})$ is weak* compact by [31, Proposition 13.8], so there exists a subnet $(\mathfrak{m}_{n_\alpha})_{\alpha\in A}$ of the sequence $(\mathfrak{m}_n)_{n\geq 1}$ that converges to a state $\mathfrak{m} \in (L^\infty(\mathbb{R}))^*$. This $\mathfrak{m}$ is in fact a left invariant mean: for $f \in L^\infty(\mathbb{R})$ and $r \in \mathbb{R}$, we have

$$\begin{aligned}\mathfrak{m}_{n_\alpha}(\tau_r(f) - f) &= \frac{1}{2n_\alpha}\int_{-n_\alpha}^{n_\alpha} f(t-r) - f(t))\mathrm{d}t \\ &= \frac{1}{2n_\alpha}\left(\int_{-n_\alpha - r}^{n_\alpha - r} f(t)\mathrm{d}t - \int_{-n_\alpha}^{n_\alpha} f(t)\mathrm{d}t\right) \\ &= \frac{1}{2n_\alpha}\left(\int_{-n_\alpha - r}^{-n_\alpha} f(t)\mathrm{d}t - \int_{n_\alpha - r}^{n_\alpha} f(t)\mathrm{d}t\right)\end{aligned}$$

by the fact that $\int_{-n_\alpha-r}^{-n_\alpha} + \int_{-n_\alpha}^{n_\alpha} = \int_{-n_\alpha-r}^{n_\alpha-r} + \int_{n_\alpha-r}^{n_\alpha}$. Hence

$$|\mathfrak{m}_{n_\alpha}(\tau_r(f) - f)| \leq \frac{1}{2n_\alpha}(|r|\|f\|_\infty + |r|\|f\|_\infty) = \frac{|r|}{n_\alpha}\|f\|_\infty \to 0.$$

Since $\mathfrak{m}_{n_\alpha}(\tau_r(f) - f) \to \mathfrak{m}(\tau_r(f) - f)$ as well by the weak* convergence, it follows that $\mathfrak{m}(\tau_r(f) - f) = 0$ or $\mathfrak{m} \circ \tau_r = \mathfrak{m}$ for all $r \in \mathbb{R}$. Hence $(\mathbb{R}, +)$ is amenable. □

Just like the previous section, this section has a secret agenda: it also wants to connect a property of a von Neumann algebra to a property of a crossed product. Note that any crossed product was in fact a fixpoint algebra by Theorem 4.18. Do you see where we are going?

Proposition 4.21. *Let $\mathscr{M} \subseteq B(\mathcal{H})$ be a von Neumann algebra and let Γ be an locally compact amenable group acting continuously on $\mathscr{M}$ by the homomorphism $\theta\colon \Gamma \to \mathrm{Aut}(\mathscr{M})$. Then there is a projection $E\colon \mathscr{M} \to \mathscr{M}_\theta^\Gamma$ of norm 1 where $\mathscr{M}_\theta^\Gamma$ is the fixpoint algebra in $\mathscr{M}$ for the action of Γ on $\mathscr{M}$. In particular, if $\mathscr{M}$ is injective, then $\mathscr{M}_\theta^\Gamma$ is injective.*

Proof. Let $\mathfrak{m}$ be a left invariant mean on $L^\infty(\Gamma)$. For fixed $T \in \mathscr{M}$ and $\xi, \eta \in \mathcal{H}$, the function

$$s \mapsto \langle \theta_s(T)\xi, \eta\rangle$$

is a continuous function on Γ bounded above by the constant $\|T\|\|\xi\|\|\eta\|$ and hence it is an element of $L^\infty(\Gamma)$. This allows us to define a sesquilinear form on $\mathcal{H}$ by

$$\langle \xi, \eta\rangle = \int_\Gamma \langle \theta_g(T)\xi, \eta\rangle \mathrm{d}\mathfrak{m}(s).$$

Since

$$\left|\int_\Gamma \langle \theta_g(T)\xi, \eta\rangle \mathrm{d}\mathfrak{m}(s)\right| \leq \|T\|\|\xi\|\|\eta\|$$

for all $\xi, \eta \in \mathcal{H}$, then by the Riesz representation theorem [14, Theorem 2.4.1] there exists a unique operator $E(T) \in B(\mathcal{H})$ such that

$$\langle E(T)\xi, \eta\rangle = \int_\Gamma \langle \theta_s(T)\xi, \eta\rangle \mathrm{d}\mathfrak{m}(s), \quad \xi, \eta \in \mathcal{H}.$$

It is easily seen that E is linear by virtue of each θ_s being a linear map $\mathscr{M} \to \mathscr{M}$ and each $E(T)$ satisfying the above property. Moreover, $\|E(T)\| \leq \|T\|$ for any $T \in \mathscr{M}$. We claim that E is our wanted projection of norm 1. Fix $T \in \mathscr{M}$ and note that the above inequality can be written as

$$\omega_{\xi,\eta}(E(T)) = \int_\Gamma \omega_{\xi,\eta}(\theta_s(T))\mathrm{d}\mathfrak{m}(s), \quad \xi, \eta \in \mathcal{H}.$$

Any $\omega \in B(\mathcal{H})_*$ is a limit in norm of sums of the vector functionals $\omega_{\xi,\eta}$. Moreover, $\|\theta_s(T)\| \leq \|T\|$ for all $s \in \Gamma$, so if $s_\alpha \to s$ in Γ, then $\theta_{s_\alpha}(T) \to \theta_s(T)$ strongly and hence ultraweakly. Therfore

$s \mapsto \omega(\theta_s(T))$ is a continuous bounded function on Γ for any $\omega \in B(\mathcal{H})_*$. From these observations it now follows that

$$\omega(E(T)) = \int_\Gamma \omega(\theta_s(T))\mathrm{d}\mathfrak{m}(s), \quad \omega \in B(\mathcal{H})_*.$$

If $\omega \in \mathscr{M}^\perp = \{\omega \in B(\mathcal{H})_* \,|\, \omega(T) = 0 \text{ for all } T \in \mathscr{M}\}$, then because $\theta_s(T) \in \mathscr{M}$ for all $s \in \Gamma$, we have $\omega(\theta_s(T)) = 0$ for all $s \in \Gamma$ and hence $\omega(E(T)) = 0$. Therefore $E(T) \in \mathscr{M}^{\perp\perp} = \mathscr{M}$ by Lemma 2.6. Moreover, for all $\omega \in B(\mathcal{H})_*$ and $s_0 \in \Gamma$, then since θ_{s_0} is a *-isomorphism and hence ultraweakly-to-ultraweakly continuous by Proposition 2.48, we have that $\omega \circ \theta_{s_0} \in B(\mathcal{H})_*$ and hence

$$\omega(\theta_{t_0}(E(T))) = \int_\Gamma \omega \circ \theta_{t_0}(\theta_s(T))\mathrm{d}\mathfrak{m}(s) = \int_\Gamma \omega(\theta_{t_0 s}(T))\mathrm{d}\mathfrak{m}(s) = \int_\Gamma \omega(\theta_s(T))\mathrm{d}\mathfrak{m}(s) = \omega(E(T))$$

for all $t_0 \in \Gamma$, where we used left invariance of $\mathfrak{m}$ at the penultimate equality. Hence we have $\theta_{t_0}(E(T)) = E(T)$ for all $t_0 \in \Gamma$. Hence all $E(T)$ are fixed by θ, so $E(T) \in \mathscr{M}_\theta^\Gamma$. Also, if $T \in \mathscr{M}_\theta^\Gamma$, then

$$\langle E(T)\xi, \eta\rangle = \int_\Gamma \langle \theta_s(T)\xi, \eta\rangle \mathrm{d}\mathfrak{m}(s) = \int_\Gamma \langle T\xi, \eta\rangle \mathrm{d}\mathfrak{m}(s) = \langle T\xi, \eta\rangle, \quad \xi, \eta \in \mathcal{H},$$

so $E(T) = T$. Hence E is a projection of $\mathscr{M}$ onto $\mathscr{M}_\theta^\Gamma$ of norm 1. □

As Christoph Waltz exclaimed near the end of the great movie *Inglourious Basterds*: "That's a bingo!"

Corollary 4.22. *If $\mathscr{M} \subseteq B(\mathcal{H})$ is a σ-finite and injective von Neumann algebra with a normal faithful state ω, then $\mathrm{R}(\mathscr{M}, \sigma_t^\omega)$ is injective.*

Proof. It follows from Proposition 4.12 that $\mathscr{M} \mathbin{\overline{\otimes}} B(L^2(\mathbb{R})) \subseteq B(\mathcal{H} \otimes L^2(\mathbb{R}))$ is injective. Therefore, there exists a projection $E \colon B(\mathcal{H} \otimes L^2(\mathbb{R})) \to \mathscr{M} \mathbin{\overline{\otimes}} B(L^2(\mathbb{R}))$ onto, of norm 1. Since $\mathrm{R}(\mathscr{M}, \sigma_t^\omega)$ can be embedded in $\mathscr{M} \mathbin{\overline{\otimes}} B(L^2(\mathbb{R}))$ by Corollary 4.17 and is the fixpoint algebra of a strongly continuous action θ_t on $\mathscr{M} \mathbin{\overline{\otimes}} B(L^2(\mathbb{R}))$ by Theorem 4.18, then by Proposition 4.21, $\mathrm{R}(\mathscr{M}, \sigma_t^\omega)$ is injective. □

CHAPTER 5

SEMIDISCRETE VON NEUMANN ALGEBRAS

The notion of semidiscreteness arises from the concept of *nuclearity* of C^*-algebras, a notion equivalent to the one of $\otimes$-nuclearity in Chapter 1 (the reader is advised to consult [4] for more information on this), and comes off as a "topological variant of approximate finite-dimensionality" at first sight. In fact, we shall not only explore the various qualities of semidiscreteness in this chapter but also prove that it is equivalent to injectivity, which will be the last goal of this project. This is *very* surprising because the two concepts look nowhere alike.

Definition 5.1. Let $\mathscr{M}$ be a von Neumann algebra. $\mathscr{M}$ is said to be *semidiscrete* if the identity mapping $\mathrm{id}_{\mathscr{M}} \colon \mathscr{M} \to \mathscr{M}$ can be approximated ultraweakly by normal, completely positive maps φ of finite rank such that $\varphi(1_{\mathscr{M}}) = 1_{\mathscr{M}}$. That is, $\mathscr{M}$ is semidiscrete if and only if there is a net $(\varphi_\alpha)_{\alpha\in A}$ in $B(\mathscr{M})$ of normal, complete positive maps of finite rank and $\varphi_\alpha(1_{\mathscr{M}}) = 1_{\mathscr{M}}$ for all $\alpha \in A$ such that

$$|\omega(\varphi_\alpha(T) - T)| \to 0, \quad T \in \mathscr{M},\ \omega \in \mathscr{M}_*,$$

or, equivalently, $\varphi_\alpha(T) \to T$ ultraweakly for all $T \in \mathscr{M}$.

As in Chapter 4, we quickly make sure that semidiscreteness is preserved by *-isomorphisms.

Proposition 5.1. *Let $\mathscr{M}$ and $\mathscr{N}$ be von Neumann algebras for which there exists a *-isomorphism $\rho\colon \mathscr{M} \to \mathscr{N}$. If $\mathscr{M}$ is semidiscrete, then $\mathscr{N}$ is semidiscrete.*

Proof. Note that ρ is necessarily unital, since $\rho(1_{\mathscr{M}})$ is a unit for $\rho(\mathscr{M}) = \mathscr{N}$. Let $(\varphi_\alpha)_{\alpha\in A}$ be a net in $B(\mathscr{M})$ satisfying the conditions of Definition 5.1. For all $\alpha \in A$, let $\psi_\alpha \colon \mathscr{N} \to \mathscr{N}$ be the linear map given by $\psi_\alpha = \rho \circ \varphi_\alpha \circ \rho^{-1}$. Let $\alpha \in A$. By Proposition 3.11, ψ_α is completely positive, and since all *-homomorphisms are contractive, $\psi_\alpha \in B(\mathscr{N})$. Moreover, by Proposition 2.48, ψ_α is normal; since ρ is unital, $\psi_\alpha(1_{\mathscr{N}}) = 1_{\mathscr{N}}$, and ψ_α clearly has finite rank, since ρ and ρ^{-1} are linear isomorphisms. Finally, for all $\omega \in \mathscr{N}_*$ and $T \in \mathscr{N}$, $\omega \circ \rho \in \mathscr{M}_*$ by Proposition 2.45, so

$$|\omega(\psi_\alpha(T) - T)| = |(\omega \circ \rho)(\varphi_\alpha(\rho^{-1}(T)) - \rho^{-1}(T))| \to 0.$$

Hence $(\psi_\alpha)_{\alpha\in A}$ is a net satisfying the conditions of Definition 5.1 for $\mathscr{N}$, so $\mathscr{N}$ is semidiscrete. □

We will need an alternate criterion for a von Neumann algebra to be semidiscrete, realizing the concept on a much more local scale. In many cases this criterion will be easier to work with than the original formulation. There are no surprises in the proof: we only work with the ultraweak topology as a locally convex topology, and the requirement of complete positivity and being finite rank is not used at all.

Proposition 5.2. *Let $\mathscr{M}$ be a von Neumann algebra. Then the following conditions are equivalent:*

(i) *$\mathscr{M}$ is semidiscrete.*
(ii) *For any $\varepsilon > 0$, any given finite set $\mathcal{F} \subseteq \mathscr{M}$ with $\mathcal{F} = \{T_1, \dots, T_n\}$ and any ultraweakly continuous functionals $\omega_1, \dots, \omega_n \in \mathscr{M}_*$, there exists a normal, completely positive map $\varphi \in B(\mathscr{M})$ of finite rank satisfying $\varphi(1_{\mathscr{M}}) = 1_{\mathscr{M}}$ such that*

$$|\omega_i(\varphi(T_i) - T_i)| < \varepsilon.$$

Proof. Assume that (i) holds, i.e. that $\mathscr{M}$ is semidiscrete, and let $(\varphi_\alpha)_{\alpha\in A}$ be a net in $B(\mathscr{M})$ as in Definition 5.1. Let $\varepsilon > 0$ be given, let $\mathcal{F} \subseteq \mathscr{M}$ be a finite set with $\mathcal{F} = \{T_1, \dots, T_n\}$ and let $\omega_1, \dots, \omega_n \in \mathscr{M}_*$. For any $i = 1, \dots, n$, we can now take $\alpha_i \in A$ such that

$$|\omega_i(\varphi_\alpha(T_i) - T_i)| < \varepsilon$$

for all $\alpha \geq \alpha_i$. Let $\alpha_0 \in A$ such that $\alpha_0 \geq \alpha_i$ for all $i = 1, \dots, n$ and let $\varphi = \varphi_{\alpha_0}$. Then clearly $\varphi \in B(\mathscr{M})$ is normal, completely positive, is of finite rank, satisfies $\varphi(1_{\mathscr{M}}) = 1_{\mathscr{M}}$ and

$$|\omega_i(\varphi(T_i) - T_i)| < \varepsilon, \quad i = 1, \dots, n$$

since $\alpha_0 \geq \alpha_i$ for all $i = 1, \dots, n$. Hence (ii) follows.

Assuming instead that (ii) holds, let $A = \{F \subseteq \mathscr{M} \,|\, F \text{ is finite}\}$ be ordered by inclusion, making it a directed set. Moreover, let $B = \{G \subseteq \mathscr{M}_* \,|\, G \text{ is finite}\}$ be ordered by inclusion as well, and let $\mathcal{C} = \{(F, G) \in A \times B \,|\, |F| = |G|\}$. We make $\mathcal{C}$ into a directed set by defining $(F_1, G_1) \leq (F_2, G_2)$ if $F_1 \subseteq F_2$ and $G_1 \subseteq G_2$. For any $(F, G) \in \mathcal{C}$, let $\varphi_{F,G} \in B(\mathscr{M})$ be given as per (ii) such that $|\omega(\varphi_{F,G}(T) - T)| < \frac{1}{|F|}$ for all $T \in F$ and $\omega \in G$. Given $T \in \mathscr{M}$, $\omega \in \mathscr{M}_*$ and $\varepsilon > 0$, take $(F_0, G_0) \in \mathcal{C}$ such that $T \in F_0$, $\omega \in G_0$ and $|F_0| > \frac{1}{\varepsilon}$. Then for all $(F, G) \in \mathcal{C}$ such that $(F, G) \geq (F_0, G_0)$, we have $|F| \geq |F_0|$ and hence

$$|\omega(\varphi_{F,G}(T) - T)| < \frac{1}{|F|} \leq \frac{1}{|F_0|} < \varepsilon.$$

Therefore $\mathscr{M}$ is semidiscrete, as the net $(\varphi_{F,G})_{(F,G)\in\mathcal{C}}$ satisfies the conditions of Definition 5.1. □

5.1 Semidiscreteness and preduals

If $\mathscr{M}$ is a von Neumann algebra and $(\varphi_\alpha)_{\alpha\in A}$ is a net in $B(\mathscr{M})$ satisfying the properties of Definition 5.1, then for any $\omega \in \mathscr{M}_*$ we have $\varphi_\alpha^*(\omega) \in \mathscr{M}_*$ as well by normality (this is Proposition 2.45), yielding a net $(\varphi_\alpha^*|_{\mathscr{M}_*})_{\alpha\in A}$ in $B(\mathscr{M}_*)$. By bringing in the canonical identification $\Lambda\colon \mathscr{M} \to (\mathscr{M}_*)^*$ of Theorem 2.7 the convergence requirement of Definition 5.1 can be written as

$$|\Lambda(T)(\varphi_\alpha^* - \mathrm{id}_{\mathscr{M}_*})(\omega)| = |\omega(\varphi_\alpha(T) - T)| \to 0, \quad T \in \mathscr{M},\ \omega \in \mathscr{M}_*.$$

It becomes useful to describe this convergence by means of the point-norm and point-weak topologies (see Appendix A for a runthrough of the definition), in which case we have $\varphi_\alpha^*|_{\mathscr{M}_*} \to \mathrm{id}_{\mathscr{M}_*}$ in the point-weak topology. The purpose of this section is to find the properties of this net in order to describe another condition equivalent to semidiscreteness.

In order of this description to be as thorough as possible, we will first need to discuss positivity properties of the maps connecting $\mathscr{M}$ to $\mathscr{M}_*$. In particular, we will need to know if complete positivity is preserved when passing to (duals of) matrix spaces, and in order to this to make sense, we need to bring in some helpful isomorphisms. Hopefully the next two paragraphs will not be too confusing.

Let $n \geq 1$ and let $\mathscr{M}$ be a von Neumann algebra. If $\phi_n\colon M_n(\mathscr{M}_*) \to M_n(\mathscr{M})_*$ is the isomorphism of Proposition 3.4, we can make $M_n(\mathscr{M}_*)$ into a Banach space by equipping it with the norm of $M_n(\mathscr{M})_*$ using ϕ_n, and we can define $\Omega_n\colon (\mathscr{M}_*)^* \odot M_n(\mathbb{C}) \to M_n(\mathscr{M}_*)^*$ by

$$\Omega_n\left(\sum_{i,j=1}^n \varphi_{ij} \otimes e_{ij}\right)(\omega) = \sum_{i,j=1}^n \varphi_{ij}(\omega_{ij}), \quad \omega = (\omega_{ij})_{i,j=1}^n \in M_n(\mathscr{M}_*),$$

where $(e_{ij})_{i,j=1}^n$ is the canonical matrix basis for $M_n(\mathbb{C})$. We claim that Ω_n is a linear isomorphism. Indeed, it is first and foremost well-defined, as $\Omega_n(w)$ is linear and bounded by $\sum_{i,j=1}^n \|\varphi_{ij}\|$ for any $w = \sum_{i,j=1}^n \varphi_{ij} \otimes e_{ij} \in (\mathscr{M}_*)^* \odot M_n(\mathbb{C})$, using Proposition 3.4. If $\Omega_n(w) = 0$ for some $w \in (\mathscr{M}_*)^* \odot M_n(\mathbb{C})$, then it is clear that $w = 0$, by checking values on matrices in $M_n(\mathscr{M}_*)$ with only one entry different from the zero functional, so Ω_n is injective. For surjectivity, let $\varphi \in M_n(\mathscr{M}_*)^*$ and for $i, j = 1, \dots, n$, define $\varphi_{ij}\colon \mathscr{M}_* \to \mathbb{C}$ by

$$\varphi_{ij}(\omega) = \varphi(\rho_{ij}(\omega)),$$

where $\rho_{ij}(\omega)$ is the matrix in $M_n(\mathscr{M}_*)$ with ω at position (i, j) and 0 everywhere else; clearly φ_{ij} is linear and bounded, and $\Omega_n(\sum_{i,j=1}^n \varphi_{ij} \otimes e_{ij}) = \varphi$. We identify $(\mathscr{M}_*)^* \odot M_n(\mathbb{C})$ with $M_n(\mathscr{M}_*)^*$ this way, and in particular the positive elements, so that an element $\varphi \in (\mathscr{M}_*)^* \odot M_n(\mathbb{C})$ is positive if and only if $\Omega_n(\varphi)$ is a positive linear functional in $M_n(\mathscr{M}_*)^*$.

Now, let $\Lambda\colon \mathscr{M} \to (\mathscr{M}_*)^*$ and $\Lambda_n\colon M_n(\mathscr{M}) \to (M_n(\mathscr{M})_*)^*$ be the canonical identifications of Theorem 2.7, and let $\mathrm{id}_n\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be the identity. Under the identification $M_n(\mathscr{M}) = \mathscr{M} \odot M_n(\mathbb{C})$, we have for all $\omega = (\omega_{ij})_{i,j=1}^n \in M_n(\mathscr{M}_*)$ that

$$\Omega_n \circ (\Lambda \otimes \mathrm{id}_n)\left(\sum_{i,j=1}^n T_{ij} \otimes e_{ij}\right)(\omega) = \sum_{i,j=1}^n \omega_{ij}(T_{ij}),$$

and

$$\phi_n^* \circ \Lambda_n((T_{ij})_{i,j=1}^n)(\omega) = \Lambda_n((T_{ij})_{i,j=1}^n)(\phi_n(\omega)) = \sum_{i,j=1}^n \omega_{ij}(T_{ij}),$$

so

$$\Lambda \otimes \mathrm{id}_n = \Omega_n^{-1} \circ \phi_n^* \circ \Lambda_n.$$

Λ_n yields a one-to-one correspondence between positive elements in $M_n(\mathscr{M})$ and positive linear functionals in $(M_n(\mathscr{M})_*)^*$. Moreover, remember from Section 3.1 that we defined $x \in M_n(\mathscr{M}_*)$ to be positive if and only if $\phi_n(x) \in M_n(\mathscr{M})_*$ was positive.

Proposition 5.3. *Let $\mathscr{M}$ be a von Neumann algebra. Then the following conditions are equivalent:*

(i) *$\mathscr{M}$ is semidiscrete.*

(ii) *There exists a net in $B(\mathscr{M}_*)$ of completely positive maps of finite rank, mapping states to states, that converges in point-norm topology to the identity mapping $\mathrm{id}_{\mathscr{M}_*}\colon \mathscr{M}_* \to \mathscr{M}_*$.*

Proof. Suppose first that $\mathscr{M}$ is semidiscrete and let $(\varphi_\alpha)_{\alpha\in A}$ be a net in $B(\mathscr{M})$ approximating the identity map on $\mathscr{M}$ as per Definition 5.1. Defining $\psi_\alpha\colon \mathscr{M}_* \to \mathscr{M}_*$ by $\psi_\alpha(\omega) = \omega \circ \varphi_\alpha = \varphi_\alpha^*(\omega)$ for all $\alpha \in A$, then all ψ_α are well-defined by Proposition 2.45, clearly linear, completely positive by Proposition 3.21 and finite rank by Lemma A.10. Moreover, if $\omega \in \mathscr{M}_*$ is a state, then $\psi_\alpha(\omega)$ is positive and $\psi_\alpha(\omega)(1_{\mathscr{M}}) = \omega(\varphi_\alpha(1_{\mathscr{M}})) = 1$, so ψ_α maps states to states. Since

$$|(\psi_\alpha(\omega) - \omega)(T)| = |\omega(\varphi_\alpha(T) - T)| \to 0$$

for all $\omega \in \mathscr{M}_*$ and $T \in \mathscr{M}$, and hence for all $\varphi \in (\mathscr{M}_*)^*$, we have $\varphi(\psi_\alpha(\omega) - \omega) \to 0$ by Theorem 2.7. Hence ψ_α converges to the identity map $\mathrm{id}_{\mathscr{M}_*}$ on $\mathscr{M}_*$ in the point-weak topology, so by letting $\mathscr{S} = \mathrm{conv}\{\psi_\alpha \,|\, \alpha \in A\}$, we have $\mathrm{id}_{\mathscr{M}_*}$ is in the point-weak closure of $\mathscr{S}$. Hence $\mathrm{id}_{\mathscr{M}_*}$ is in the point-norm closure of $\mathscr{S}$ as well by Corollary A.8. It can be checked easily that $\mathscr{S}$ itself consists of completely positive maps of finite rank that map states to states, hence yielding (ii).

Assume instead that (ii) holds and let $(\chi_\alpha)_{\alpha\in A}$ be a net in $B(\mathscr{M}_*)$ satisfying the conditions of (ii). For each $\alpha \in A$, note that by considering the dual map $\chi_\alpha^*\colon (\mathscr{M}_*)^* \to (\mathscr{M}_*)^*$, we can define a map $\varphi_\alpha\colon \mathscr{M} \to \mathscr{M}$ by

$$\varphi_\alpha = \Lambda^{-1} \circ \chi_\alpha^* \circ \Lambda,$$

where $\Lambda\colon \mathscr{M} \to (\mathscr{M}_*)^*$ is the canonical identification from Theorem 2.7. Since χ_α^* has finite rank and Λ is a linear isomorphism, φ_α also has finite rank. φ_α is clearly ultraweakly-to-ultraweakly continuous by Corollary 2.8.

To prove that φ_α is in fact completely positive, let $n \geq 1$. We will use what we know about the maps Λ_n, ϕ_n and Ω_n as well as $\mathrm{id}_n\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$, as defined in the discussion before the statement of this proposition. Note that since χ_α is completely positive, then $\phi_n \circ \chi_\alpha^{(n)} \circ \phi_n^{-1}\colon M_n(\mathscr{M})_* \to M_n(\mathscr{M})_*$ is positive as well. Hence

$$(\phi_n^*)^{-1} \circ (\chi_\alpha^{(n)})^* \circ \phi_n^*\colon (M_n(\mathscr{M})_*)^* \to (M_n(\mathscr{M})_*)^*$$

maps positive linear functionals in $(M_n(\mathscr{M})_*)^*$ to positive linear functionals, so

$$\Lambda_n^{-1} \circ (\phi_n^*)^{-1} \circ (\chi_\alpha^{(n)})^* \circ \phi_n^* \circ \Lambda_n$$

is a positive map $M_n(\mathscr{M}) \to M_n(\mathscr{M})$. For all $\omega = (\omega_{ij})_{i,j=1}^n \in M_n(\mathscr{M}_*)$, then

$$\begin{aligned}\Omega_n \circ (\chi_\alpha^* \otimes \mathrm{id}_n)\left(\sum_{i,j=1}^n \varphi_{ij} \otimes e_{ij}\right)(\omega) &= \sum_{i,j=1}^n \varphi_{ij}(\chi_\alpha(\omega_{ij}) \\ &= \Omega_n\left(\sum_{i,j=1}^n \chi_\alpha(\varphi_{ij} \otimes e_{ij}\right)(\chi_\alpha^{(n)}(\omega)) \\ &= (\chi_\alpha^{(n)})^* \circ \Omega_n\left(\sum_{i,j=1}^n \varphi_{ij} \otimes e_{ij}\right)(\omega),\end{aligned}$$

so $\Omega_n \circ (\chi_\alpha^* \otimes \mathrm{id}_n) \circ \Omega_n^{-1} = (\chi_\alpha^{(n)})^*$. As

$$\Lambda_n^{-1} \circ (\phi_n^*)^{-1} \circ (\chi_\alpha^{(n)})^* \circ \phi_n^* \circ \Lambda_n = (\Lambda^{-1} \otimes \mathrm{id}_n) \circ (\chi_\alpha^* \otimes \mathrm{id}_n) \circ (\Lambda \otimes \mathrm{id}_n) = \varphi_\alpha^{(n)}.$$

Hence $\varphi_\alpha^{(n)}$ is positive, so φ_α is completely positive. Finally, let $\omega \in \mathscr{M}_*$ and note that

$$(\Lambda \circ \varphi_\alpha)(1_{\mathscr{M}}))(\omega) = (\chi_\alpha^* \circ \Lambda)(1_{\mathscr{M}})(\omega) = \chi_\alpha(\omega)(1_{\mathscr{M}}).$$

By Theorem 2.40, $\omega = \sum_{i=1}^4 \lambda_i\omega_i$ where $\lambda_i \in \mathbb{C}$ and ω_i is an ultraweakly continuous state. Hence

$$\chi_\alpha(\omega)(1_{\mathscr{M}}) = \sum_{i=1}^4 \lambda_i\chi_\alpha(\omega_i)(1_{\mathscr{M}}) = \sum_{i=1}^4 \lambda_i = \omega(1_{\mathscr{M}}) = \Lambda(1_{\mathscr{M}})(\omega),$$

as χ_α sends normal states to normal states. Hence $\Lambda(\varphi_\alpha(1_{\mathscr{M}})) = \Lambda(1_{\mathscr{M}})$, so $\varphi_\alpha(1_{\mathscr{M}}) = 1_{\mathscr{M}}$. Finally, for $T \in \mathscr{M}$ and $\omega \in \mathscr{M}_*$, we have

$$\omega(\varphi_\alpha(T)) = \Lambda(\varphi_\alpha(T))(\omega) = \chi_\alpha^*(\Lambda(T))(\omega) = \chi_\alpha(\omega)(T),$$

so

$$|\omega(\varphi_\alpha(T) - T)| = |(\chi_\alpha(\omega) - \omega)(T)| \le \|\chi_\alpha(\omega) - \omega\|\|T\| \to 0.$$

Hence $\mathscr{M}$ is semidiscrete, as $(\varphi_\alpha)_{\alpha\in A}$ satisfies the wanted properties. □

Similar to the local condition of Proposition 5.2 equivalent to semidiscrete, we can derive a consequence of condition (ii) above.

Corollary 5.4. *Let $\mathscr{M}$ be a semidiscrete von Neumann algebra. Then for any $\omega_1, \dots, \omega_n \in \mathscr{M}_*$ and $\varepsilon > 0$, there exists a normal, completely positive map $\varphi \in B(\mathscr{M})$ of finite rank, satisfying $\varphi(1_{\mathscr{M}}) = 1_{\mathscr{M}}$, such that*

$$\|\omega_i \circ \varphi - \omega_i\| < \varepsilon, \quad i = 1, \dots, n.$$

Proof. By Proposition 5.3, there exists a net $(\chi_\alpha)_{\alpha\in A}$ in $B(\mathscr{M}_*)$ having the properties mentioned in condition (ii) of the proposition. Using this net, it is easy to find a normal, completely positive map $\chi \in B(\mathscr{M}_*)$ of finite rank that sends normal states to normal states such that $\|\chi(\omega_i) - \omega_i\| < \varepsilon$ for all $i = 1, \dots, n$ (the method used in Proposition 5.2 can easily be applied here). As in the proof of Proposition 5.3, one can find that $\varphi \in B(\mathscr{M})$ given by $\varphi = \Lambda^{-1} \circ \chi^* \circ \Lambda$ where $\Lambda\colon \mathscr{M} \to (\mathscr{M}_*)^*$ is the canonical identification of Theorem 2.7 has every property mentioned in the statement of this proposition. □

The final theorem of this section is perhaps a bit of a cheat, as one of the implications depends on a result that we have not proved yet (even though we will). The reason that it is put here is simply because the notation and mindset used in the proof is very much in keeping with the methods and ideas used previously in this section, and one might have forgotten them all once we actually have all the information we need to prove the theorem. We *do* however have the knowledge needed to prove most of it, so here it is.

Theorem 5.5. *Let $\mathcal{A}$ be a C^*-algebra. Then the following are equivalent:*

(i) *$\mathcal{A}^{**}$ is semidiscrete.*

(ii) *The identity map $\mathcal{A}^* \to \mathcal{A}^*$ is the point-weak* limit of a net of completely positive finite-rank contractions, i.e. there exists a net $(\rho_\alpha)_{\alpha\in A}$ of completely positive finite-rank contractions $\mathcal{A}^* \to \mathcal{A}^*$ such that $\rho_\alpha(\chi) \to \chi$ in the weak*-topology for all $\chi \in \mathcal{A}^*$.*

(iii) *For any C^*-algebra $\mathcal{B}$, any completely positive contraction $\delta\colon \mathcal{B} \to \mathcal{A}^*$ is the point-weak* limit of a net of completely positive finite-rank contractions.*

(iv) *For any C^*-algebra $\mathcal{B}$, any state in $S(\mathcal{A}\odot\mathcal{B})$ is the weak* limit of states in $\mathcal{A}^*\odot\mathcal{B}^*\cap S(\mathcal{A}\odot\mathcal{B})$ (see page 26).*

(v) *$\mathcal{A}$ is $\otimes$-nuclear.*

Proof. (v) $\Rightarrow$ (i) follows from Theorem 5.19, requiring the knowledge that semidiscreteness is the same as injectivity (in other words, stick around for a proof). Let $\mathscr{M} = \mathcal{A}^{**}$ and assume that $\mathscr{M}$ is semidiscrete. If $(\varphi_\alpha)_{\alpha\in A}$ is a net in $B(\mathscr{M})$ approximating the identity map on $\mathscr{M}$ as per Definition 5.1, recall that in the proof of Proposition 5.3 – specifically (i) $\Rightarrow$ (ii) – we found a net $(\psi_\alpha)_{\alpha\in A}$ in $B(\mathscr{M}_*)$ of completely positive finite-rank maps, sending states to states, that converged to the identity map $\mathscr{M}_* \to \mathscr{M}_*$ in the point-weak topology. If $(\varphi_\alpha)_{\alpha\in A}$ and $(\psi_\alpha)_{\alpha\in A}$ are such nets of $B(\mathscr{M})$ and $B(\mathscr{M}_*)$ respectively found in the proof of (i) $\Rightarrow$ (ii), then by letting $\Omega\colon \mathscr{M}_* \to \mathcal{A}^*$ be the isometric isomorphism of Proposition 3.24 we now define $\rho_\alpha\colon \mathcal{A}^* \to \mathcal{A}^*$ by $\rho_\alpha = \Omega\circ\psi_\alpha\circ\Omega^{-1}$ for all $\alpha \in A$. Then each ρ_α is contained in $B(\mathcal{A}^*)$, has finite rank and is completely positive. For $\chi \in \mathcal{A}^*$ and $a \in \mathcal{A}$, we have

$$\|\rho_\alpha(\chi)\| = \|\psi_\alpha \circ \Omega^{-1}(\chi)\circ\iota\| \leq \|\Omega^{-1}(\chi)\|\varphi_\alpha\|\|\iota\| \leq \|\chi\|.$$

This proves that each ρ_α is contractive. Finally,

$$\rho_\alpha(\chi)(a) = \Omega^{-1}(\chi)(\varphi_\alpha(\iota(a)) \to \Omega^{-1}(\chi)(\iota(a)) = \chi(a)$$

for all $\chi \in \mathcal{A}^*$ and $a \in \mathcal{A}$, so we have (ii). If $\mathcal{B}$ is a C^*-algebra and $\delta\colon \mathcal{B} \to \mathcal{A}^*$ is a completely positive contraction, then note that $\rho_\alpha\circ\delta\colon \mathcal{B}\to\mathcal{A}^*$ is completely positive, contractive and has finite rank for all $\alpha \in \mathcal{A}$ and that $\rho_\alpha(\delta(b)) \to \delta(b)$ in the weak*-topology for all $b \in \mathcal{B}$, so (iii) follows from (ii).

Assume (iii). Let $\mathcal{B}$ be a C^*-algebra and let $\varphi \in S(\mathcal{A}\odot\mathcal{B})$. Define $\delta\colon \mathcal{B}\to\mathcal{A}^*$ by

$$\delta(b)(a) = \varphi(a\otimes b).$$

As $|\delta(b)(a)| \leq 1$ for all $a \in (\mathcal{A})_1$ and $b \in (\mathcal{B})_1$, then $\|\delta(b)\| \leq 1$ for all $b \in (\mathcal{B})_1$. Therefore δ is a contraction. For $b_1,\dots,b_n \in \mathcal{B}$ and $a_1,\dots,a_n \in \mathcal{A}$, write $x = \sum_{i=1}^n a_i\otimes b_i$ and note

$$\sum_{i,j=1}^n \delta(b_i^*b_j)(a_i^*a_j) = \varphi(x^*x) \geq 0.$$

Thus δ is completely positive by Proposition 3.22. Hence $\delta = \rho^*\circ\delta$ is the point-weak* limit of a net $(\delta_\alpha)_{\alpha\in A}$ of completely positive contractions $\mathcal{B}\to\mathcal{A}^*$ of finite rank by the assumption.

Let $\varepsilon > 0$. Since $\|\varphi\|_{\mathrm{alg}} = 1$, there exist $a \in (\mathcal{A})_1$ and $b \in (\mathcal{B})_1$ such that $|\delta(b)(a)| = |\varphi(a\otimes b)| \geq 1 - \frac{\varepsilon}{2}$. Taking $\alpha_0 \in A$ such that $\alpha \geq \alpha_0$ implies $|\delta(b)(a) - \delta_\alpha(b)(a)| < \frac{\varepsilon}{2}$, we now see that $\alpha \geq \alpha_0$ implies

$$\|\delta_\alpha\| \geq |\delta_\alpha(b)(a)| > |\delta'(b)(a)| - \frac{\varepsilon}{2} \geq 1-\varepsilon.$$

Hence $\|\delta_\alpha\| \to 1$. Defining $\delta'_\alpha = \|\delta_\alpha\|^{-1}\delta_\alpha$ for large enough α, it follows that $\delta'_\alpha \to \delta$ in the point-weak* topology. Each δ'_α is then a completely positive map of norm 1 and of finite rank. By universality of the tensor product, then from each δ'_α we can derive a linear functional $\varphi_\alpha\colon \mathcal{A}\odot\mathcal{B}\to\mathbb{C}$ that uniquely satisfies

$$\varphi_\alpha(a\otimes b) = \delta'_\alpha(b)(a), \quad a\in\mathcal{A},\ b\in\mathcal{B}.$$

For any $x = \sum_{i=1}^n a_i\otimes b_i \in \mathcal{A}\odot\mathcal{B}$, Proposition 3.22 tells us that

$$\varphi'_\alpha(x^*x) = \sum_{i,j=1}^n \delta'_\alpha(b_i^*b_j)(a_i^*a_j) \geq 0,$$

so φ'_α is algebraically positive. Clearly $\|\varphi_\alpha\|_{\mathrm{alg}} \leq 1$. For any $\varepsilon > 0$, let $b \in (\mathcal{B})_1$ such that $\|\delta'_\alpha(b)\| + \frac{\varepsilon}{2} \geq 1$, and let $a \in (\mathcal{A})_1$ such that $|\delta'_\alpha(b)(a)| + \frac{\varepsilon}{2} \geq \|\delta'_\alpha(b)\|$. Then

$$|\varphi_\alpha(a\otimes b)| = |\delta'_\alpha(b)(a)| \geq 1-\varepsilon,$$

so $\|\varphi_\alpha\|_{\mathrm{alg}} = 1$. Hence $\varphi_\alpha \in S(\mathcal{A} \odot \mathcal{B})$.

But wait, there's more! For any δ'_α then Lemma A.9 yields $\varphi_1, \dots, \varphi_n \in \mathcal{B}^*$ and $\psi_1, \dots, \psi_n \in \mathcal{A}^*$ such that

$$\delta'_\alpha(b) = \sum_{i=1}^{n} \varphi_i(b)\psi_i, \quad b \in \mathcal{B}.$$

Defining $\chi = \sum_{i=1}^{n} \psi_i \odot \varphi_i$, then $\chi \in \mathcal{A}^* \odot \mathcal{B}^*$. We also see that for $x = \sum_{j=1}^{m} a_j \otimes b_j \in \mathcal{A} \odot \mathcal{B}$, we have

$$\varphi_\alpha(x) = \sum_{j=1}^{m} \delta'_\alpha(b_j)(a_j) = \sum_{j=1}^{m}\sum_{i=1}^{n} \psi_i(a_j)\varphi_i(b_j) = \sum_{j=1}^{m} \chi(a_j \otimes b_j) = \chi(x),$$

so $\varphi_\alpha \in \mathcal{A}^* \odot \mathcal{B}^* \cap S(\mathcal{A} \odot \mathcal{B}) = \mathbb{M}(\mathcal{A}, \mathcal{B})$ (see page 26). Finally, for all $x \in \mathcal{A} \odot \mathcal{B}$, we see that

$$\varphi_\alpha(x) = \sum_{i=1}^{n} \delta'_\alpha(b_i)(a_i) \to \sum_{i=1}^{n} \delta(b_i)(a_i) = \varphi(x),$$

hence (iv).

For (iv) $\Rightarrow$ (v), we step outside the proof for just a moment. Assume first that $\mathcal{A}$ is non-unital and let $\tilde{\mathcal{A}}$ denote the unitization of $\mathcal{A}$. We shall soon see that $(\tilde{\mathcal{A}})^{**}$ is semidiscrete if and only if $\mathcal{A}^{**}$ is semidiscrete; this will follow from Corollary 3.15 and Proposition 5.7, as $\mathbb{C}$ is semidiscrete. Hence (iv) holds for $\tilde{\mathcal{A}}$ as well. Now let $\mathcal{B}$ be a C^*-algebra; (iv) then implies that

$$\|x\|_{\max} = \sup\{\varphi(x^*x) \mid \varphi \in S(\mathcal{A} \odot \mathcal{B})\} = \sup\{\varphi(x^*x) \mid \varphi \in \mathbb{M}(\mathcal{A}, \mathcal{B})\}, \quad x \in \mathcal{A} \odot \mathcal{B}.$$

(See page 25 for an explanation of the first equality.) If $\mathcal{B}$ is unital, then Corollary 1.48, yields that $\|x\|_{\max} = \|x\|_{\min}$ for all $x \in \tilde{\mathcal{A}} \odot \mathcal{B}$. Theorem 1.49 now yields that $\mathcal{A} \odot \mathcal{B}$ has a unique C^*-norm. If $\mathcal{B}$ is non-unital, we similarly have $\|x\|_{\max} = \|x\|_{\min}$ for all $x \in \tilde{\mathcal{A}} \odot \tilde{\mathcal{B}}$, so $\mathcal{A} \odot \mathcal{B}$ has a unique C^*-norm by the same theorem. Hence we conclude that $\mathcal{A}$ is $\otimes$-nuclear. If $\mathcal{A}$ is unital, the same considerations (but with no need to pass to unitizations) yield that $\mathcal{A}$ is $\otimes$-nuclear. □

Thus semidiscreteness is inseparably connected to the notion of $\otimes$-nuclearity which will help out a great deal in the future, should we for instance want to prove that $\otimes$-nuclearity is preserved by well-known C^*-algebra constructions. We close out the section by one of the consequences of the above theorem, for which Nathanial Brown notes in [3] that a C^*-algebra proof cannot be easily derived ("good luck" are his exact words).

Corollary 5.6. *Let $\mathcal{A}$ be a C^*-algebra with a closed two-sided ideal $\mathfrak{J}$. Then $\mathcal{A}$ is $\otimes$-nuclear if and only if $\mathfrak{J}$ and $\mathcal{A}/\mathfrak{J}$ are $\otimes$-nuclear.*

Proof. As $\mathcal{A}^{**} \cong \mathfrak{J}^{**} \oplus (\mathcal{A}/\mathfrak{J})^{**}$ by Proposition 3.14, the result follows from Propositions 5.7 and 5.1 and Theorem 5.5. □

5.2 The construction of semidiscrete von Neumann algebras

The purpose of the next two sections will be to show that the typical von Neumann algebra constructions preserve semidiscreteness, keeping in mind that we have to justify the "illegal" use of the statement that semidiscreteness is equivalent to injectivity (which we haven't yet proved) in the proof of Theorem 5.5 and Corollary 5.6. The constructions that we will investigate are therefore the same as in Section 4.2, but do not think that the statement have no reasons for existence in their own right. The proofs are in a way much more delicate than those in the aforementioned section, even though they are much longer.

Proposition 5.7. *Let $(\mathscr{M}_i)_{i\in I}$ be a family of von Neumann algebras. Then $\mathscr{M} = \bigoplus_{i\in I} \mathscr{M}_i$ is semidiscrete if and only if $\mathscr{M}_i$ is semidiscrete for all $i \in I$.*

Proof. We prove first that $\mathscr{M}$ semidiscrete implies that all $\mathscr{M}_i$ are semidiscrete. Let $\varphi\colon \mathscr{M} \to \mathscr{M}$ be a completely positive map such that $\varphi(1_{\mathscr{M}}) = 1_{\mathscr{M}}$. Let $i_0 \in I$ be fixed. Define $\mathscr{N} = \bigoplus_{i\in I} \mathscr{M}_{i_0}$, and let $\Delta\colon \mathscr{M}_{i_0} \to \mathscr{N}$ be the diagonal mapping given by $\Delta(T)(\xi_i)_{i\in I} = (T\xi_i)_{i\in I}$. Note that Δ is a

*-homomorphism and hence completely positive by Proposition 3.11, as well as normal by Corollary 2.51, satisfying $\Delta(1_{\mathscr{M}_{i_0}}) = 1_{\mathscr{N}}$. Furthermore let $\rho \in (\mathscr{M}_{i_0})_*$ be a fixed ultraweakly continuous state. Define maps $\theta_i \colon \mathscr{M}_{i_0} \to \mathscr{M}_i$ for all $i \in I$ by

$$\theta_i(T) = \begin{cases} T & \text{for } i = i_0 \\ \rho(T)1_{\mathscr{M}_i} & \text{for } i \neq i_0 \end{cases}$$

and yet another map $\theta = \bigoplus_{i\in I} \theta_i \colon \mathscr{N} \to \mathscr{M}$ by

$$\theta((T_i)_{i\in I}) = (\theta_i(T_i))_{i\in I}.$$

θ_i is completely positive and normal for all $i \in I$: in the case $i \neq i_0$ this is clear, and since θ_i is the composition of ρ and the clearly normal *-homomorphism $\mathbb{C} \to \mathscr{M}_i$ given by $\lambda \mapsto \lambda 1_{\mathscr{M}_i}$ for $i \neq i_0$, it should be clear as well. Thus θ is completely positive and normal by Corollary 3.17 and additionally satisfies $\theta(1_{\mathscr{N}}) = 1_{\mathscr{M}}$. Finally, let $\pi \colon \mathscr{M} \to \mathscr{M}_{i_0}$ be the projection of $\mathscr{M}$ onto $\mathscr{M}_{i_0}$. π is then a normal *-homomorphism by Proposition 2.52 and thus completely positive, and $\pi(1_{\mathscr{M}}) = 1_{\mathscr{M}_{i_0}}$. We can now define a map $\varphi_{i_0} \colon \mathscr{M}_{i_0} \to \mathscr{M}_{i_0}$ by

$$\varphi_{i_0} = \pi \circ \varphi \circ \theta \circ \Delta.$$

Hence if $\varphi \colon \mathscr{M} \to \mathscr{M}$ is normal, completely positive and has finite rank with $\varphi(1_{\mathscr{M}}) = 1_{\mathscr{M}}$, then the same holds for φ_{i_0}.

Assume now that $\mathscr{M}$ is semidiscrete and let $(\varphi_\alpha)_{\alpha\in A}$ be a net in $B(\mathscr{M})$ approximating the identity map on $\mathscr{M}$ as per Definition 5.1. On the grounds of what we just defined and proved, then for $i_0 \in I$ the net $((\varphi_\alpha)_{i_0})_{\alpha\in A}$ in $B(\mathscr{M}_{i_0})$ is a net of completely positive, normal maps of finite rank satisfying $(\varphi_\alpha)_{i_0}(1_{\mathscr{M}_{i_0}}) = 1_{\mathscr{M}_{i_0}}$ for all $\alpha \in A$. Additionally, for any given $\omega \in (\mathscr{M}_{i_0})_*$ define $\tilde{\omega} = \omega \circ \pi \in (\mathscr{M}_{i_0})_*$, using Proposition 2.45 as π is normal. By defining $\tilde{T} = \theta \circ \Delta(T) \in \mathscr{M}$ and noting that $\pi \circ \theta \circ \Delta = \mathrm{id}_{\mathscr{M}_{i_0}}$, then

$$\tilde{\omega}(\varphi_\alpha(\tilde{T}) - \tilde{T}) = \omega(\pi(\varphi_\alpha(\tilde{T}) - \tilde{T}))) = \omega((\varphi_\alpha)_{i_0}(T) - T).$$

As $|\tilde{\omega}(\varphi_\alpha(\tilde{T}) - \tilde{T})| \to 0$ by assumption, it follows that $\mathscr{M}_{i_0}$ is semidiscrete.

For the converse statement, assume that $\mathscr{M}_i$ is semidiscrete for all $i \in I$ and let $J \subseteq I$ be a finite subset of I. Define $J' = I \setminus J$ and let $\mathscr{N}_1 = \bigoplus_{i\in J} \mathscr{M}_i$ and $\mathscr{N}_2 = \bigoplus_{i\in J'} \mathscr{M}_i$. For any $i \in J$, let $\varphi_i \in B(\mathscr{M}_i)$ be a completely positive, normal mapping such that $\varphi_i(1_{\mathscr{M}_i}) = 1_{\mathscr{M}_i}$. Define $\varphi' \colon \mathscr{N}_1 \to \mathscr{N}_1$ by

$$\varphi'((T_i)_{i\in J}) = (\varphi_i(T_i))_{i\in J}, \quad (T_i)_{i\in J} \in \mathscr{N}_1.$$

Then φ' is normal and completely positive and $\varphi'(1_{\mathscr{N}_1}) = 1_{\mathscr{N}_1}$ by Corollary 3.17.

Choose an ultraweakly continuous state $\rho \in \mathscr{M}_*$, let $\theta_1 \colon \mathscr{M} \to \mathscr{N}_1$ and $\theta_2 \colon \mathscr{M} \to \mathscr{N}_2$ be the projections, and let $\vartheta_1 \colon \mathscr{N}_1 \to \mathscr{M}$ and $\vartheta_2 \colon \mathscr{N}_2 \to \mathscr{M}$ be the inclusions. θ_1, θ_2, ϑ_1 and ϑ_2 are normal and completely positive by Propositions 2.52 and 3.11 for all $i \in I$, since they are *-homomorphisms.

Now, define a map $\varphi \colon \mathscr{M} \to \mathscr{M}$ by

$$\varphi = \vartheta_1 \circ \varphi' \circ \theta_1 + \vartheta_2 \circ \theta_2 \circ \kappa \circ \rho,$$

where $\kappa \colon \mathbb{C} \to \mathscr{M}$ is defined by $\kappa(\lambda) = \lambda 1_{\mathscr{M}}$. Again κ is normal and completely positive. It then follows that φ is normal and completely positive, satisfying $\varphi(1_{\mathscr{M}}) = 1_{\mathscr{M}}$. Moreover, if the φ_i for all $i \in J$ have finite rank, then φ' also has finite rank, so as $\vartheta_2 \circ \theta_2 \circ \kappa \circ \rho$ has image contained in the linear span of $\vartheta_2(\theta_2(1_{\mathscr{M}}))$, it follows that φ has finite rank.

Now let $\varepsilon > 0$, $T^1, \ldots, T^n \in \mathscr{M}$ and $\omega^1, \ldots, \omega^n \in \mathscr{M}_*$ be given, where $T^p = (T_i^p)_{i\in I}$ with $T_i^p \in \mathscr{M}_i$ for all $i \in I$ and $p = 1, \ldots, n$. Since $\mathscr{M}_* \cong \bigoplus_{i\in I} (\mathscr{M}_i)_*$ by Proposition 2.57, ω^p corresponds to a family $(\omega_i^p)_{i\in I}$ where $\omega_i^p \in (\mathscr{M}_i)_*$ for all $i \in I$ and $p = 1, \ldots, n$ and

$$\|\omega^p\| = \sum_{i\in I} \|\omega_i^p\| < \infty, \quad p = 1, \ldots, n.$$

Hence there exists a finite subset $J \subseteq I$ such that

$$\sum_{i\notin J} \|\omega_i^p\| < \varepsilon, \quad p = 1, \ldots, n.$$

Let λ be the cardinality of J. For $i \in J$ we can by Proposition 5.2 find normal, completely positive maps $\varphi_i \in B(\mathscr{M}_i)$ of finite rank that satisfy $\varphi_i(1_{\mathscr{M}_i}) = 1_{\mathscr{M}_i}$, such that

$$|\omega_i^p(\varphi_i(T_i^p) - T_i^p)| < \frac{\varepsilon}{\lambda}, \quad i \in J,\ p = 1, \ldots, n.$$

By defining $\varphi \in B(\mathscr{M})$ by means of the φ_i for $i \in J$ as we did above, we then find that

$$\begin{aligned} |\omega^p(\varphi(T^p) - T^p)| &= \left| \sum_{i \in J} \omega_i^p(\varphi_i(T_i^p) - T_i^p) + \sum_{i \notin J} \omega_i^p(1_{\mathscr{M}_i})\rho(T^p) \right| \\ &\leq \sum_{i \in J} |\omega_i^p(\varphi_i(T_i^p) - T_i^p)| + |\rho(T^p)| \sum_{i \notin J} \|\omega_i^p\| \\ &< \varepsilon + \|T^p\|\varepsilon \\ &= (1 + \|T^p\|)\varepsilon \end{aligned}$$

for all $p = 1, \ldots, n$, it follows from Proposition 5.2 that $\mathscr{M}$ is semidiscrete. □

For the reduced von Neumann algebra case, note that if $T \in \mathscr{M}_P$ for a von Neumann algebra $\mathscr{M}$ and a projection $P \in \mathscr{M}$ and there is an $S \in \mathscr{M}$ such that $PS|_{P(\mathcal{H})} = T$, we have $PSP = PTP$. Hence PTP is an operator in $\mathscr{M}$ for all $T \in \mathscr{M}_P$.

Lemma 5.8. *Let $\mathcal{H}$ be a Hilbert space, let $\mathscr{M} \subseteq B(\mathcal{H})$ be a von Neumann algebra and let $P \in \mathscr{M}$ be a projection. Then the maps $\iota\colon \mathscr{M}_P \to \mathscr{M}$ and $\pi\colon \mathscr{M} \to \mathscr{M}_P$ given by*

$$\iota(S) = PSP, \quad \pi(T) = PT|_{P(\mathcal{H})}, \quad S \in \mathscr{M}_P,\ T \in \mathscr{M}$$

are normal and completely positive. Moreover, if $\mathscr{S} \subseteq \mathscr{M}$ is a finite-dimensional subspace, then $\pi(\mathscr{S})$ is finite-dimensional.

Proof. Note that ι and π are clearly linear, bounded and positive; the last property follows from the equations

$$\langle PSP\xi, \xi \rangle = \langle SP\xi, P\xi \rangle \geq 0, \quad \langle PT|_{P(\mathcal{H})}\eta, \eta \rangle = \langle T\eta, \eta \rangle \geq 0,$$

where $\xi \in \mathcal{H}$ and $\eta \in P(\mathcal{H})$, and $S \in \mathscr{M}_P$ and $T \in \mathscr{M}$ are positive.

We now check normality. Therefore, let $\omega_1 \in \mathscr{M}_*$ and $\omega_2 \in (\mathscr{M}_P)_*$ and pick square-summable sequences $(\xi_n')_{n\geq 1}$, $(\eta_n')_{n \geq 1}$ in $P(\mathcal{H})$ and $(\xi_n)_{n \geq 1}$, $(\eta_n)_{n\geq 1}$ in $\mathcal{H}$ such that

$$\omega_1(S) = \sum_{n=1}^{\infty} \langle S\xi_n', \eta_n' \rangle, \quad \omega_2(T) = \sum_{n=1}^{\infty} \langle T\xi_n, \eta_n \rangle, \quad S \in \mathscr{M}_P,\ T \in \mathscr{M},$$

using Theorem 2.7 and Proposition 2.2. As

$$\omega_1(\iota(S)) = \sum_{n=1}^{\infty} SP\xi_n, P\eta_n \rangle, \quad \omega_2(\pi(T)) = \sum_{n=1}^{\infty} \langle T\xi_n', \eta_n' \rangle, \quad S \in \mathscr{M}_P,\ T \in \mathscr{M},$$

it clearly follows that $\omega_1 \circ \iota \in (\mathscr{M}_P)_*$ and $\omega_2 \circ \pi \in \mathscr{M}_*$. Proposition 2.45 then tells us that ι and π are normal.

To see that ι and π are completely positive, let $n \geq 1$, let $S = (S_{ij})_{i,j=1}^n \in M_n(\mathscr{M}_P)$ and $T = (T_{ij})_{i,j=1}^n \in M_n(\mathscr{M})$ be positive matrices $\xi = (\xi_1, \ldots, \xi_n) \in \mathcal{H}^n$ and $\eta = (\eta_1, \ldots, \eta_n) \in P(\mathcal{H})^n$. Then

$$\langle \iota^{(n)}(S)\xi, \xi \rangle = \sum_{i,j=1}^{n} \langle PS_{ij}P\xi_j, \xi_i \rangle = \sum_{i,j=1}^{n} \langle S_{ij}P\xi_j, P\xi_i \rangle = \langle SP\xi, P\xi \rangle \geq 0,$$

where $P\xi = (P\xi_1, \ldots, P\xi_n) \in P(\mathcal{H})^n$ and

$$\langle \pi^{(n)}(T)\eta, \eta \rangle = \sum_{i,j=1}^{n} \langle T_{ij}\eta_j, \eta_i \rangle = \langle T\eta, \eta \rangle \geq 0.$$

Hence ι and π are completely positive.

For the final statement, assume that $\mathscr{S} \subseteq \mathscr{M}$ is a finite-dimensional subspace with a vector basis $T_1, \ldots, T_n \in \mathscr{S}$. Then it is clear that the operators $PT_i|_{P(\mathcal{H})} \in \mathscr{M}_P$ (where $i = 1, \ldots, n$) span the subspace $\pi(\mathscr{S})$; hence $\pi(\mathscr{S})$ is finite-dimensional. □

Proposition 5.9. *Let $\mathscr{M} \subseteq B(\mathcal{H})$ be a semidiscrete von Neumann algebra and let $P \in \mathscr{M}$ be a projection. Then the reduced von Neumann algebra $\mathscr{M}_P$ is semidiscrete.*

Proof. Let $\varepsilon > 0$, $T_1, \ldots, T_n \in \mathscr{M}$ and $\omega_1, \ldots, \omega_n \in \mathscr{M}_{P*}$ be given, and fix an ultraweakly continuous state $\rho \in \mathscr{M}_{P*}$. By Theorem 2.7 and Proposition 2.2, then for all $i = 1, \ldots, n$ we have square-summable sequences $(\xi_n^i)_{n\geq 1}$ and $(\eta_n^i)_{n\geq 1}$ in $P(\mathcal{H})$ such that

$$\omega_i(T) = \sum_{n=1}^{\infty} \langle T\xi_n^i, \eta_n^i \rangle, \quad T \in \mathscr{M}_P;$$

hence by defining $\omega_i' \colon \mathscr{M} \to \mathbb{C}$ for each i by

$$\omega_i'(T) = \sum_{n=1}^{\infty} \langle T\xi_n^i, \eta_n^i \rangle, \quad T \in \mathscr{M},$$

we obtain linear functionals $\omega_i' \in \mathscr{M}_*$; moreover, for all $T \in \mathscr{M}$, we have $\omega_i'(T) = \omega_i(PT|_{P(\mathcal{H})})$ and $\omega_i'(T) = \omega_i'(TP)$ for all $T \in \mathscr{M}$ since the sequences consist of elements of $P(\mathcal{H})$. Since $\mathscr{M}$ is semidiscrete, then Proposition 5.2 yields a normal, completely positive mapping $\varphi \in B(\mathscr{M})$ of finite rank with $\varphi(1_{\mathscr{M}}) = 1_{\mathscr{M}}$, additionally satisfying

$$\left|\omega_i'\left(\varphi(PT_iP + \rho(PT_i|_{P(\mathcal{H})})(1_{\mathscr{M}} - P)) - (PT_iP + \rho(PT_i|_{P(\mathcal{H})})(1_{\mathscr{M}} - P))\right)\right| < \varepsilon, \quad i = 1, \ldots, n,$$

i.e. we are approximating on the operators $PT_iP + \rho(T_i)(1_{\mathscr{M}} - P) \in \mathscr{M}$. Define $\psi \in B(\mathscr{M}_P)$ by

$$\psi(T) = P\varphi(PTP + \rho(T)(1_{\mathscr{M}} - P))|_{P(\mathcal{H})}, \quad T \in \mathscr{M}_P.$$

Since $P\tilde{T}P$ is uniquely determined by $T \in \mathscr{M}_P$, ψ is well-defined. Moreover, since $P1_{P(\mathcal{H})}P = P$, then $\psi(1_{P(\mathcal{H})}) = 1_{P(\mathcal{H})}$. ψ clearly has finite rank by Lemma 5.8. From the same lemma, we see that the map $\mathscr{M}_P \to \mathscr{M}$, $T \mapsto PTP$ is normal and completely positive. Since ρ is ultraweakly continuous and completely positive by Proposition 3.12 and the map $\mathbb{C} \to \mathscr{M}$ given by $\lambda \mapsto \lambda(1_{\mathscr{M}} - P)$ is clearly a normal *-homomorphism, then Propositions 3.11 and 3.5 yield that the map

$$T \mapsto PTP + \rho(T)(1_{\mathscr{M}} - P), \quad T \in \mathscr{M}_P,$$

is normal and completely positive. Hence Lemma 5.8 yields that the same holds for ψ. For the grand finale, then for all $i = 1, \ldots, n$ we have

$$\begin{aligned}
&\quad |\omega_i(\psi(PT_i|_{P(\mathcal{H})}) - PT_i|_{P(\mathcal{H})})| \\
&= \left|\omega_i\left(P\varphi(PT_iP + \rho(PT_i|_{P(\mathcal{H})})(1_{\mathscr{M}} - P))|_{P(\mathcal{H})} - PT_i|_{P(\mathcal{H})}\right)\right| \\
&= \left|\omega_i'\left(\varphi(PT_iP + \rho(PT_i|_{P(\mathcal{H})})(1_{\mathscr{M}} - P)) - PT_iP\right)\right| \\
&= \left|\omega_i'\left(\varphi(PT_iP + \rho(PT_i|_{P(\mathcal{H})})(1_{\mathscr{M}} - P)) - (PT_iP + \rho(PT_i|_{P(\mathcal{H})})(1_{\mathscr{M}} - P))P\right)\right| \\
&= \left|\omega_i'\left(\varphi(PT_iP + \rho(PT_i|_{P(\mathcal{H})})(1_{\mathscr{M}} - P)) - (PT_iP + \rho(PT_i|_{P(\mathcal{H})})(1_{\mathscr{M}} - P))\right)\right| \\
&< \varepsilon.
\end{aligned}$$

Hence $\mathscr{M}_P$ is semidiscrete by Proposition 5.2. □

Proposition 5.10. *Let $\mathscr{M}$ and $\mathscr{N}$ be von Neumann algebras. Then $\mathscr{M} \overline{\otimes} \mathscr{N}$ is semidiscrete if and only if $\mathscr{M}$ and $\mathscr{N}$ are semidiscrete.*

Proof. Assume that $\mathscr{M}$ and $\mathscr{N}$ are semidiscrete. If $\varphi \in B(\mathscr{M})$ and $\psi \in B(\mathscr{N})$ are normal, completely positive mappings, then there exists a normal, completely positive map $\varphi \otimes \psi \in B(\mathscr{M} \overline{\otimes} \mathscr{N})$ satisfying

$$\varphi \otimes \psi(S \otimes T) = \varphi(S) \otimes \psi(T)$$

for all $S \in \mathscr{M}$ and $T \in \mathscr{N}$ by Corollary 3.18. Moreover, if φ and ψ satisfy $\varphi(1_{\mathscr{M}}) = 1_{\mathscr{M}}$ and $\psi(1_{\mathscr{N}}) = 1_{\mathscr{N}}$, then $(\varphi \otimes \psi)(1_{\mathscr{M} \overline{\otimes} \mathscr{N}}) = 1_{\mathscr{M}} \otimes 1_{\mathscr{N}} = 1_{\mathscr{M} \overline{\otimes} \mathscr{N}}$, and if φ and ψ are of finite rank, then $\varphi \otimes \psi$ has finite rank as well, as finite-dimensional subspaces of $B(\mathscr{M} \overline{\otimes} \mathscr{N})$ are ultraweakly closed [14, Theorem 1.2.17]. Moreover, if φ and ψ are of the above form, then $\|\varphi \otimes \psi\| = 1$ by Proposition 3.9.

Let $\varepsilon > 0$, $T_1, \ldots, T_n \in \mathscr{M} \overline{\otimes} \mathscr{N}$ and $\omega_1, \ldots, \omega_n \in (\mathscr{M} \overline{\otimes} \mathscr{N})_*$ be given. We will show that φ and ψ with the properties above can be chosen in a way such that

$$|\omega_i((\varphi \otimes \psi)T_i - T_i)| < \varepsilon, \quad i = 1, \ldots, n.$$

Suppose first that we have found $\varphi \otimes \psi$ such that the above inequality holds where all ω_i are of the form $\alpha_i \otimes \beta_i$ for $\alpha_i \in \mathscr{M}_*$ and $\beta_i \in \mathscr{N}_*$ (see page 76 for an explanation of their construction). For new $\varepsilon > 0$, $T_1, \ldots, T_n \in \mathscr{M} \overline{\otimes} \mathscr{N}$ and $\omega_1, \ldots, \omega_n \in (\mathscr{M} \overline{\otimes} \mathscr{N})_*$, let $M = \max\{\|T_i\| \mid i = 1, \ldots, n\}$. Because $\mathscr{M}_* \odot \mathscr{N}_*$ is norm-dense in $(\mathscr{M} \overline{\otimes} \mathscr{N})_*$, we have

$$\left\| \omega_i - \sum_{j=1}^{m_i} \alpha_i^j \otimes \beta_i^j \right\| < \frac{\varepsilon}{4M}$$

for appropriately chosen $\alpha_i^j \in \mathscr{M}_*$ and $\beta_i^j \in \mathscr{N}_*$, where $j = 1, \ldots, m_i$, $i = 1, \ldots, n$. Letting $m = \sum_{i=1}^n m_i$ and choosing $\varphi \in B(\mathscr{M})$ and $\psi \in B(\mathscr{N})$ with the above properties such that

$$|(\alpha_i^j \otimes \beta_i^j)((\varphi \otimes \psi)(T_i) - T_i)| < \frac{\varepsilon}{2m}, \quad i = 1, \ldots, n, \ j = 1, \ldots, m_i,$$

we then have for all $i = 1, \ldots, n$ that

$$\begin{aligned} |\omega_i((\varphi \otimes \psi)T_i - T_i)| &\leq \left\| \omega_i - \sum_{j=1}^{m_i} \alpha_i^j \otimes \beta_i^j \right\| \|(\varphi \otimes \psi)T_i - T_i\| + \sum_{j=1}^{m_i} \|(\alpha_i^j \otimes \beta_i^j)(\varphi \otimes \psi)T_i - T_i\| \\ &\leq \frac{\varepsilon}{4M} \cdot 2M + m_i \cdot \frac{\varepsilon}{2m} \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

proving the result in the general case, so that by Proposition 5.2, $\mathscr{M} \overline{\otimes} \mathscr{N}$ is semidiscrete.

To prove the result for ω_i of the form $\alpha_i \otimes \beta_i$ for $\alpha_i \in \mathscr{M}_*$ and $\beta_i \in \mathscr{N}_*$, note that since $\mathscr{M}$ and $\mathscr{N}$ are semidiscrete, Corollary 5.4 yields normal, completely positive maps $\varphi \in B(\mathscr{M})$ and $\psi \in B(\mathscr{N})$ of finite rank with $\varphi(1_{\mathscr{M}}) = 1_{\mathscr{M}}$ and $\psi(1_{\mathscr{N}}) = 1_{\mathscr{N}}$ such that

$$\|\alpha_i \circ \varphi - \alpha_i\| < \frac{\varepsilon}{2K}, \quad \|\beta_i \circ \psi - \beta_i\| < \frac{\varepsilon}{2K}, \quad i = 1, \ldots, n,$$

where

$$K = \max\{\|\alpha_i\| + \|\beta_i\| \mid i = 1, \ldots, n\} \cdot \max\{\|T_i\| \mid i = 1, \ldots, n\}.$$

Note that for $S \in \mathscr{M}$ and $T \in \mathscr{N}$, then

$$\alpha_i \otimes \beta_i((\varphi \otimes \psi)(S \otimes T)) = \alpha_i(\varphi(S))\beta_i(\psi(T)) = (\alpha_i \circ \varphi) \otimes (\beta_i \circ \psi)(S \otimes T),$$

so by uniqueness, the ultraweakly continuous functionals $(\alpha_i \otimes \beta_i) \circ (\varphi \otimes \psi)$ and $(\alpha_i \circ \varphi) \otimes (\beta_i \circ \psi)$ are equal for all $i = 1, \ldots, n$, with the help of Proposition 2.45. Hence for all $i = 1, \ldots, n$, we see that

$$\begin{aligned} |(\alpha_i \otimes \beta_i)((\varphi \otimes \psi)T_i - T_i)| &\leq \|T_i\| \|(\alpha_i \circ \varphi) \otimes (\beta_i \circ \psi) - (\alpha_i \otimes \beta_i)\| \\ &\leq \|T_i\| \left(\|(\alpha_i \circ \varphi) \otimes (\beta_i \circ \psi) - (\alpha_i \circ \varphi) \otimes \beta_i\| + \|(\alpha_i \circ \varphi) \otimes \beta_i - (\alpha_i \otimes \beta_i)\| \right) \\ &\leq \|T_i\| \left(\|\alpha_i\| \|\beta_i \circ \psi - \beta_i\| + \|\alpha_i \circ \varphi - \alpha_i\| \|\beta_i\| \right) \\ &< \frac{\varepsilon}{2K} \|T_i\| (\|\alpha_i\| + \|\beta_i\|) \\ &\leq \varepsilon, \end{aligned}$$

completing the proof of the first implication.

Assume now that $\mathscr{M} \mathbin{\overline{\otimes}} \mathscr{N}$ is semidiscrete. Let $\mathrm{id}_{\mathscr{M}} \colon \mathscr{M} \to \mathscr{M}$ be the identity map and let $\rho \in \mathscr{N}_*$ be a fixed ultraweakly normal state. By Proposition 3.12 and Corollary 3.18, there exists a normal, completely positive map $\mathrm{id}_{\mathscr{M}} \otimes \rho \colon \mathscr{M} \mathbin{\overline{\otimes}} \mathscr{N} \to \mathscr{M} \mathbin{\overline{\otimes}} \mathbb{C}$ such that

$$(\mathrm{id}_{\mathscr{M}} \otimes \rho)(S \otimes T) = S \otimes \rho(T), \quad S \in \mathscr{M},\ T \in \mathscr{N}.$$

As $\mathscr{M} \mathbin{\overline{\otimes}} \mathbb{C} = \mathscr{M} \otimes \mathbb{C}$ by Lemma 1.34, then by letting $\pi \colon \mathscr{M} \to \mathscr{M} \otimes \mathbb{C}$ be given by $\pi(T) = T \otimes 1$ it is clear that π is a *-isomorphism. Letting $J \colon \mathscr{M} \to \mathscr{M} \mathbin{\overline{\otimes}} \mathscr{N}$ be given by $J(T) = T \otimes 1_{\mathscr{N}}$, it is clear from the tensor product operator calculus (see page 8) that J is a *-homomorphism. Hence if $\varphi \in B(\mathscr{M} \mathbin{\overline{\otimes}} \mathscr{N})$ is normal and completely positive, it defines a normal and completely positive map $\varphi' \colon \mathscr{M} \to \mathscr{M}$ given by $\varphi' = \pi^{-1} \circ (\mathrm{id}_{\mathscr{M}} \otimes \rho) \circ \varphi \circ J$, using Proposition 3.11. If φ has finite rank and $\varphi(1_{\mathscr{M} \overline{\otimes} \mathscr{N}}) = 1_{\mathscr{M} \overline{\otimes} \mathscr{N}}$, then clearly φ' also has finite rank and maps $1_{\mathscr{M}}$ to $1_{\mathscr{M}}$.

Let $\beta \colon \mathbb{C} \to \mathbb{C}$ denote the identity. For any positive functional $\omega \in \mathscr{M}_*$, Corollary 3.18 yields a positive linear functional $\omega \otimes \beta \colon \mathscr{M} \mathbin{\overline{\otimes}} \mathbb{C} \to \mathbb{C}$. Since

$$(\omega \otimes \beta) \circ (\mathrm{id}_{\mathscr{M}} \otimes \rho)(S \otimes T) = \omega(S)\rho(T) = (\omega \otimes \rho)(S \otimes T)$$

for all $S \in \mathscr{M}$ and $T \in \mathscr{N}$, and $(\omega \otimes \beta) \circ (\mathrm{id}_{\mathscr{M}} \otimes \rho) \in (\mathscr{M} \mathbin{\overline{\otimes}} \mathscr{N})_*$, it follows from uniqueness of $\omega \otimes \rho$, as noted in the remark before Proposition 3.19, that

$$(\omega \otimes \beta) \circ (\mathrm{id}_{\mathscr{M}} \otimes \rho) = \omega \otimes \rho.$$

Using the fact that any $\omega \in \mathscr{M}_*$ is a linear combination of four positive ultraweakly continuous linear functionals (Theorem 2.40), the above equality holds for arbitrary $\omega \in \mathscr{M}_*$. For any given $S \in \mathscr{M}$, note that

$$(\mathrm{id}_{\mathscr{M}} \otimes \rho)(\varphi(T \otimes 1_{\mathscr{N}})) = \pi(\varphi'(T)) = \varphi'(T) \otimes 1 = (\mathrm{id}_{\mathscr{M}} \otimes \rho)(\varphi'(T) \otimes 1_{\mathscr{N}}).$$

Hence if any $\omega \in \mathscr{M}_*$ is given additionally, one sees that

$$\begin{aligned}
(\omega \otimes \rho)(\varphi(T \otimes 1_{\mathscr{N}}) - T \otimes 1_{\mathscr{N}}) &= ((\omega \otimes \beta) \circ (\mathrm{id}_{\mathscr{M}} \otimes \rho))(\varphi(T \otimes 1_{\mathscr{N}}) - T \otimes 1_{\mathscr{N}}) \\
&= ((\omega \otimes \beta) \circ (\mathrm{id}_{\mathscr{M}} \otimes \rho))(\varphi'(T) \otimes 1_{\mathscr{N}} - T \otimes 1_{\mathscr{N}}) \\
&= (\omega \otimes \rho)((\varphi'(T) - T) \otimes 1_{\mathscr{N}}) \\
&= \omega(\varphi'(T) - T)\rho(1_{\mathscr{N}}) \\
&= \omega(\varphi'(T) - T).
\end{aligned}$$

Now, given $\varepsilon > 0$, $\omega_1, \dots, \omega_n \in \mathscr{M}_*$ and $T_1, \dots, T_n \in \mathscr{M}$, then by Proposition 5.2, the semidiscreteness of $\mathscr{M} \mathbin{\overline{\otimes}} \mathscr{N}$ yields a normal, completely positive map $\varphi \in B(\mathscr{M} \mathbin{\overline{\otimes}} \mathscr{N})$ of finite rank, satisfying $\varphi(1_{\mathscr{M} \overline{\otimes} \mathscr{N}}) = 1_{\mathscr{M} \overline{\otimes} \mathscr{N}}$ and

$$|(\omega_i \otimes \rho)(\varphi(T_i \otimes 1_{\mathscr{N}}) - T_i \otimes 1_{\mathscr{N}})| < \varepsilon, \quad i = 1, \dots, n.$$

As proved before, φ induces a map $\varphi' \in B(\mathscr{M})$ as above, satisfying

$$|\omega_i(\varphi'(T_i) - T_i)| = |(\omega_i \otimes \rho)(\varphi(T_i \otimes 1_{\mathscr{N}}) - T_i \otimes 1_{\mathscr{N}})| < \varepsilon, \quad i = 1, \dots, n.$$

Hence $\mathscr{M}$ is semidiscrete by Proposition 5.2. In a similar manner, one sees that $\mathscr{N}$ is semidiscrete. □

Proposition 5.11. *The von Neumann algebra $B(\mathcal{H})$ is semidiscrete for any Hilbert space $\mathcal{H}$.*

Proof. Let $(\mathcal{H}_\alpha)_{\alpha \in A}$ be the family of all finite-dimensional subspaces of $\mathcal{H}$, and for each $\alpha \in A$, let P_α be the orthogonal projection onto $\mathcal{H}_\alpha$. We make A into a directed set by defining $\alpha \leq \beta$ for $\alpha, \beta \in A$ if and only if $\mathcal{H}_\alpha \subseteq \mathcal{H}_\beta$ or equivalently $P_\alpha \leq P_\beta$. Let $\rho \in B(\mathcal{H})_*$ be a fixed normal state. For $\alpha \in A$, define a map $\varphi_\alpha \in B(B(\mathcal{H}))$ by

$$\varphi_\alpha(T) = P_\alpha T P_\alpha + \rho(T)(1_{\mathcal{H}} - P_\alpha), \quad T \in B(\mathcal{H}).$$

This map is normal and completely positive which can be deduced as follows. Clearly the map $T \mapsto \rho(T)(1_{\mathcal{H}} - P_\alpha)$ for $T \in B(\mathcal{H})$ is completely positive by Propositions 3.11 and 3.12 as it is the composition of the positive functional ρ and the *-homomorphism $\mathbb{C} \to B(\mathcal{H})$ given by $\lambda \mapsto \lambda(1_{\mathcal{H}} - P_\alpha)$. Moreover, the aforementioned *-homomorphism is normal and ρ is ultraweakly continuous by assumption, so normality follows. The map $T \mapsto P_\alpha T P_\alpha$ is clearly ultraweakly-to-ultraweakly continuous and hence normal. To see that it is completely positive, let $n \geq 1$ and let $T = (T_{ij})_{i,j=1}^n \in M_n(B(\mathcal{H}))$ be a positive matrix and note that

$$\begin{pmatrix} P_\alpha T_{11} P_\alpha & P_\alpha T_{12} P_\alpha & \cdots & P_\alpha T_{1n} P_\alpha \\ P_\alpha T_{21} P_\alpha & P_\alpha T_{22} P_\alpha & \cdots & P_\alpha T_{2n} P_\alpha \\ \vdots & \vdots & & \vdots \\ P_\alpha T_{n1} P_\alpha & P_\alpha T_{n2} P_\alpha & \cdots & P_\alpha T_{nn} P_\alpha \end{pmatrix} = \begin{pmatrix} P_\alpha & 0 & \cdots & 0 \\ 0 & P_\alpha & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & P_\alpha \end{pmatrix}^* T \begin{pmatrix} P_\alpha & 0 & \cdots & 0 \\ 0 & P_\alpha & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & P_\alpha \end{pmatrix}$$

is positive. This implies complete positivity of $T \mapsto P_\alpha T P_\alpha$. The sum of the two maps is normal and completely positive as well.

We note that φ_α has finite rank and $\varphi_\alpha(1_\mathcal{H}) = 1_\mathcal{H}$. Let $\omega \in B(\mathcal{H})_*$ and $T \in B(\mathcal{H})$, we know from Proposition 2.2 that $\omega = \sum_{i=1}^\infty \omega_{\xi_i,\eta_i}$ for sequences $(\xi_i)_{i\geq 1}$ and $(\eta_i)_{i\geq 1}$ in $\mathcal{H}$ such that $\sum_{i=1}^\infty \|\xi_i\|^2 < \infty$ and $\sum_{i=1}^\infty \|\eta_i\|^2 < \infty$. Let $\varepsilon > 0$, choose $n \geq 1$ such that

$$\sum_{i=n+1}^\infty \|\xi_i\|^2 < \sqrt{\frac{\varepsilon}{3\|T\|}}, \qquad \sum_{i=n+1}^\infty \|\eta_i\|^2 < \sqrt{\frac{\varepsilon}{3\|T\|}}$$

and choose $\alpha_0 \in A$ such that $\xi_i \in P_{\alpha_0}(\mathcal{H})$ and $T\xi_i \in P_{\alpha_0}(\mathcal{H})$ for all $i = 1, \ldots, n$. Then for all $\alpha \geq \alpha_0$ we find that

$$\begin{aligned} |\omega(\varphi_\alpha(T) - T)| &= \left| \sum_{i=1}^\infty \langle (P_\alpha T P_\alpha + \rho(T)(1 - P_\alpha) - T)\xi_i, \eta_i \rangle \right| \\ &= \left| \sum_{i=n+1}^\infty \langle (P_\alpha T P_\alpha + \rho(T)(1 - P_\alpha) - T)\xi_i, \eta_i \rangle \right| \\ &\leq 3\|T\| \sum_{i=n+1}^\infty \|\xi_i\|\|\eta_i\| \\ &\leq 3\|T\| \left[\sum_{i=n+1}^\infty \|\xi_i\|^2 \right]^{1/2} \left[\sum_{i=n+1}^\infty \|\eta_i\|^2 \right]^{1/2} \\ &< \varepsilon, \end{aligned}$$

so that $\lim_{\alpha\in A} \omega(\varphi_\alpha(T)) = \lim_{\alpha \in A} \omega(T)$. Hence $B(\mathcal{H})$ is semidiscrete. □

The last theorem will, regrettably, be stated without proof. It contains one of the most useful conditions equivalent to semidiscreteness, and the proofs of the next section depend extremely much on it.

Theorem 5.12. *Let $\mathscr{M}$ be a von Neumann algebra. Then the following are equivalent:*

(i) *$\mathscr{M}$ is semidiscrete.*

(ii) *For any $S_1, \ldots, S_n \in \mathscr{M}$ and $T_1, \ldots, T_n \in \mathscr{M}'$ it holds that*

$$\left\| \sum_{i=1}^n S_i T_i \right\| \leq \left\| \sum_{i=1} S_i \otimes T_i \right\|_{\min},$$

i.e. the map $\eta\colon \mathscr{M} \odot \mathscr{M}' \to B(\mathcal{H})$ satisfying $\eta(S \otimes T) = ST$ is contractive with respect to the minimal norm.

In particular, $\mathscr{M}$ is semidiscrete if and only if $\mathscr{M}'$ is semidiscrete.

Proof. Omitted. See [28, Theorem 4.11]. □

5.3 The equivalence of semidiscreteness and injectivity

We now embark on perhaps the most deep collection of theorems that this project can offer – just in time. The purpose here is, true to the title, to prove that semidiscreteness is equivalent to injectivity, and the proof takes us through results of all the previous chapters. Hilbert-Schmidt operators and continuous crossed products also make a surprise visit, only emphasizing that the proof is tremendously nontrivial.

Before going to the first big result, we will prove a nice result concerning the state space of the C^*-algebra $B(\mathcal{H})$.

Lemma 5.13. *Let $\mathcal{H}$ be a Hilbert space. Then the set of weakly continuous states is weak*-dense in $S(B(\mathcal{H}))$. "Weakly" can be replaced by "ultraweakly".*

Proof. Let $T \in B(\mathcal{H})_{\mathrm{sa}}$ and note that $\langle T\xi, \xi\rangle \geq 0$ for all $\xi \in \mathcal{H}$ with $\|\xi\| = 1$ implies $T \geq 0$. Then it follows from Proposition 0.7 and Lemma A.4 that any $\varphi \in S(B(\mathcal{H}))$ is the weak*-limit of states of the form $\omega_{\xi_1} + \ldots + \omega_{\xi_n}$ for $\xi_1, \ldots, \xi_n \in \mathcal{H}$. The last statement follows immediately. □

It should be noted now that if a von Neumann algebra is finite (Definition 4.8) and σ-finite (Definition 2.11) then there exists a normal faithful tracial state $\tau\colon \mathscr{M} \to \mathbb{C}$. A proof is given in [1, Corollary III.2.5.8].

Proposition 5.14. *Let $\mathscr{M} \subseteq B(\mathcal{H})$ be a finite, σ-finite and injective von Neumann algebra with a normal faithful tracial state τ on $\mathscr{M}$ (see e.g. [1, Corollary III.2.5.8]). Let $S_1, \ldots, S_n, T_1, \ldots, T_n \in \mathscr{M}$ and $\varepsilon > 0$ be given. Then there is a positive Hilbert-Schmidt operator $P \in B(\mathcal{H})$ such that*

(i) $\|P\|_2 = 1$;
(ii) $|\langle S_iP, P\rangle_2 - \tau(S_i)| \leq \varepsilon$ *for all* $i = 1, \ldots, n$;
(iii) $\|T_iP - PT_i\|_2 \leq \varepsilon$ *for all* $i = 1, \ldots, n$.

Proof. Since every element in $\mathscr{M}$ is a linear combination of unitary operators, we may as well assume that $T_1, \ldots, T_n$ are unitaries (after proving the result for unitaries, the general case follows from choosing a smaller ε than the one given).

Let $E\colon B(\mathcal{H}) \to \mathscr{M}$ be a projection of norm 1. Then $\tau \circ E$ is a state on $B(\mathcal{H})$. By Lemma 5.13, we can find a net $(\varphi_\alpha)_{\alpha\in A}$ of normal states on $B(\mathcal{H})$ such that $\varphi_\alpha \to \tau \circ E$ in the weak* topology. For all $i = 1, \ldots, n$ and $\alpha \in A$, we have $T_i \cdot \varphi_\alpha - \varphi_\alpha \cdot T_i \in B(\mathcal{H})_*$ by Lemma 2.36 and Theorem 2.40, and hence for $S \in B(\mathcal{H})$ we find that

$$(T_i \cdot \varphi_\alpha - \varphi_\alpha \cdot T_i)(S) = \varphi_\alpha(ST_i - T_iS) \to \tau(E(ST_i - T_iS)) = \tau(E(S)T_i - T_iE(S)) = 0$$

by Tomiyama's theorem and the fact that τ is a trace. Therefore $T_i \cdot \varphi_\alpha - \varphi_\alpha \cdot T_i \to 0$ in the weak topology on $B(\mathcal{H})_*$, since the space of bounded linear functionals on $B(\mathcal{H})_*$ can be identified with $B(\mathcal{H})$ by Proposition 2.5. Note that by "weak topology", we do *not* mean "weak operator topology" but the coarsest topology such that all bounded linear functionals on $B(\mathcal{H})_*$ are continuous.

We now consider the vector space $V = (B(\mathcal{H})_*)^n \oplus B(\mathcal{H})^*$ equipped with two topologies σ_1 and σ_2. They are given as follows:

- σ_1 is the product of the weak topology on $B(\mathcal{H})_*$ and the weak* topology on $B(\mathcal{H})^*$.
- σ_2 is the product of the norm topology on $B(\mathcal{H})_*$ and the weak* topology on $B(\mathcal{H})^*$.

Note that σ_2 is finer than σ_1. Any linear functional ω on V that is continuous with respect to σ_1 can be written in the form

$$\omega(\psi_1, \ldots, \psi_{n+1}) = \sum_{i=1}^{n+1} \omega_i(\psi_i), \quad \psi_1, \ldots, \psi_n \in B(\mathcal{H})_*,\ \psi_{n+1} \in B(\mathcal{H})^*$$

where $\omega_i\colon B(\mathcal{H})_* \to \mathbb{C}$ for $i = 1, \ldots, n$ are linear functionals on $B(\mathcal{H})_*$ continuous with respect to the weak topology and $\omega_{n+1}\colon B(\mathcal{H})^* \to \mathbb{C}$ is a weak*-continuous linear functional. This can formulated similarly for functionals that are continuous with respect to σ_2. Because $(B(\mathcal{H})_*)^*$ precisely consists of the linear functionals that are continuous with respect to the weak topology on $B(\mathcal{H})_*$, it follows that the set of continuous linear functionals on (V,σ_1) and (V,σ_2) are the same. Hence it follows for any convex set $\mathscr{S} \subseteq V$, that the closures of $\mathscr{S}$ under σ_1 and σ_2 are equal (see e.g. [22, Theorems 3.10 and 3.12]).

Define $\Phi_\alpha \in V$ for $\alpha \in A$ and $\Phi \in V$ by

$$\Phi_\alpha = (T_1 \cdot \varphi_\alpha - \varphi_\alpha \cdot T_1, \ldots, T_n \cdot \varphi_\alpha - \varphi_\alpha \cdot T_n, \varphi_\alpha), \quad \Phi = (0, \ldots, 0, \tau \circ E),$$

and let $\mathscr{S}$ be the convex hull of the set of all Φ_α for $\alpha \in A$, i.e. the set of finite convex combinations of Φ_α's. Because $\mathscr{S} \subseteq V$ is convex and Φ is contained in the closure of $\mathscr{S}$ in the σ_1 topology, it is also contained in the closure of $\mathscr{S}$ in the σ_2 topology. Let $\alpha_1, \ldots, \alpha_n \in A$ and $\lambda_1, \ldots, \lambda_n \in [0,1]$ such that $\sum_{i=1}^n \lambda_i = 1$. If $\varphi = \sum_{i=1}^n \lambda_i \varphi_{\alpha_i}$, then any element of $\mathscr{S}$ is of the form

$$\lambda_1 \Phi_{\alpha_1} + \ldots + \lambda_n \Phi_{\alpha_n} = (T_1 \cdot \varphi - \varphi \cdot T_1, \ldots, T_n \cdot \varphi - \varphi \cdot T_n, \varphi).$$

As Φ is now contained in the closure of $\mathscr{S}$ in the σ_2 topology, the above observation yields a net $(\psi_\beta) \subseteq B(\mathcal{H})_*$ of convex combinations of various φ_α such that $\|T_i \cdot \psi_\beta - \psi_\beta \cdot T_i\| \to 0$ for all $i = 1, \ldots, n$ and $\psi_\beta \to \tau \circ E$ in the weak*-topology on $B(\mathcal{H})^*$. That is not all, however: since all φ_α are ultraweakly continuous states and $S(B(\mathcal{H}))$ is convex, it follows that all ψ_β are ultraweakly continuous states as well. The existence of the above net thus yields a state $\psi \in B(\mathcal{H})_*$ such that $\|T_i \cdot \psi - \psi \cdot T_i\| \leq \varepsilon^2$ and $|\psi(S_i) - \tau(E(S_i))| < \varepsilon$ for all $i = 1, \ldots, n$. By Theorem B.16, there is a unique positive trace class operator $R \in \mathscr{T}(\mathcal{H})$ with $\|R\|_1 = \|\psi\| = 1$ and $\psi(T) = \mathrm{tr}\,(RT)$ for all $T \in B(\mathcal{H})$.

We now claim that $P = R^{1/2}$ is the desired Hilbert-Schmidt operator. P is a positive Hilbert-Schmidt operator by Proposition B.8, with

$$\|P\|_2 = \|R\|_1^{1/2} = 1.$$

Furthermore, for all $T \in B(\mathcal{H})$, TP is Hilbert-Schmidt by Proposition B.7 and

$$\langle TP, P \rangle_2 = \mathrm{tr}\,(PTP) = \mathrm{tr}\,(P^2 T) = \mathrm{tr}\,(RT) = \psi(T)$$

by Corollary B.15. In particular,

$$|\langle S_i P, P \rangle_2 - \tau(S_i)| = |\psi(S_i) - \tau(E(S_i))| < \varepsilon$$

for all $i = 1, \ldots, n$. Hence P satisfies the conditions (i) and (ii).

Let $i = 1, \ldots, n$. Since T_i is unitary, we have $\|T_i P - P T_i\|_2 = \|T_i P T_i^* - P\|_2$ by Proposition B.7. The Powers-Størmer inequality (Proposition B.21) then yields

$$\|T_i P T_i^* - P\|_2 \leq \|T_i P^2 T_i^* - P^2\|_1^{1/2},$$

as $T_i P^2 T_i^* = (T_i P T_i^*)^2$. Moreover, $T_i P^2 T_i^*$ is a trace class operator, so

$$\mathrm{tr}\,(T_i P^2 T_i^* T) = \mathrm{tr}\,(R T_i T T_i^*) = \psi(T_i T T_i^*) = (T_i^* \cdot \psi \cdot T_i)(T), \quad T \in B(\mathcal{H}).$$

As $T_i^* \cdot \psi \cdot T_i \in B(\mathcal{H})_*$ by Lemma 2.36 and Theorem 2.40, we then have

$$(T_i^* \cdot \psi \cdot T_i - \psi)(T) = \mathrm{tr}\,((T_i P^2 T_i^* - P^2)T), \quad T \in B(\mathcal{H}).$$

Therefore $\|T_i^* \cdot \psi \cdot T_i - \psi\| = \|T_i^* P^2 T_i - P^2\|_1$ by Theorem B.16, so

$$\|T_i P^2 T_i^* - P^2\|_1^{1/2} = \|T_i^* \cdot \psi \cdot T_i - \psi\|^{1/2} = \|T_i^* \cdot (\psi \cdot T_i) - T_i^* \cdot (T_i \cdot \psi)\|^{1/2} \leq \|\psi \cdot T_i - T_i \cdot \psi\|^{1/2} < \varepsilon.$$

Hence

$$\|T_i P - P T_i\|_2 = \|T_i P T_i^* - P\|_2 \leq \|T_i P^2 T_i^* - P^2\|_1^{1/2} < \varepsilon$$

for all $i = 1, \ldots, n$, so (iii) is satisfied as well. □

In the next big theorem, we will need the notion of a conjugate von Neumann algebra about which Section B.2 should provide sufficient information. The reason that we do not relegate the following lemma (in its own right, it is really a theorem) to that section is that we might as well keep all results concerning tracial states together, as well as the fact that it lays some of the groundwork for a branch of von Neumann algebra theory called *Tomita-Takesaki theory*. We merely seek to underline its importance by putting it here with the other very serious theorems of this section.

Lemma 5.15 (The commutation theorem)**.** *Let $\mathscr{M}$ be a von Neumann algebra allowing for a faithful normal tracial state $\tau\colon \mathscr{M} \to \mathbb{C}$. Then there exists a Hilbert space $\mathcal{H}$ such that $\mathscr{M}$ is *-isomorphic to a von Neumann algebra $\mathscr{N} \subseteq B(\mathcal{H})$, a cyclic vector $\xi_0 \in \mathcal{H}$ for $\mathscr{N}$ and a conjugate linear isometry $J\colon \mathcal{H} \to \mathcal{H}$ that satisfies*

(i) $J^2 = 1_{\mathcal{H}}$;
(ii) *the map $\alpha\colon B(\mathcal{H}) \to B(\mathcal{H})$ given by $\alpha(T) = JTJ$ is a well-defined conjugate linear isomorphism;*
(iii) *$\alpha|_{\mathscr{N}}$ is a conjugate linear unital *-algebra homomorphism.*
(iv) *$J\mathscr{N}J = \mathscr{N}'$, so $\alpha|_{\mathscr{N}}$ is in fact an isomorphism onto $\mathscr{N}'$;*
(v) *if $\overline{\mathscr{N}} \subseteq B(\overline{\mathcal{H}})$ denotes the conjugate von Neumann algebra of $\mathscr{N}$, then the map $\overline{\mathscr{N}} \to \mathscr{N}'$ given by $\overline{T} \mapsto JTJ$ for $T \in \mathscr{N}$ is a unital *-isomorphism.*

Proof. By (the proof of) Proposition 2.58, the GNS triple $(\pi, \mathcal{H}, \xi_0)$ corresponding to τ consists of a faithful normal representation $\pi\colon \mathscr{M} \to B(\mathcal{H})$, and moreover ξ_0 is a cyclic and separating vector for the von Neumann algebra $\pi(\mathscr{M})$. Let $\mathscr{N} = \pi(\mathscr{M})$.

Because ξ_0 is cyclic, $\mathscr{N}\xi_0$ is dense in $\mathcal{H}$, and because ξ_0 is separating, the map $\mathscr{N} \to \mathscr{N}\xi_0$ given by $T \mapsto T\xi_0$ is a bijection. Hence we can transfer the *-algebra structure of $\mathscr{N}$ to $\mathscr{N}\xi_0$ by defining

$$(S\xi_0)(T\xi_0) := (ST)\xi_0, \quad (T\xi_0)^* := T^*\xi_0, \quad S, T \in \mathscr{N}.$$

Since

$$\|T\xi_0\|^2 = \langle T\xi_0, T\xi_0\rangle = \langle T^*T\xi_0, \xi_0\rangle = \tau(T^*T) = \tau(TT^*) = \langle TT^*\xi_0, \xi_0\rangle = \|T^*\xi_0\|^2$$

because τ is a trace, then by Proposition A.1, the *-operation on $\mathscr{N}\xi_0$ extends to a conjugate linear isometry $J\colon \mathcal{H} \to \mathcal{H}$. We have $J^2 = 1_{\mathcal{H}}$, by continuity of J and the fact that $J^2S\xi_0 = S\xi_0$ for all $S \in \mathscr{N}$.

For (ii) note that α is first of all well-defined: for all $T \in B(\mathcal{H})$, JTJ is linear and

$$\|JTJ\xi\| = \|TJ\xi\| \leq \|T\|\|J\xi\| = \|T\|\|\xi\|.$$

α is conjugate linear, and moreover injective, since $JTJ = 0$ implies $T = J(JTJ)J = 0$, and surjective, since $\alpha(JTJ) = T$ for all $T \in B(\mathcal{H})$. Hence α is a conjugate linear isomorphism, and (ii) is obtained.

For $S, T, V \in \mathscr{N}$, we have

$$\langle JTJS\xi_0, V\xi_0\rangle = \langle ST^*\xi_0, V\xi_0\rangle = \tau(V^*ST^*) = \tau(T^*V^*S) = \langle S\xi_0, VT\xi_0\rangle = \langle S\xi_0, JT^*JV\xi_0\rangle,$$

so by continuity, $\alpha(T)^* = \alpha(T^*)$ for all $T \in \mathscr{N}$. Also, for $S, T \in \mathscr{N}$, we have $(JSJ)(JTJ) = J(ST)J$, so α is multiplicative as well, and that α is unital is clear. Hence (iii) follows.

(iv) is the one that will require the most work. For $T \in \mathscr{N}$, then for all $S, A \in \mathscr{N}$ we have

$$(JTJ)SA\xi_0 = SAT^*\xi_0 = S(JT)A^*\xi_0 = S(JTJ)A\xi_0,$$

so $(JTJ)S = S(JTJ)$ and hence $\alpha(\mathscr{N}) = J\mathscr{N}J \subseteq \mathscr{N}'$. Now, for $S' \in \mathscr{N}'$, note that for $T \in \mathscr{N}$, we have

$$\langle JS'\xi_0, T\xi_0\rangle = \langle T^*(S')^*\xi_0, \xi_0\rangle = \langle (S')^*T^*\xi_0, \xi_0\rangle = \langle \xi_0, TS'\xi_0\rangle = \langle \xi_0, S'T\xi_0\rangle = \langle (S')^*\xi_0, T\xi_0\rangle.$$

Hence

$$JS'\xi_0 = (S')^*\xi_0, \quad S' \in \mathscr{N}'$$

by a continuity and density argument. To show $\mathscr{N}' \subseteq J\mathscr{N}J$, we can instead prove $J\mathscr{N}'J \subseteq \mathscr{N} = \mathscr{N}''$ since $J^2 = 1_{\mathcal{H}}$. Therefore let $S', T' \in \mathscr{N}'$. We will show that $JS'J$ and T' commute, so that $JS'J \subseteq \mathscr{N}'' = \mathscr{N}$. For any $T \in \mathscr{N}$, we have

$$(JS'J)T'(T\xi_0) = (JS'J)TT'\xi_0 = JS'(JTJ)(JT'\xi_0) = JS'\alpha(T)(T')^*\xi_0$$

by what we proved above. Since S', $\alpha(T)$ and $(T')^*$ are contained in $\mathscr{N}'$ then $S'\alpha(T)(T')^* \in \mathscr{N}'$, so we now find that

$$J(S'\alpha(T)(T')^*)\xi_0 = T'\alpha(T^*)(S')^*\xi_0 = T'JT^*J(S')^*\xi_0 = T'JT^*S'\xi_0 = T'JS'T^*\xi_0 = T'(JS'J)T\xi_0.$$

Hence $(JS'J)T' = T'(JS'J)$ by continuity, so equality follows.

For (v), we prove in Section B.2 that the map $T \mapsto \overline{T}$ is a conjugate linear, multiplicative, unital and adjoint-preserving isomorphism. Restricting its inverse to $\overline{\mathscr{N}}$ and composing with α yields the desired *-isomorphism, since the composition of two conjugate linear maps is linear. $\square$

We now proceed to a really great result, reaping the harvest we have sown with Theorem 5.14.

Theorem 5.16. *Let $\mathscr{M}$ be a σ-finite, finite and injective von Neumann algebra with a faithful normal tracial state $\tau\colon \mathscr{M} \to \mathbb{C}$. Then $\mathscr{M}$ is semidiscrete.*

Proof. Let $\mathcal{H}$ be the Hilbert space and $\mathscr{N} \subseteq B(\mathcal{H})$ the von Neumann algebra *-isomorphic to $\mathscr{M}$ of Lemma 5.15 with cyclic vector $\xi_0 \in \mathcal{H}$, with $J\colon \mathcal{H} \to \mathcal{H}$ being the conjugate linear isometry. It will suffice to prove that $\mathscr{N}$ is semidiscrete. Letting $\overline{\mathscr{N}} \subseteq B(\overline{\mathcal{H}})$ denote the conjugate von Neumann algebra of $\mathscr{N}$, the map $\overline{\mathscr{N}} \to \mathscr{N}'$ given by $\overline{T} \mapsto JTJ$ for $T \in \mathscr{N}$ is an isometric *-isomorphism by Lemma 5.15. Hence it induces a *-isomorphism $\mathscr{N} \otimes_{\min} \overline{\mathscr{N}} \to \mathscr{N} \otimes_{\min} \mathscr{N}'$ that must be isometric. To prove that $\mathscr{N}$ is semidiscrete is equivalent to proving, by Theorem 5.12, that the *-homomorphism $\eta\colon \mathscr{N} \odot \mathscr{N}' \to B(\mathcal{H})$ given by

$$\eta\left(\sum_{j=1}^{n} S_j \otimes T_j\right) = \sum_{j=1}^{n} S_j T_j$$

(see Proposition 1.17) is contractive with respect to $\|\cdot\|_{\min}$ on $\mathscr{N} \odot \mathscr{N}'$. From the above considerations, this is equivalent to proving that the *-homomorphism $\eta'\colon \mathscr{N} \odot \overline{\mathscr{N}} \to B(\mathcal{H})$ given by

$$\eta'\left(\sum_{j=1}^{n} S_j \otimes \overline{T_j}\right) = \sum_{j=1}^{n} S_j J T_j J$$

is contractive with respect to $\|\cdot\|_{\min}$ on $\mathscr{N} \odot \mathscr{N}'$, by means of the isometric *-isomorphism $\mathscr{N} \otimes_{\min} \overline{\mathscr{N}} \to \mathscr{N} \otimes_{\min} \mathscr{N}'$.

To show this, note that since ξ_0 is cyclic for $\mathscr{N}$, it follows that it is cyclic for $\eta'(\mathscr{N} \odot \overline{\mathscr{N}})$ as well because $\eta'(T \odot 1_{\overline{\mathcal{H}}}) = T$ for all $T \in \mathscr{N}$. It will therefore suffice to prove that $\omega_{\xi_0} \circ \eta'$ is contractive with respect to $\|\cdot\|_{\min}$. To do this, we will regard τ as a tracial state on $\mathscr{N}$.

For $S_1, \ldots, S_n \in \mathscr{N}$ and $\overline{T_1}, \ldots, \overline{T_n} \in \overline{\mathscr{N}}$ we find

$$\omega_{\xi_0}(\eta'(\sum_{j=1}^{n} S_j \otimes \overline{T_j})) = \left\langle \sum_{j=1}^{n} S_j J T_j J \xi_0, \xi_0 \right\rangle = \left\langle \sum_{j=1}^{n} S_j T_j^* \xi_0, \xi_0 \right\rangle = \tau\left(\sum_{j=1}^{n} S_j T_j^*\right).$$

Thus we have to prove

$$\left|\tau\left(\sum_{j=1}^{n} S_j T_j^*\right)\right| \leq \left\|\sum_{i=1}^{n} S_j \otimes \overline{T_j}\right\|_{\min}.$$

Let $\varepsilon > 0$ be given. From Proposition 5.14, we obtain the existence of a Hilbert-Schmidt operator $P \in B(\mathcal{H})$ such that $\|P\|_2 = 1$ and

$$\left|\left\langle \sum_{j=1}^{n} S_j T_j^* P, P \right\rangle_2 - \tau\left(\sum_{j=1}^{n} S_j T_j^*\right)\right| \leq \varepsilon, \quad \sum_{j=1}^{n} \|T_j^* P - P T_j^*\|_2 \|S_j\| \leq \varepsilon.$$

To see this, apply Proposition 5.14 to the operators $S_1T_1^*, \ldots, S_nT_n^*$ and $\|S_1\|T_1^*, \ldots, \|S_n\|T_n^*$ of $\mathscr{N}$. Now we obtain

$$\begin{aligned}
\left|\tau\left(\sum_{j=1}^n S_jT_j^*\right)\right| &\leq \left|\left\langle \sum_{j=1}^n S_jT_j^*P, P\right\rangle_2\right| + \varepsilon \\
&\leq \left\|\sum_{j=1}^n S_jT_j^*P\right\|_2 \|P\|_2 + \varepsilon \\
&\leq \left\|\sum_{j=1}^n S_jPT_j^* + \sum_{j=1}^n S_j(T_j^*P - PT_j^*)\right\|_2 + \varepsilon \\
&\leq \left\|\sum_{j=1}^n S_jPT_j^*\right\|_2 + \sum_{j=1}^n \|S_j(T_j^*P - PT_j^*)\|_2 + \varepsilon \\
&\leq \left\|\sum_{j=1}^n S_jPT_j^*\right\|_2 + \sum_{j=1}^n \|S_j\|\|(T_j^*P - PT_j^*)\|_2 + \varepsilon \\
&\leq \left\|\sum_{j=1}^n S_jPT_j^*\right\|_2 + 2\varepsilon.
\end{aligned}$$

By Proposition B.19, P corresponds to a unit vector $\xi \in \mathcal{H} \otimes \overline{\mathcal{H}}$. For any $j = 1, \ldots, n$, then by Proposition B.20, S_jP corresponds to $(S_j \otimes 1_{\overline{\mathcal{H}}})\xi$ and hence $S_jPT_j^*$ corresponds to the vector

$$(1_{\mathcal{H}} \otimes \overline{T_j})(S_j \otimes 1_{\overline{\mathcal{H}}})\xi = (S_j \otimes \overline{T_j})\xi.$$

Hence

$$\left\|\sum_{j=1}^n S_jPT_j^*\right\|_2 = \left\|\sum_{j=1}^n (S_j \otimes \overline{T_j})\xi\right\|_{B(\mathcal{H}\otimes\overline{\mathcal{H}})} \leq \left\|\sum_{j=1}^n S_j \otimes \overline{T_j}\right\|_{B(\mathcal{H}\otimes\overline{\mathcal{H}})}.$$

Therefore

$$\left|\tau\left(\sum_{j=1}^n S_jT_j^*\right)\right| \leq \left\|\sum_{j=1}^n S_j \otimes \overline{T_j}\right\|_{B(\mathcal{H}\otimes\overline{\mathcal{H}})} + 2\varepsilon = \left\|\sum_{j=1}^n S_j \otimes \overline{T_j}\right\|_{\min} + 2\varepsilon,$$

as the identity maps on $\mathscr{N}$ and $\overline{\mathscr{N}}$ are faithful representations. Since ε was arbitrary, it follows that

$$\left|\tau\left(\sum_{j=1}^n S_jT_j^*\right)\right| \leq \left\|\sum_{j=1}^n S_j \otimes \overline{T_j}\right\|_{\min}.$$

This completes the proof. □

It is now time to call in the continuous crossed product from Chapter 4, giving us another criterion for a σ-finite von Neumann algebra to be semidiscrete.

Theorem 5.17. *Let $\mathscr{M} \subseteq B(\mathcal{H})$ be a σ-finite von Neumann algebra with a faithful normal state $\omega \in \mathscr{M}_*$ and assume that $\mathrm{R}(\mathscr{M}, \sigma_t^\omega)$ is semidiscrete. Then $\mathscr{M}$ is semidiscrete.*

Proof. We will prove that $\mathscr{M}$ satisfies the condition of Theorem 5.12. We remember first that the continuous crossed product $\mathrm{R}(\mathscr{M}, \sigma_t^\omega)$ is the von Neumann algebra in $B(L^2(\mathbb{R}), \mathcal{H})$ generated by elements of the form $\pi(T)$ for $T \in \mathscr{M}$ and $\lambda(t)$ for $t \in \mathbb{R}$, where

$$(\pi(T)f)(s) = \sigma_s^\omega(T)f(s), \quad f \in C_c(\mathbb{R}, \mathcal{H}), \; s \in \mathbb{R}$$

and

$$(\lambda(t)f)(s) = f(t^{-1}s), \quad f \in C_c(\Gamma, \mathcal{H}), s \in \Gamma.$$

Since $\mathrm{R}(\mathscr{M}, \sigma_t^\omega)$ embeds into $\mathscr{M} \mathbin{\overline{\otimes}} B(L^2(\mathbb{R}))$ by Corollary 4.17 and can hence be viewed as a subset, it follows that $\mathscr{M}' \otimes 1_{L^2(\mathbb{R})} \subseteq \mathrm{R}(\mathscr{M}, \sigma_t^\omega)'$ by Proposition 1.35.

Assuming that $\mathrm{R}(\mathscr{M}, \sigma_t^\omega)$ is semidiscrete, then it follows from Theorem 5.12 that

$$\left\| \sum_{i=1}^n \pi(S_i)\pi'(T_i) \right\| \le \left\| \sum_{i=1}^n \pi(S_i) \otimes \pi'(T_i) \right\|_{\min}$$

for all $S_1, \dots, S_n \in \mathscr{M}$ and $T_1, \dots, T_n \in \mathscr{M}'$, where

$$\pi' \colon \mathscr{M}' \to B(\mathcal{H}) \mathbin{\overline{\otimes}} B(L^2(\mathbb{R})) = B(\mathcal{H} \otimes L^2(\mathbb{R})) = B(L^2(\mathbb{R}, \mathcal{H}))$$

is given by $\pi'(T) = T \otimes 1_{L^2(\mathbb{R})}$. $\pi \colon \mathscr{M} \to B(L^2(\mathbb{R}, \mathcal{H}))$ is by construction a faithful representation of $\mathscr{M}$ and $\pi' \colon \mathscr{M}' \to B(L^2(\mathbb{R}, \mathcal{H}))$ is clearly faithful. Therefore Theorem 1.43(ii) tells us that

$$\left\| \sum_{i=1}^n S_i \otimes T_i \right\|_{\min} = \left\| \sum_{i=1}^n \pi(S_i) \otimes \pi'(T_i) \right\|_{B(L^2(\mathbb{R},\mathcal{H}) \otimes L^2(\mathbb{R},\mathcal{H}))} = \left\| \sum_{i=1}^n \pi(S_i) \otimes \pi'(T_i) \right\|_{\min},$$

where the last equality follows from noting that the identity maps on $B(L^2(\mathbb{R}, \mathcal{H}))$ are faithful representations. Hence we have

$$\left\| \sum_{i=1}^n \pi(S_i)\pi'(T_i) \right\| \le \left\| \sum_{i=1}^n S_i \otimes T_i \right\|_{\min}$$

for all $S_1, \dots, S_n \in \mathscr{M}$ and $T_1, \dots, T_n \in \mathscr{M}'$. Hence if we prove

$$\left\| \sum_{i=1}^n S_i T_i \right\| \le \left\| \sum_{i=1}^n \pi(S_i)\pi'(T_i) \right\|$$

for all such operators, it follows from Theorem 5.12 that $\mathscr{M}$ is semidiscrete.

If $\|\sum_{i=1}^n S_i T_i\| = 0$, then the inequality is trivial, so we can assume that $\|\sum_{i=1}^n S_i T_i\| \neq 0$. Take $\xi \in \mathcal{H}$ with $\|\xi\| = 1$ that satisfies $\sum_{i=1}^n S_i T_i \xi \neq 0$. As the function $t \mapsto \sigma_t^\omega(S)$ for $t \in \mathbb{R}$ is strongly continuous for all $S \in \mathscr{M}$, it follows that the function

$$\beta \colon t \mapsto \left\| \sum_{i=1}^n \sigma_{-t}^\omega(S_i) T_i \xi \right\|, \quad t \in \mathbb{R}$$

is continuous. Moreover, since σ^ω is a group homomorphism, then σ_0^ω is the identity mapping on $\mathscr{M}$, so

$$\beta(0) = \left\| \sum_{i=1}^n S_i T_i \xi \right\|.$$

Let $0 < \varepsilon < \beta(0)$; by continuity of β, we can take $\delta > 0$ such that $|t| < \delta$ implies $\beta(0) - \beta(t) < \varepsilon$. Hence $\beta(t) > \beta(0) - \varepsilon > 0$ for all $|t| < \delta$.

Let $g \in C_c(\mathbb{R})$ be your favourite continuous function with support contained in the interval $(-\delta, \delta)$ and $\|g\|_2 = 1$, and define $f \in C_c(\mathbb{R}, \mathcal{H})$ by $f(t) = g(t)\xi$ for $t \in \mathbb{R}$, so that $f = \xi \otimes g$ when seen as an element of $\mathcal{H} \otimes L^2(\mathbb{R})$ by Proposition 4.13. For any $T \in \mathscr{M}'$, we hence have $\pi'(T)f = T\xi \otimes g$ or $(\pi'(T)f)(t) = g(t)T\xi$ for all $t \in \mathbb{R}$. Therefore

$$(\pi(S)\pi'(T)f)(t) = g(t)\sigma_t^\omega(S)T\xi, \quad S \in \mathscr{M},\ T \in \mathscr{M}',\ t \in \mathbb{R}.$$

Note that $\pi(S)\pi'(T)f \in C_c(\mathbb{R}, \mathcal{H})$, so this also holds for all finite linear combinations of functions $\pi(S)\pi'(T)f$ for $S \in \mathscr{M}$ and $T \in \mathscr{M}'$. Additionally,

$$\|f\|^2 = \langle f, f \rangle = \int_{\mathbb{R}} |g(t)|^2 \|\xi\|^2 \mathrm{d}t = \|g\|_2^2 = 1,$$

so $\|f\| = 1$. This groundwork finally yields

$$\begin{aligned}
\left\|\sum_{i=1}^n \pi(S_i)\pi'(T_i)\right\|^2 &\geq \left\|\sum_{i=1}^n \pi(S_i)\pi'(T_i)f\right\|^2 \\
&= \int_{\mathbb{R}} \left\|\sum_{i=1}^n (\pi(S_i)\pi'(T_i)f)(t)\right\|^2 \mathrm{d}t \\
&= \int_{\mathbb{R}} \left\|\sum_{i=1}^n \sigma_t^\omega(S_i)T_i\xi\right\|^2 |g(t)|^2 \mathrm{d}t \\
&= \int_{-\delta}^{\delta} \beta(t)^2 |g(t)|^2 \mathrm{d}t \\
&\geq \int_{-\delta}^{\delta} (\beta(0)-\varepsilon)^2 |g(t)|^2 \mathrm{d}t \\
&= (\beta(0)-\varepsilon)^2 \int_{-\delta}^{\delta} |g(t)|^2 \mathrm{d}t = (\beta(0)-\varepsilon)^2.
\end{aligned}$$

Since this holds for arbitrary $0 < \varepsilon < \beta(0)$, we conclude that

$$\left\|\sum_{i=1}^n S_i T_i \xi\right\| = \beta(0) \leq \left\|\sum_{i=1}^n \pi(S_i)\pi'(T_i)\right\|.$$

Taking the supremum over all $\xi \in \mathcal{H}$ with $\|\xi\| = 1$, we finally obtain the inequality

$$\left\|\sum_{i=1}^n S_i T_i\right\| = \left\|\sum_{i=1}^n \pi(S_i)\pi'(T_i)\right\| \leq \left\|\sum_{i=1}^n S_i \otimes T_i\right\|_{\min}$$

for all $S_1, \ldots, S_n \in \mathscr{M}$ and $T_1, \ldots, T_n \in \mathscr{M}'$, so $\mathscr{M}$ is semidiscrete. □

We now finally have all we need to jump to *la grande finale* of this project that all of our previous achievements have been working towards. Not surprisingly, it uses a variety of different results, making the proof a tribute in some way to what we have proved up until now.

Theorem 5.18. *Let $\mathscr{M}$ be a von Neumann algebra. Then $\mathscr{M}$ is injective if and only if $\mathscr{M}$ is semidiscrete.*

Proof. By Proposition 2.59,

$$\mathscr{M} \cong \bigoplus_{\alpha \in A} (\mathscr{M}_{P_\alpha} \mathbin{\overline{\otimes}} B(\mathcal{K}_\alpha))_{Q_\alpha},$$

where $\mathcal{K}_\alpha$ is a Hilbert space, $P_\alpha \in \mathscr{M}$ and $Q_\alpha \in \mathscr{M}_{P_\alpha} \mathbin{\overline{\otimes}} B(\mathcal{K}_\alpha)$ are projections, and $\mathscr{M}_{P_\alpha}$ is σ-finite for all $\alpha \in A$. By Propositions 4.1, 4.7, 4.2, 4.12 and 4.8, we see that $\mathscr{M}$ is injective if and only if $\mathscr{M}_{P_\alpha}$ is injective for all $\alpha \in A$. By Propositions 5.1, 5.7, 5.11, 5.10 and 5.9, we see that $\mathscr{M}$ is semidiscrete if and only if $\mathscr{M}_{P_\alpha}$ is semidiscrete for all $\alpha \in A$. Since each $\mathscr{M}_{P_\alpha}$ is σ-finite, we see that it suffices to prove the result for σ-finite von Neumann algebras, so we can from here onward assume that $\mathscr{M}$ is σ-finite. By Propositions 2.58, 4.1 and 5.1, we can furthermore assume that $\mathscr{M} \subseteq B(\mathcal{H})$ has a faithful normal state $\omega \in \mathscr{M}_*$ and a cyclic and separating unit vector $\xi \in \mathcal{H}$. It follows from Proposition 2.21 that ξ is also cyclic and separating for $\mathscr{M}'$.

Assume first that $\mathscr{M}$ is semidiscrete. Defining the map $\theta \colon \mathscr{M} \to (\mathscr{M}')^*$ by

$$\theta(T)(T') = \langle TT'\xi, \xi\rangle,$$

Proposition 3.23 tells us that θ is completely positive. The map $\mathscr{M} \times \mathscr{M}' \to \mathbb{C}$ given by $(T, T') \mapsto \theta(T)(T')$ is bilinear and hence induces a unique linear functional $\varphi \colon \mathscr{M} \odot \mathscr{M}' \to \mathbb{C}$. For all $S_1, \ldots, S_n \in \mathscr{M}$ and $T_1, \ldots, T_n \in \mathscr{M}'$, we then have

$$\varphi\left(\left(\sum_{i=1}^n S_i \otimes T_i\right)^* \left(\sum_{j=1}^n S_j \otimes T_j\right)\right) = \sum_{i,j=1}^n \langle S_i^* S_j T_i^* T_j \xi, \xi\rangle = \theta^{(n)}((S_i^* S_j)_{i,j=1}^n)((T_i^* T_j)_{i,j=1}^n) \geq 0,$$

so $\varphi(x^*x) \geq 0$ for all $x \in \mathscr{M} \odot \mathscr{M}'$. If $\eta\colon \mathscr{M} \odot \mathscr{M}' \to B(\mathcal{H})$ is the map given by

$$\eta\left(\sum_{i=1}^n S_i \otimes T_i'\right) = \sum_{i=1}^n S_i T_i',$$

then $\varphi = \omega_\xi \circ \eta$. Since $\mathscr{M}$ is semidiscrete, Theorem 5.12 yields that η is contractive with respect to $\|\cdot\|_{\min}$, so φ is contractive with respect to $\|\cdot\|_{\min}$ as well. Hence φ extends to a positive contractive linear functional $\tilde{\varphi}\colon \mathscr{M} \otimes_{\min} \mathscr{M}' \to \mathbb{C}$ by Proposition A.1 (positivity follows from continuity and algebraic positivity). As $\varphi(1_\mathcal{H} \otimes 1_\mathcal{H}) = 1$, $\tilde{\varphi}$ is then a state by [31, Theorem 11.5].

Since $\mathscr{M} \otimes_{\min} \mathscr{M}'$ embeds naturally into $B(\mathcal{H}) \otimes_{\min} \mathscr{M}'$ by [4, Proposition 3.6.1], the Hahn-Banach theorem allows for an extension of $\tilde{\varphi}$ to a state $\Phi \in S(B(\mathcal{H}) \otimes_{\min} \mathscr{M}')$ by Lemma 2.41. Define $\tilde{\theta}\colon B(\mathcal{H}) \to (\mathscr{M}')^*$ by

$$\tilde{\theta}(T)(T') = \Phi(T \otimes T').$$

Note that for $S \in \mathscr{M}$, we have

$$\tilde{\theta}(S)(T') = \Phi(S \otimes T') = \varphi(S \otimes T') = \theta(S)(T'),$$

and more specifically that $\tilde{\theta}(1_\mathcal{H})(T') = \theta(1_\mathcal{H})(T') = \omega_\xi(T')$ for all $T' \in \mathscr{M}'$. For positive $T \in B(\mathcal{H})$ and $T' \in \mathscr{M}'$, then $T = S^*S$ and $T' = (S')^*S'$ for $S \in B(\mathcal{H})$ and $S' \in \mathscr{M}'$, so that

$$\tilde{\theta}(T)(T') = \Phi((S \otimes S')^*(S \otimes S')) \geq 0.$$

Hence $\tilde{\theta}(T)$ is a positive linear functional on $\mathscr{M}'$ for all positive $T \in B(\mathcal{H})$. If $T \in B(\mathcal{H})$ is non-zero and positive with $\lambda = \|T\|$, then $0 \leq \lambda^{-1}T \leq 1_\mathcal{H}$, and hence $\tilde{\theta}(\lambda^{-1}T) \leq \tilde{\theta}(1_\mathcal{H}) = \omega_\xi$. Using the notation of Proposition 3.23, $\tilde{\theta}(\lambda^{-1}T)$ is then contained in

$$C_\xi = \{\varphi \in (\mathscr{M}')^* \mid 0 \leq \varphi \leq \omega_\xi\},$$

so that $\tilde{\theta}(T)$ and hence $\tilde{\theta}(B(\mathcal{H}))$ is contained in the complex linear span E of C_ξ, since any operator in a unital C^*-algebra is a finite linear combination of positive operators [31, Theorem 11.2].

For all $T_1, \ldots, T_n \in B(\mathcal{H})$ and $T_1', \ldots, T_n' \in \mathscr{M}'$, we see that

$$\sum_{i,j=1}^n \tilde{\theta}(T_i^*T_j)(T_i'^*T_j') = \sum_{i,j=1}^n \Phi((T_i \otimes T_i')^*(T_j \otimes T_j')) = \Phi\left(\left(\sum_{i=1}^n T_i \otimes T_i'\right)^*\left(\sum_{i=1}^n T_i \otimes T_i'\right)\right) \geq 0,$$

so Proposition 3.22 tells us that $\tilde{\theta}$ is completely positive. Moreover, we proved in Proposition 3.23 that $\theta\colon \mathscr{M} \to E$ is a completely positive linear isomorphism with a completely positive inverse. Defining $E\colon B(\mathcal{H}) \to \mathscr{M}$ by

$$E = \theta^{-1} \circ \tilde{\theta},$$

E is then completely positive and for $T \in \mathscr{M}$, $E(T) = T$. Hence $\mathscr{M}$ is injective.

For the converse, assume that $\mathscr{M}$ is injective. By Corollary 4.22, the continuous crossed product $\mathscr{N} = \mathrm{R}(\mathscr{M}, \sigma_t^\omega)$ is injective. Moreover, $\mathscr{N}$ is a semifinite von Neumann algebra by Theorem 4.19. Hence the identity of $\mathscr{N}$ is a semifinite projection. By [10, Corollary III.2.4.2], there exists a finite projection $P \in \mathscr{N}$ that has the same central support as the identity; therefore $C_P = 1_\mathscr{N}$. Hence $\mathscr{N}_P$ is finite and also injective by Proposition 4.8.

To see that $\mathscr{N}_P$ is semidiscrete, note that by repeating the handling of Proposition 2.59 in beginning of the proof, we only need to prove that all σ-finite reduced von Neumann algebras of $\mathscr{N}_P$ are semidiscrete. But by Proposition 4.8, any reduced von Neumann algebra of $\mathscr{N}_P$ is injective and by [10, Proposition I.6.8.11], any such is also finite. Hence Theorem 5.16 implies that any σ-finite reduced von Neumann algebra of $\mathscr{N}_P$ is semidiscrete, so $\mathscr{N}_P$ is semidiscrete itself.

We now want to pass back to $\mathscr{N}$. Since $C_P = 1_\mathscr{N}$, then by Lemma 2.55, $\mathscr{N}$ is isomorphic to a reduced von Neumann algebra of $\mathscr{N}_P \mathbin{\overline{\otimes}} B(\mathcal{K})$ for some Hilbert space $\mathcal{K}$. Propositions 5.11 and 5.10 yield that $\mathscr{N}_P \mathbin{\overline{\otimes}} B(\mathcal{K})$ is semidiscrete, in turn yielding that $\mathscr{N}$ is semidiscrete by Propositions 5.9 and 5.1. Theorem 5.17 now tells us that $\mathscr{M}$ is semidiscrete, completing the proof. □

And there we go. As promised in Theorem 5.5, we bring a theorem that puts the final piece of this humongous jigsaw puzzle into place. The proof somewhat resembles the proof of the previous theorem.

Theorem 5.19. *Let $\mathcal{A}$ be a $\otimes$-nuclear C^*-algebra. Then $\mathcal{A}^{**}$ is injective.*

Proof. Let $\varphi \in S(\mathcal{A})$ and let $(\mathcal{H}_\varphi, \pi_\varphi, \xi_\varphi)$ be its associated GNS triple. Note that $\varphi = \omega_{\xi_\varphi} \circ \pi_\varphi$ by construction. Letting $C = \{\psi \in \mathcal{A}^* \mid 0 \leq \psi \leq \varphi\}$ and E be the complex linear span of C, we define $\theta_\varphi \colon \pi_\varphi(\mathcal{A})' \to E$ by

$$\theta_\varphi(T)(a) = \langle T\pi_\varphi(a)\xi_\varphi, \xi_\varphi\rangle, \quad T' \in \pi_\varphi(\mathcal{A})', \quad a \in \mathcal{A}.$$

By Proposition 3.23, θ_φ is completely positive with a completely positive inverse.

Let $\mathcal{B} = \mathcal{A} \odot \pi_\varphi(\mathcal{A})'$. The map $(a, T) \mapsto \theta_\varphi(T)(a)$ for $a \in \mathcal{A}$ and $T \in \pi_\varphi(\mathcal{A})'$ is bilinear and hence induces a linear functional $\psi \colon \mathcal{B} \to \mathbb{C}$ such that $\psi(a \otimes T) = \theta_\varphi(T)(a)$. For all $a_1, \dots, a_n \in \mathcal{A}$ and $T_1, \dots, T_n \in \mathscr{M}'$, we then have

$$\psi\left(\left(\sum_{i=1}^n a_i \otimes T_i\right)^* \left(\sum_{j=1}^n a_j \otimes T_j\right)\right) = \sum_{i,j=1}^n \langle \pi_\varphi(a_i^* a_j) T_i^* T_j \xi_\varphi, \xi_\varphi\rangle = \theta_\varphi^{(n)}((T_i^* T_j)_{i,j=1}^n)((a_i^* a_j)_{i,j=1}^n) \geq 0,$$

so $\psi(x^*x) \geq 0$ for all $x \in \mathcal{B}$. Moreover, by Proposition 1.17 there is an induced representation $\Omega \colon \mathcal{B} \to B(\mathcal{H}_\varphi)$ such that

$$\Omega(a \otimes T) = T\pi_\varphi(a), \quad a \in \mathcal{A}, \ T \in \pi_\varphi(\mathcal{A})'.$$

As $\|\Omega(x)\| \leq \|x\|_{\max}$ for all $x \in \mathcal{B}$, it follows for $x = \sum_{i=1}^n a_i \otimes T_i \in \mathcal{B}$ that

$$|\psi(x)| = \left|\sum_{i=1}^n \theta_\varphi(T_i)(a_i)\right| \leq \left|\left\langle \sum_{i=1}^n T\pi_\varphi(a)\xi_\varphi, \xi_\varphi \right\rangle\right| \leq \|\sum_{i=1}^n T\pi_\varphi(a)\|\|\xi_\varphi\|^2 = \|\Omega(x)\| \leq \|x\|_{\max},$$

so ψ is $\|\cdot\|_{\max}$-continuous and therefore $\|\cdot\|_{\min}$-continuous, since $\mathcal{A}$ is $\otimes$-nuclear. Therefore ψ extends to a positive contractive linear functional $\tilde{\psi} \colon \mathcal{A} \otimes_{\min} \pi_\varphi(\mathcal{A})' \to \mathbb{C}$ by Proposition A.1 (again, positivity follows from continuity algebraic positivity of ψ).

Since $\mathcal{A} \otimes_{\min} \pi_\varphi(\mathcal{A})'$ embeds naturally into $\mathcal{A} \otimes_{\min} B(\mathcal{H}_\varphi)$ by Theorem 1.43, the Hahn-Banach theorem allows for an extension of $\tilde{\psi}$ to a contractive linear functional $\Psi \in \mathcal{A} \otimes_{\min} B(\mathcal{H}_\varphi)$ by Lemma 2.41. Define $\tilde{\theta}_\varphi \colon B(\mathcal{H}) \to \mathcal{A}^*$ by

$$\tilde{\theta}_\varphi(T)(a) = \Psi(a \otimes T).$$

For all $a \in \mathcal{A}$, we then have

$$\tilde{\theta}_\varphi(T)(a) = \Psi(a \otimes T) = \varphi(a \otimes T) = \theta_\varphi(T)(a),$$

and more specifically that $\tilde{\theta}_\varphi(1_{\mathcal{H}_\varphi})(a) = \theta_\varphi(1_{\mathcal{H}_\varphi})(a) = \varphi(a)$ for all $a \in \mathcal{A}$. For positive $T \in B(\mathcal{H})$ and $a \in \mathcal{A}$, then $T = S^*S$ and $a = b^*b$ for some $S \in B(\mathcal{H})$ and $b \in \mathcal{A}$, so that $\tilde{\theta}_\varphi(T)(a) = \Psi((b \otimes S)^*(b \otimes S)) \geq 0$. Hence $\tilde{\theta}_\varphi(T)$ is a positive linear functional on $\mathcal{A}$ for all positive $T \in B(\mathcal{H})$. Finally, if $T \in B(\mathcal{H}_\varphi)$ is non-zero and positive with $\lambda = \|T\|$, then $0 \leq \lambda^{-1}T \leq 1_{\mathcal{H}_\varphi}$, and hence $\tilde{\theta}_\varphi(\lambda^{-1}T) \leq \tilde{\theta}_\varphi(1_{\mathcal{H}}) = \varphi$. Hence $\tilde{\theta}_\varphi(\lambda^{-1}T)$ is then contained in C so that $\tilde{\theta}_\varphi(T)$ and hence $\tilde{\theta}_\varphi(B(\mathcal{H}))$ is contained in E, since any operator in $B(\mathcal{H})$ is a finite linear combination of positive operators [31, Theorem 11.2]. Finally, for all $T_1, \dots, T_n \in B(\mathcal{H})$ and $a_1, \dots, a_n \in \mathcal{A}$ we note that

$$\sum_{i,j=1}^n \tilde{\theta}_\varphi(T_i^* T_j)(a_i^* a_j) = \sum_{i,j=1}^n \Psi((a_i \otimes T_i)^*(a_j \otimes T_j)) = \Psi\left(\left(\sum_{i=1}^n a_i \otimes T_i\right)^* \left(\sum_{i=1}^n a_i \otimes T_i\right)\right) \geq 0,$$

so Proposition 3.22 tells us that $\tilde{\theta}_\varphi$ is completely positive. Defining $E \colon B(\mathcal{H}_\varphi) \to \pi_\varphi(\mathcal{A})'$ by

$$E = \theta_\varphi^{-1} \circ \tilde{\theta}_\varphi,$$

E is then completely positive and for $T \in \pi_\varphi(\mathcal{A})'$, $E(T) = T$. Therefore $\pi_\varphi(\mathcal{A})'$ is injective and hence semidiscrete, so Theorem 5.12 tells us that $\pi_\varphi(\mathcal{A})''$ is injective.

Since $\varphi \in S(\mathcal{A})$ was arbitrary and π_φ is nondegenerate in every case, then for all $\varphi \in S(\mathcal{A})$ there exists a surjective normal *-homomorphism $\rho_\varphi \colon \mathcal{A}^{**} \to \pi_\varphi(\mathcal{A})''$ such that $\pi_\varphi = \rho_\varphi \circ \iota$, where ι denotes the inclusion $\mathcal{A} \to \mathcal{A}^{**}$; this follows from the discussion after the proof of Theorem 2.63. If we can prove that the family $(\rho_\varphi)_{\varphi \in S(\mathcal{A})}$ is separating, then Corollary 4.10 will yield that $\mathcal{A}^{**}$ is injective.

Therefore assume that $\rho_\varphi(T) = 0$ for all $\varphi \in S(\mathcal{A})$. Let $(x_\alpha)_{\alpha \in A}$ be a net in $\mathcal{A}$ such that $\iota(x_\alpha) \to T$ ultraweakly. Then

$$\pi_\varphi(x_\alpha) = \rho_\varphi(\iota(x_\alpha)) \to \rho_\varphi(T) = 0$$

ultraweakly for all $\varphi \in S(\mathcal{A})$, and therefore

$$\varphi(x_\alpha) = \langle \pi_\varphi(x_\alpha)\xi_\varphi, \xi_\varphi \rangle \to 0.$$

By Theorem 2.34, we then see that $\psi(x_\alpha) \to 0$ for all $\psi \in \mathcal{A}^*$. Note that because $\Omega \colon (\mathcal{A}^{**})_* \to \mathcal{A}^*$ given by

$$\Omega(\omega)(a) = \omega(\iota(a))$$

is an isometric isomorphism (see page 62), then for all $\omega \in (\mathcal{A}^{**})_*$, we have $\omega(\iota(x_\alpha)) \to 0$ for all $\omega \in (\mathcal{A}^{**})_*$. Therefore $\iota(x_\alpha) \to 0$ ultraweakly, and hence $T = 0$, completing the proof. $\square$

This concludes the main part of the project.

Appendix A

Topological and algebraic properties of Banach space operators

All sorts of small and useful results are needed in the main parts of the project, and this chapter is devoted to proving them. The range of results here is quite wide, and no connection between the sections is intended. Hopefully most readers won't find the proofs too trivial.

1.1 Operator extensions

Throughout the project, we need the important fact that any bounded operator on normed spaces extends naturally to their completions. A proof is given here, along with an important corollary.

Proposition A.1. *Let V and W be normed spaces and let $T\colon V \to W$ be a bounded linear operator. If $\mathfrak{X}$ and $\mathfrak{Y}$ are Banach spaces and $\varphi_V\colon V \to \mathfrak{X}$ and $\varphi_W\colon W \to \mathfrak{Y}$ are linear isometric maps with dense range, then there is a unique bounded linear operator $\tilde{T} \in B(\mathfrak{X}, \mathfrak{Y})$ such that $\tilde{T} \circ \varphi_V = \varphi_W \circ T$, i.e. the following diagram commutes:*

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\varphi_V"'] & W \arrow[d, "\varphi_W"] \\ \mathfrak{X} \arrow[r, dotted, "\tilde{T}"'] & \mathfrak{Y} \end{tikzcd}$$

The extension satisfies $\|\tilde{T}\| = \|T\|$ and the following statements hold:

(i) *If there exists $c > 0$ such that $\|Ty\| = c\|y\|$ for all $x \in V$, then $\|\tilde{T}x\| = c\|x\|$ for all $x \in \mathfrak{X}$.*

(ii) *If T is surjective and bounded below, then $\tilde{T}$ is surjective as well.*

(iii) *If T is a surjective isometry, then $\tilde{T}$ is also a surjective isometry.*

(iv) *If T is not linear but conjugate linear, then $\tilde{T}$ is conjugate linear.*

(v) *If V and W are inner product spaces, $\mathfrak{X}$ and $\mathfrak{Y}$ are Hilbert spaces and T is isometric, then $\tilde{T}$ preserves inner products.*

(vi) *If V and W are normed algebras (resp. normed *-algebras), $\mathfrak{X}$ and $\mathfrak{Y}$ are Banach algebras (resp. Banach *-algebras) and φ_V, φ_W and T are homomorphisms (resp. *-homomorphisms), then $\tilde{T}$ is a homomorphism (resp. *-homomorphism) as well.*

Proof. Uniqueness of the extension is clear: if $\tilde{T}_1$ and $\tilde{T}_2$ are bounded linear operators $\mathfrak{X} \to \mathfrak{Y}$ satisfying $\tilde{T}_1 \circ \varphi_V = \tilde{T}_2 \circ \varphi_V = \varphi_W \circ T$, then by picking a sequence $(x_n)_{n\geq 1}$ in V for any given $x \in \mathfrak{X}$ such that $\varphi_V(x_n) \to x$, continuity of $\tilde{T}_1$ and $\tilde{T}_2$ yields

$$\tilde{T}_1(x) = \lim_{n\to\infty} \tilde{T}_1\varphi_V(x_n) = \lim_{n\to\infty} \tilde{T}_2\varphi_V(x_n) = \tilde{T}_2(x).$$

The big question here is how to define $\tilde{T}$ in the first place. Since we want $\tilde{T}$ to be continuous and $\varphi_V(V)$ is dense in $\mathfrak{X}$, we may try to define

$$\tilde{T}(x) = \lim_{n\to\infty} \varphi_W(Tx_n)$$

where $(x_n)_{n\geq 1}$ is a sequence of V such that $\varphi_V(x_n) \to x$, but it is not at all clear that it is well-defined: if there is a limit at all, is it then independent of the choice of sequence?

To address this, let $x \in \mathfrak{X}$ and let $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$ be sequences in V such that $\varphi_V(x_n) \to x$ and $\varphi_V(y_n) \to x$. First and foremost, $(\varphi_V(x_n))$ is a Cauchy sequence in $\mathfrak{X}$, so $(x_n)_{n\geq 1}$ is a Cauchy sequence in V since φ_V is an isometry. Therefore, as

$$\|\varphi_W(Tx_n) - \varphi_W(Tx_m)\| = \|Tx_n - Tx_m\| \leq \|T\|\|x_n - x_m\|$$

for $n, m \geq 1$, $(\varphi_W(Tx_n))_{n\geq 1}$ is a Cauchy sequence in $\mathfrak{Y}$, and hence converges to some $\tilde{x} \in \mathfrak{Y}$ by completeness. Same goes for $(\varphi_W(Ty_n))_{n\geq 1}$ that then converges to some $\tilde{y} \in \mathfrak{Y}$. Since

$$\|\varphi_W(Tx_n) - \varphi_W(Ty_n)\| \leq \|T\|\|x_n - y_n\| = \|T\|\|\varphi_V(x_n) - \varphi_V(y_n)\|$$

for $n \geq 1$, it follows from boundedness of T that

$$\|\tilde{x} - \tilde{y}\| \leq \|\tilde{x} - \varphi_W(Tx_n)\| + \|T\|\|\varphi_V(x_n) - \varphi_V(y_n)\| + \|\tilde{y} - \varphi_W(Ty_n)\| \to 0$$

for $n \to \infty$. Hence $\tilde{x} = \tilde{y}$. Hence we can define $\tilde{T}$ this way by $\tilde{T}x = \tilde{x}$. Note for $y \in V$ that $\tilde{T}\varphi_V(y) = \varphi_W(Ty)$, so $\tilde{T}$ also satisfies the needed equation. Moreover, $\tilde{T}$ is linear. Indeed, for $x, y \in \mathfrak{X}$ and sequences $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$ in V such that $\varphi_V(x_n) \to x$ and $\varphi_V(y_n) \to y$, then $\varphi_V(x_n + y_n) \to x + y$. Hence

$$\varphi_W(T(x_n + y_n)) = \varphi_W(T(x_n)) + \varphi_W(T(y_n)) \to \tilde{T}x + \tilde{T}y,$$

so $\tilde{T}(x + y) = \tilde{T}x + \tilde{T}y$. Similarly one proves that $\tilde{T}(\lambda x) = \lambda\tilde{T}x$ for $x \in \mathfrak{X}$ and $\lambda \in \mathbb{C}$ if T is linear, and if T is conjugate linear, then conjugate linearity of $\tilde{T}$ follows in the same way, proving (iv) once the general statement is proved.

Now note that

$$\|\varphi_W(Ty)\| = \|Ty\| \leq \|T\|\|y\| = \|T\|\|\varphi_W(y)\|$$

for all $y \in V$. Let $x \in \mathfrak{X}$ and take some sequence $(x_n)_{n\geq 1}$ in V such that $\varphi_V(x_n) \to x$. Since we then have $\|\varphi_W(Tx_n)\| \leq \|T\|\|\varphi_V(x_n)\|$ for all $n \geq 1$, we obtain $\|\tilde{T}x\| \leq \|T\|\|x\|$. Moreover, for any $x \in V$, we have

$$\|Tx\| = \|\varphi_W(Tx)\| = \|\tilde{T}\varphi_V(x)\| \leq \|\tilde{T}\|\|x\|,$$

so we finally have $\|\tilde{T}\| = \|T\|$. This concludes the proof of the general statement.

If $\|Ty\| = c\|y\|$ for some $c > 0$ and all $y \in V$, then for any $x \in \mathfrak{X}$, if $\varphi_V(x_n) \to x$ for some sequence $(x_n)_{n\geq 1}$ in V, then

$$\|\tilde{T}x\| = \lim_{n\to\infty} \|\varphi_W(Tx_n)\| = c \lim_{n\to\infty} \|x_n\| = c \lim_{n\to\infty} \|\varphi_V(x_n)\| = c\|x\|,$$

proving (i). If T is surjective and bounded below, i.e. $\|Tx\| \geq c\|x\|$ for some $c > 0$, and $y \in \mathfrak{Y}$, then $\varphi_W(y_n) \to y$ for some sequence $(y_n)_{n\geq 1}$ in W. Take $x_n \in V$ for all $n \geq 1$ such that $Tx_n = y_n$. Then $\varphi_W(Tx_n) \to y$. As

$$\|\varphi_V(x_n) - \varphi_V(x_m)\| = \|x_n - x_m\| \leq c^{-1}\|y_n - y_m\| = c^{-1}\|\varphi_W(y_n) - \varphi_W(y_n)\|$$

for all $m, n \geq 1$, we see that $(\varphi_V(x_n))_{n\geq 1}$ is a Cauchy sequence, hence converging to some $x \in \mathfrak{X}$ that must satisfy $\tilde{T}x = y$. Hence $\tilde{T}$ is surjective, proving (iii). (v) is a consequence of the polarization identity.

Finally, assume that the conditions of (vi) are satisfied and let $x, y \in \mathfrak{X}$. Then $\varphi_V(x_n) \to x$ and $\varphi_V(y_n) \to x$ for sequences $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$ in V, so $\varphi_V(x_n y_n) \to xy$ since the convergent sequences are necessarily bounded. Then

$$\tilde{T}(xy) = \lim_{n\to\infty} \varphi_W(T(x_n y_n)) = \lim_{n\to\infty} [\varphi_W(Tx_n)\varphi_W(Ty_n)] = \tilde{T}x\tilde{T}y.$$

If the spaces in question are *-algebras and the maps are *-homomorphisms, note that

$$\|\varphi_V(x_n^*) - \varphi(x^*)\| \leq \|x_n^* - x^*\| = \|x_n - x\| \to 0,$$

since the involution is isometric, so that

$$\tilde{T}x^* = \lim_{n\to\infty} \varphi_W(T(x_n^*)) = \lim_{n\to\infty} \varphi_W(T(x_n))^* = (\tilde{T}x)^*,$$

proving (vi). $\square$

Corollary A.2. *Let $\mathfrak{X}$ be a Banach space with a dense subspace V. Let $\varphi\colon V\times V\to\mathbb{C}$ be a sesquilinear form for which there exists $C\geq 0$ such that*

$$|\varphi(x,y)|\leq k\|x\|\|y\|,\quad x,y\in V.$$

(We say that φ is bounded *by k.) Then φ extends uniquely to a sesquilinear form $\tilde{\varphi}\colon\mathfrak{X}\times\mathfrak{X}\to\mathbb{C}$ such that $|\tilde{\varphi}(x,y)|\leq k\|x\|\|y\|$ for all $x,y\in\mathfrak{X}$. Moreover:*

(i) *If φ is Hermitian, i.e. $\varphi(x,y)=\overline{\varphi(y,x)}$ for all $x,y\in V$, then $\tilde{\varphi}$ is Hermitian.*
(ii) *If φ is positive, i.e. $\varphi(x,x)\geq 0$ for all $x\in V$, then $\tilde{\varphi}$ is positive.*

Proof. Fix $y\in V$ and note that the map $V\to\mathbb{C}$ given by $x\mapsto\varphi(x,y)$ is a linear functional on V, bounded by $k\|y\|$. Proposition A.1 tells us that there exists a unique bounded linear functional $\varphi_y\colon\mathfrak{X}\to\mathbb{C}$ such that $\varphi_y(x)=\varphi(x,y)$ for all $x\in V$, also satisfying $\|\varphi_y\|\leq k\|y\|$. Hence we obtain a map $\Omega\colon V\to\mathfrak{X}^*$ given by $\Omega(y)=\varphi_y$. For any given $y_1,y_2\in V$ and $\lambda_1,\lambda_2\in\mathbb{C}$ we have

$$\Omega(\lambda_1y_1+\lambda_2y_2)(x)=\varphi(x,\lambda_1y_1+\lambda_2y_2)=\overline{\lambda_1}\varphi(x,y_1)+\overline{\lambda_2}\varphi(x,y_2)=\overline{\lambda_1}\varphi_{y_1}(x)+\overline{\lambda_2}\varphi_{y_2}(x)$$

for all $x\in V$, so uniqueness of φ_y yields that Ω is in fact conjugate linear. Moreover,

$$|\Omega(y)(x)|=|\varphi(x,y)|\leq k\|x\|\|y\|,\quad x\in\mathfrak{X},\ y\in V,$$

so $\|\Omega\|\leq k$. Proposition A.1 now says that Ω extends to a unique conjugate linear operator $\tilde{\Omega}\colon\mathfrak{X}\to\mathfrak{X}^*$ that uniquely satisfies $\tilde{\Omega}(y)=\varphi_y$ for all $y\in V$ and also satisfies $\|\tilde{\Omega}\|\leq k$. Define $\tilde{\varphi}\colon\mathfrak{X}\times\mathfrak{X}\to\mathbb{C}$ by $\tilde{\varphi}(x,y)=\tilde{\Omega}(y)(x)$. Then $\tilde{\varphi}$ is sesquilinear and

$$|\tilde{\varphi}(x,y)|=|\tilde{\Omega}(y)(x)|\leq\|\tilde{\Omega}(y)\|\|x\|\leq k\|x\|\|y\|$$

for all $x,y\in\mathfrak{X}$. Finally, $\tilde{\varphi}(x,y)=\tilde{\Omega}(y)(x)=\varphi_y(x)=\varphi(x,y)$ for all $x,y\in V$, so $\tilde{\varphi}$ extends φ.

For uniqueness, let $\psi\colon\mathfrak{X}\times\mathfrak{X}\to\mathbb{C}$ be another sesquilinear form that extends φ and is bounded by k. For any $y\in\mathfrak{X}$, note that $\psi_y\colon\mathfrak{X}\to\mathbb{C}$ given by $\psi_y(x)=\psi(x,y)$ is a bounded linear functional on $\mathfrak{X}$. Note now that $\psi_y(x)=\psi(x,y)=\varphi(x,y)=\varphi_y(x)$ for all $x,y\in V$. Defining $\Psi\colon\mathfrak{X}\to\mathfrak{X}^*$ by $\Psi(y)=\psi_y$, Ψ is continuous as it satisfies $\|\Psi(y)\|\leq k\|y\|$ for all $y\in\mathfrak{X}$; moreover $\Psi(y)=\varphi_y=\tilde{\Omega}(y)$ for all $y\in V$. Hence $\Psi=\tilde{\Omega}$ by continuity, and therefore

$$\tilde{\varphi}(x,y)=\tilde{\Omega}(y)(x)=\Psi(y)(x)=\psi(x,y)$$

for all $x,y\in\mathfrak{X}$, so $\tilde{\varphi}$ is uniquely determined by the boundedness and sesquilinearity of φ.

Note that if $x,y\in\mathfrak{X}$ and $x_n\to x$ and $y_n\to y$ for sequences $(x_n)_{n\geq1}$ and $(y_n)_{n\geq1}$ in $\mathfrak{X}$, then

$$|\varphi(x_n,y_n)-\varphi(x,y)|\leq k(\|x_n-x\|\|y_n\|+\|x\|\|y_n-y\|)\to 0$$

as $(\|y_n\|)_{n\geq1}$ is bounded, so $\varphi(x_n,y_n)\to\varphi(x,y)$. For $x,y\in\mathfrak{X}$ choose sequences $(x_n)_{n\geq1}$ and $(y_n)_{n\geq1}$ in V such that $x_n\to x$ and $y_n\to x$. If φ is Hermitian, then

$$\tilde{\varphi}(x,y)=\lim_{n\to\infty}\varphi(x_n,y_n)=\overline{\lim_{n\to\infty}\varphi(y_n,x_n)}=\overline{\tilde{\varphi}(y,x)},$$

and thus $\tilde{\varphi}$ is Hermitian as well. If φ is positive, $\tilde{\varphi}(x,x)=\lim_{n\to\infty}\varphi(x_n,x_n)\geq 0$, so $\tilde{\varphi}$ is also positive, completing the proof. ☐

1.2 A property of the weak* topology on state spaces

Recall that for any Banach space $\mathfrak{X}$ that the *weak* topology* on $\mathfrak{X}^*$ is the locally convex Hausdorff topology given by the separating family of seminorms given by $\varphi\mapsto|\varphi(x)|$ for $x\in\mathfrak{X}$. Hence in $\mathfrak{X}^*$ a net $(\varphi_\alpha)_{\alpha\in A}$ converges to φ in the weak* topology if and only if $\varphi_\alpha(x)\to\varphi(x)$ for all $x\in\mathfrak{X}$.

We use the opportunity to give a characterisation of weak* continuous linear functionals.

Lemma A.3. *Let $\mathfrak{X}$ be a Banach space and let $\psi\colon\mathfrak{X}^*\to\mathbb{C}$ be a linear functional. Then ψ is weak* continuous if and only if there exists an $x\in\mathfrak{X}$ such that $\psi(\varphi)=\varphi(x)$ for all $\varphi\in\mathfrak{X}^*$.*

Proof. Assume that ψ is weak* continuous. Then there exist $x_1, \dots, x_n \in \mathfrak{X}$ and $\varepsilon > 0$ such that

$$\{\varphi \in \mathfrak{X}^* \mid |\varphi(x_i)| < \varepsilon,\ 1 \le i \le n\} \subseteq \{\varphi \in \mathfrak{X}^* \mid |\psi(\varphi)| < 1\}.$$

By defining a linear functional $\hat{x}\colon \mathfrak{X}^* \to \mathbb{C}$ by $\hat{x}(\varphi) = \varphi(x)$, then if $\hat{x_i}(\varphi) = 0$ for all $1 \le i \le n$ we see that $\hat{x_i}(\lambda\varphi) = 0$ for all $\lambda \in \mathbb{C}$ and $1 \le i \le n$. This implies $|\lambda||\psi(\varphi)| < 1$, so $\psi(\varphi) = 0$. Hence

$$\bigcup_{i=1}^{n} \ker \hat{x_i} \subseteq \ker \psi.$$

Defining a map $\pi\colon \mathfrak{X}^* \to \mathbb{C}^n$ by $\pi(\varphi) = (\varphi(x_1), \dots, \varphi(x_n))$, then $\ker \pi = \bigcup_{i=1}^{n} \ker \hat{x_i} \subseteq \ker \psi$, so π induces a linear map $\gamma\colon \pi(\mathfrak{X}^*) \to \mathbb{C}$ given by $\gamma(\pi(\varphi)) = \psi(\varphi)$. Since $\pi(\mathfrak{X}^*)$ is a complete subspace of the Hilbert space $\mathbb{C}^n$, the Riesz representation theorem [14, Theorem 2.3.1] provides a $\lambda = (\lambda_1, \dots, \lambda_n) \in \pi(\mathfrak{X}^*) \subseteq \mathbb{C}^n$ such that $\gamma(\pi(\varphi)) = \langle \pi(\varphi), \lambda \rangle$. Hence

$$\psi(\varphi) = \sum_{i=1}^{n} \lambda_i \varphi(x_i) = \varphi\left(\sum_{i=1}^{n} \lambda_i x_i\right).$$

The other implication is trivial. □

The above lemma is all the more effective when combined with the Hahn-Banach separation theorem for the weak* topology as the next lemma amply demonstrates.

Lemma A.4. *Let $\mathcal{A}$ be a unital C^*-algebra and let $\mathfrak{X} \subseteq S(\mathcal{A})$. Suppose that it holds for all $a \in \mathcal{A}_{\mathrm{sa}}$ that $\varphi(a) \ge 0$ for all $\varphi \in \mathfrak{X}$ implies $a \in \mathcal{A}_+$. Then the weak* closure $\mathscr{S}$ of the convex hull of $\mathfrak{X}$ is equal to $S(\mathcal{A})$, i.e. the convex hull of $\mathfrak{X}$ is weak*-dense in $\mathcal{A}$.*

Proof. Since $S(\mathcal{A})$ is weak*-compact and convex by [31, Proposition 13.8], it is clear that $\mathscr{S} \subseteq S(\mathcal{A})$ no matter what. Suppose that there exists $\varphi \in S(\mathcal{A})$ such that $\varphi \notin \mathscr{S}$. By the Hahn-Banach separation theorem for locally convex topological vector spaces [14, Corollary 1.2.12] and Lemma A.3, there is a weak*-continuous linear functional on $\mathcal{A}^*$, thus an $a \in \mathcal{A}$, and a real number λ such that

$$\mathrm{Re}\psi(a) \le \lambda < \mathrm{Re}\varphi(a), \quad \psi \in \mathscr{S}.$$

Letting $b = \frac{1}{2}(a + a^*)$, we have $\psi(b) = \frac{1}{2}(\psi(a) + \overline{\psi(a)}) = \mathrm{Re}\psi(a)$ for all $\psi \in S(\mathcal{A})$ so that

$$\psi(b) \le \lambda < \varphi(b), \quad \psi \in \mathscr{S}.$$

As $\lambda 1_{\mathcal{B}} - b \in \mathcal{A}_{\mathrm{sa}}$ and $\psi(\lambda 1_{\mathcal{B}} - b) = \lambda - \psi(b) \ge 0$ for all $\psi \in \mathfrak{X}$, we have $\lambda 1_{\mathcal{B}} - b \in \mathcal{A}_+$. As φ is a state, it follows that $\varphi(b) \le \lambda$, a contradiction. Hence $\mathscr{S} = S(\mathcal{A})$. □

1.3 The point-norm and point-weak topology

Another pair of locally convex topologies become useful in Chapter 5. Here they are.

Definition A.1. Let $\mathfrak{X}$ be a Banach space. The *point-norm topology* on $B(\mathfrak{X})$ is the locally convex Hausdorff topology generated by the separating family of seminorms

$$T \mapsto \|Tx\|, \quad x \in \mathfrak{X},$$

and the *point-weak topology* on $B(\mathfrak{X})$ is the locally convex Hausdorff topology likewise generated by the separating family of seminorms given by

$$T \mapsto |\varphi(Tx)|, \quad x \in \mathfrak{X},\ \varphi \in \mathfrak{X}^*.$$

Note that if $\mathfrak{X}$ is a Hilbert space, then the point-norm topology and point-weak topology are respectively just the strong operator topology and the weak operator topology on $B(\mathfrak{X})$ by the Riesz representation theorem [13, Theorem 5.25]. Similarly, for an arbitrary Banach space $\mathfrak{X}$, if $(T_\alpha)_{\alpha \in A}$ is a net in $B(\mathfrak{X})$, $T \in B(\mathfrak{X})$ and $T_\alpha \to T$ in the point-norm topology, then $T_\alpha \to T$ in the point-weak topology.

Proposition A.5. *Let $\mathfrak{X}$ be a Banach space. If $\omega\colon B(\mathfrak{X}) \to \mathbb{C}$ is a linear functional, then the following are equivalent:*

(i) *ω is continuous with respect to the point-norm topology.*
(ii) *ω is continuous with respect to the point-weak topology.*
(iii) *There exist $x_1, \ldots, x_n \in \mathfrak{X}$ and $\varphi_1, \ldots, \varphi_n \in \mathfrak{X}^*$ such that*

$$\omega(T) = \sum_{i=1}^{n} \varphi_i(Tx_i), \quad T \in B(\mathfrak{X}).$$

Proof. The implications (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) are obvious. Assume that ω is continuous with respect to the point-norm topology, and take $C > 0$ and $x_1, \ldots, x_n \in \mathfrak{X}$ such that

$$|\omega(T)| \leq C \sum_{i=1}^{n} \|Tx_i\|, \quad T \in B(\mathfrak{X}) \tag{A.1}$$

by [13, Proposition 5.15]. Equip the vector space $\mathfrak{X}^n = \mathfrak{X} \oplus \ldots \oplus \mathfrak{X}$ with the norm $\|\cdot\|_{\text{taxi}}$, i.e.

$$\|(y_1, \ldots, y_n)\|_{\text{taxi}} = \sum_{i=1}^{n} \|y_i\|, \quad y_1, \ldots, y_n \in \mathfrak{X}$$

(the reason for the name is because the norm is often called the *taxicab norm*) and let

$$\mathfrak{Y} = \{(Tx_1, \ldots, Tx_n) \,|\, T \in B(\mathfrak{X})\}.$$

$\mathfrak{Y}$ is then a subspace of $\mathfrak{X}^n$. Define $\Phi_0 \colon \mathfrak{Y} \to \mathbb{C}$ by

$$\Phi_0((Tx_1, \ldots, Tx_n)) = \omega(T), \quad T \in B(\mathfrak{X}).$$

Φ_0 is then well-defined by (A.1), linear and $\|\Phi_0\| \leq C$. By the Hahn-Banach theorem [13, Theorem 5.7], there exists a linear functional $\Phi \colon \mathfrak{X}^n \to \mathbb{C}$ such that $\Phi|_{\mathfrak{Y}} = \Phi_0$ and $\|\Phi\| = \|\Phi_0\| \leq C$. Define a linear functional $\varphi_i \in \mathfrak{X}^*$ for all $i = 1, \ldots, n$ by

$$\varphi_i(x) = \Phi(\iota_i(x)),$$

where $\iota_i \colon \mathfrak{X} \to \mathfrak{X}^n$ is the inclusion into the i'th copy of $\mathfrak{X}$ in $\mathfrak{X}^n$. Then

$$\omega(T) = \Phi((Tx_1, \ldots, Tx_n)) = \sum_{i=1}^{n} \varphi_i(Tx_i), \quad T \in B(\mathfrak{X}),$$

establishing (i) $\Rightarrow$ (iii). □

Proposition A.6. *Let $\mathfrak{X}$ be a locally convex topological vector space and let $\mathfrak{Y} \subseteq \mathfrak{X}$ be a convex subset. Then the following are equivalent:*

(i) *$y \in \overline{\mathfrak{Y}}$.*
(ii) *There exists a net $(y_\alpha)_{\alpha \in A}$ in $\mathfrak{Y}$ such that $\varphi(y_\alpha) \to \varphi(y)$ for all continuous linear functionals φ on $\mathfrak{X}$.*

Proof. (i) $\Rightarrow$ (ii) is obvious. Assume for the converse that (i) does not hold. Since $\overline{\mathfrak{Y}}$ is convex, then from the Hahn-Banach separation theorem for locally convex topological vector spaces (cf. [14, Corollary 1.2.12]), it follows that there exists a continuous linear functional $\varphi \colon \mathfrak{X} \to \mathbb{C}$ and $\lambda \in \mathbb{R}$ such that

$$\mathrm{Re}\varphi(x) > \lambda \geq \mathrm{Re}\varphi(y), \quad y \in \overline{\mathfrak{Y}}.$$

In particular, $\mathrm{Re}\varphi(y_\alpha) \leq \lambda$ for any net $(y_\alpha)_{\alpha \in A}$ in $\mathfrak{Y}$, so $\mathrm{Re}\varphi(y_\alpha)$ cannot converge to $\mathrm{Re}\varphi(y)$ as that would imply $\mathrm{Re}\varphi(y) \leq \lambda$. Hence $\varphi(y_\alpha) \not\to \varphi(y)$ for any net $(y_\alpha)_{\alpha \in A}$ in $\mathfrak{Y}$, so (ii) does not hold either. □

The next result is really the essential one. Everybody uses it all the time, even in their sleep.

Theorem A.7. *Let $\mathfrak{X}$ be a vector space equipped with two locally convex topologies τ_1 and τ_2. Assume that the set of linear functionals on $\mathfrak{X}$ that are continuous with respect to τ_1 coincides with the set of linear functionals continuous with respect to τ_2. Then for any convex subset $\mathfrak{Y} \subseteq \mathfrak{X}$, the closures of $\mathfrak{Y}$ with respect to τ_1 and τ_2 are equal.*

Proof. For any $i = 1, 2$, Proposition A.6 tells us that $y \in \overline{\mathfrak{Y}}^{\tau_i}$ if only if there exists a net $(y_\alpha)_{\alpha \in A}$ such that $\varphi(y_\alpha) \to \varphi(y)$ for all τ_i-continuous linear functionals φ on $\mathfrak{X}$. As the sets of τ_1-continuous and τ_2-continuous linear functionals on $\mathfrak{X}$ coincide, the result follows. □

The connection to the topologies defined in this section follows immediately from Proposition A.5:

Corollary A.8. *For any Banach space $\mathfrak{X}$ and any convex subset $\mathscr{S} \subseteq B(\mathfrak{X})$, the point-norm and point-weak closures of $\mathscr{S}$ coincide.*

1.4 Everything you always wanted to know about finite rank operators

This section contains a lot of small results about finite rank operators, including a C^*-algebraic proof of the fact that any finite rank operator on a Hilbert space has finite spectrum. Lemmas abound!

In the general Banach space case, two results are all we will need.

Lemma A.9. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach spaces and let $\psi\colon \mathcal{A} \to \mathcal{B}$ be a bounded linear map of finite rank. Then there exist $b_1, \ldots, b_n \in \mathcal{B}$ and $\varphi_1, \ldots, \varphi_n \in \mathcal{A}^*$ such that*

$$\psi(a) = \sum_{i=1}^{n} \varphi_i(a) b_i, \quad a \in \mathcal{A}.$$

Proof. Let $\{b_1, \ldots, b_n\}$ be a vector basis for $\psi(\mathcal{A})$. It is then clear that ψ is of the above form, so we only need to prove that the φ_i are linear and bounded. Linearity is easy, since for $a, b \in \mathcal{A}$ and $\lambda_1, \lambda_2 \in \mathbb{C}$, then

$$\sum_{i=1}^{n} (\lambda_1 \varphi_i(a) + \lambda_2 \varphi_i(b)) b_i = \lambda_1 \psi(a) + \lambda_2 \psi(b) = \psi(\lambda_1 a + \lambda_2 b) = \sum_{i=1}^{n} \varphi_i(\lambda_1 a + \lambda_2 b) b_i,$$

yielding

$$\varphi_i(\lambda_1 a + \lambda_2 b) = \lambda_1 \varphi_i(a) + \lambda_2 \varphi_i(b)$$

for all $i = 1, \ldots, n$ by linear independence of the b_i. Define $\omega_1\colon \mathcal{A} \to \mathbb{C}^n$ by $\omega_1(a) = (\varphi_1(a), \ldots, \varphi_n(a))$ and $\omega_2\colon \mathbb{C}^n \to \varphi(\mathcal{A})$ by $\omega_2(\lambda_1, \ldots, \lambda_n) = \sum_{i=1}^{n} \lambda_i b_i$. Then $\psi = \omega_2 \circ \omega_1$. As ω_2 is clearly a bounded isomorphism with respect to the $\|\cdot\|_1$-norm, it follows from the Open Mapping Theorem [13, Corollary 5.11] that ω_2^{-1} is bounded. Hence for all $i = 1, \ldots, n$ and $a \in \mathcal{A}$, it follows that

$$|\varphi_i(a)| \leq \|\omega_1(a)\|_1 = \|\omega_2^{-1}(\psi(a))\|_1 \leq \|\omega_2^{-1}\| \|\psi\| \|a\|,$$

so all the φ_i are bounded; hence $\varphi_i \in \mathcal{A}^*$ for all $i = 1, \ldots, n$. □

Note that this implies, along with the Riesz representation theorem [14, Theorem 2.3.1], that any finite rank operator on Hilbert space is a finite sum of elementary operators, as found in Section 2.2.

Lemma A.10. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach spaces and let $\psi\colon \mathcal{A} \to \mathcal{B}$ be a bounded linear map of finite rank. Then $\psi^*\colon \mathcal{B}^* \to \mathcal{A}^*$ is of finite rank as well.*

Proof. Using Lemma A.9, there exist $a_1, \ldots, a_n \in \mathcal{A}$, $\varphi_1, \ldots, \varphi_n \in \mathcal{A}^*$ such that

$$\psi(a) = \sum_{j=1}^{n} \varphi_j(a) a_j, \quad a \in \mathcal{A}.$$

Hence for $\omega \in \mathcal{A}^*$ and $a \in \mathcal{A}$, we have

$$\psi^*(\omega)(a) = \omega(\psi(a)) = \sum_{j=1}^{n} \omega(a_j) \varphi_j(a).$$

Therefore $\psi^*(\omega) = \sum_{j=1}^{n} \omega(a_j) \varphi_j$, so ψ^* has finite rank. □

Recall that if $\mathcal{A}$ is a unital C^*-algebra and $p \in \mathcal{A}$ is a non-zero projection, then $p\mathcal{A}p$ is a C^*-algebra with unit p.

Lemma A.11. *Let $\mathcal{A}$ be a unital C^*-algebra and let $p \in \mathcal{A}$ be a projection different from 0 and $1_\mathcal{A}$. Then $x \in p\mathcal{A}p$ and $y \in (1_\mathcal{A} - p)\mathcal{A}(1_\mathcal{A} - p)$ are invertible if and only if $x + y$ is invertible in $\mathcal{A}$.*

Proof. If $x + y$ is invertible in $\mathcal{A}$ with inverse $z \in \mathcal{A}$, then note that $(x + y)p = xp = px = p(x + y)$, so that $(pzp)x = pz(x + y)p = p$ and $x(pzp) = p(x + y)zp = p$, so that x is invertible with inverse $pzp \in p\mathcal{A}p$. In a similar manner, one proves that y is invertible with inverse $(1_\mathcal{A} - p)z(1_\mathcal{A} - p)$. For the converse statement, let $a \in p\mathcal{A}p$ be the inverse of x and $b \in (1_\mathcal{A} - p)\mathcal{A}(1_\mathcal{A} - p)$ be the inverse of y. Then it is easy to see that $a + b \in \mathcal{A}$ is the inverse of $x + y$. □

Lemma A.12. *Let $\mathcal{A}$ be a unital C^*-algebra, let $p \in \mathcal{A}$ be a projection different from 0 and $1_\mathcal{A}$ and let $x \in p\mathcal{A}p$, so that $x = pxp$. Then*

$$\sigma_\mathcal{A}(x) = \sigma_{p\mathcal{A}p}(x) \cup \{0\}.$$

Proof. For any $\lambda \in \mathbb{C}$ we have

$$\begin{aligned} \lambda \notin \sigma_{p\mathcal{A}p}(x) \text{ and } \lambda \neq 0 &\Leftrightarrow x - \lambda p \text{ is invertible in } p\mathcal{A}p \text{ and } \lambda \neq 0 \\ &\Leftrightarrow x - \lambda p - \lambda(1_\mathcal{A} - p) \text{ is invertible in } \mathcal{A} \\ &\Leftrightarrow x - \lambda 1_\mathcal{A} \text{ is invertible in } \mathcal{A} \\ &\Leftrightarrow \lambda \notin \sigma_\mathcal{A}(x). \end{aligned}$$

At the second biconditional, we used Lemma A.11 along with the fact that $-\lambda(1_\mathcal{A} - p)$ is invertible in $(1_\mathcal{A} - p)\mathcal{A}(1_\mathcal{A} - p)$ for all $\lambda \neq 0$. □

Lemma A.13. *For a Hilbert space $\mathcal{H}$ and any non-zero projection $P \in B(\mathcal{H})$, the restriction map $PB(\mathcal{H})P \to B(P(\mathcal{H}))$ is a unital *-isomorphism.*

Proof. The reader will hopefully forgive the ramshackle proof. Define $\varphi \colon PB(\mathcal{H})P \to B(P(\mathcal{H}))$ by $\varphi(T) = T|_\mathfrak{X}$ where $\mathfrak{X} = P(\mathcal{H})$. φ is clearly additive and unital. Moreover, for $S, T \in PB(\mathcal{H})P$, we have $PSP = S$ and $PTP = T$ and therefore

$$\varphi(ST) = ST|_\mathfrak{X} = (PSP)(PTP)|_\mathfrak{X} = (PSP)|_\mathfrak{X}(PTP)|_\mathfrak{X} = S|_\mathfrak{X}T|_\mathfrak{X} = \varphi(S)\varphi(T),$$

so φ is multiplicative. For any $\xi, \eta \in P(\mathcal{H})$ and $S \in PB(\mathcal{H})P$, note that

$$\langle \xi, \varphi(S^*)\eta = \langle \xi, S^*\eta\rangle = \langle S\xi, \eta\rangle = \langle \varphi(S)\xi, \eta\rangle,$$

so $\varphi(S^*) = \varphi(S)^*$. Therefore φ is a unital *-homomorphism. φ is surjective; indeed if $T \in B(P(\mathcal{H}))$, then the operator $PTP \in PB(\mathcal{H})P$ has image T under φ. Finally, if $S|_\mathfrak{X} = T|_\mathfrak{X}$ for $S, T \in PB(\mathcal{H})P$, then $S\xi = SP\xi = TP\xi = T\xi$ for all $\xi \in \mathcal{H}$, so φ is injective. □

Lemma A.14. *For any $n \geq 1$ and $A \in M_n(\mathbb{C})$, $\sigma(a)$ is the set of eigenvalues of A.*

Proof. Let $I \in M_n(\mathbb{C})$ denote the identity matrix. For $\lambda \in \mathbb{C}$, $\lambda I - A$ is not invertible if and only if it is not injective. Hence $\lambda I - A$ is not invertible if and only if there exists a non-zero vector $x \in \mathbb{C}^n$ such that $(\lambda I - A)x = 0$ or $Ax = \lambda x$, i.e. if λ is an eigenvalue of A. □

Recall that if a Hilbert space $\mathcal{H}$ is finite-dimensional, then it is isometrically isomorphic to $\mathbb{C}^n$ for some $n \geq 1$. Thus $B(\mathcal{H}) \cong M_n(\mathbb{C})$. As any complex $n \times n$ matrix has finitely many eigenvalues, it then follows that any $T \in B(\mathcal{H})$ has finite spectrum since unital *-isomorphisms preserve spectra [31, Corollary 9.3].

Lemma A.15. *A finite rank operator T on a Hilbert space $\mathcal{H}$ has finite spectrum.*

Proof. We can assume that $T \neq 0$. Let $\xi_1, \ldots, \xi_n$ be an orthonormal basis for $T(\mathcal{H})$, and let $\mathcal{H}_0$ be the linear span of the vectors $\xi_1, \ldots, \xi_n$ and $T^*\xi_1, \ldots, T^*\xi_n$. Then $\mathcal{H}_0$ is a Hilbert space and $N = \dim \mathcal{H}_0 < \infty$. Note that for all $\xi \in \mathcal{H}$, $T\xi \in \mathcal{H}_0$. Moreover, since there exist $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ such that $T\xi = \sum_{i=1}^n \lambda_i \xi_i$, then $T^*T\xi \in \mathcal{H}_0$. If P is the orthogonal projection onto $\mathcal{H}_0$, it is clear that $PT = T$. If $\xi \in \mathcal{H}_0^\perp$, then

$$0 = \langle T^*T\xi, \xi\rangle = \langle T\xi, T\xi\rangle = \|T\xi\|^2.$$

Thus $T(1_\mathcal{H} - P) = 0$, so $TP = T$ and hence $PTP = T$. As $PB(\mathcal{H})P \cong B(\mathcal{H}_0) \cong M_N(\mathbb{C})$ by Lemma A.13, we see that $\sigma_{PB(\mathcal{H})P}(T)$ is finite. Hence Lemma A.12 tells us that T has finite spectrum. □

1.5 Separable C^*-algebras

Recall that a metric space is *separable* if it has a countable dense subset. The last result of this section is needed in Chapter 3 to prove a criterion equal to positivity of a matrix with C^*-algebra entries.

Lemma A.16. *Any non-empty subset $\mathscr{S}$ of a separable metric space $\mathfrak{X}$ with metric d is separable.*

Proof. Let $\{x_n\}_{n\geq 1}$ be a countable dense subset of $\mathfrak{X}$. Let $\{q_m\}_{m\geq 1}$ be an enumeration of the positive rational numbers, and let $\mathscr{S}_{n,m} = \{x \in \mathscr{S} \,|\, d(x, x_n) < q_m\}$. For any $n, m \geq 1$, let $x_{n,m}$ be a point of $\mathscr{S}_{n,m}$ if $\mathscr{S}_{n,m}$ is non-empty. The collection of all $x_{n,m}$ is clearly non-empty – otherwise $\mathfrak{X}$ would not be separable – and moreover countable. Now let $x \in \mathscr{S}$ and let $\varepsilon > 0$. Then there exists $n \geq 1$ such that $d(x, x_n) < \frac{\varepsilon}{4}$ and $m \geq 1$ such that $\frac{\varepsilon}{4} < q_m < \frac{\varepsilon}{2}$. As $x \in \mathscr{S}_{n,m}$, then $\mathscr{S}_{n,m}$ is non-empty and hence $d(x, x_{n,m}) < d(x, x_n) + d(x_n, x_{n,m}) < \frac{\varepsilon}{4} + \frac{\varepsilon}{2} < \varepsilon$. Hence the collection of all $x_{n,m}$ is a countable dense subset of $\mathscr{S}$. □

The above metric space result then has the following application.

Proposition A.17. *Let $\mathcal{A}$ be a separable C^*-algebra. Then $\mathcal{A}$ has a faithful state.*

Proof. Since $\mathcal{A}$ is separable, then the preceding lemma tells us that $\mathcal{B} = \{a \in \mathcal{A}_+ \,|\, \|a\| = 1\}$ has a countable dense subset $\{a_n \,|\, n \geq 1\}$. For each $n \geq 1$, choose $\varphi_n \in S(\mathcal{A})$ such that $\varphi_n(a_n) = \|a_n\| = 1$ by Theorem 0.2. Define $\varphi\colon \mathcal{A} \to \mathbb{C}$ by

$$\varphi(a) = \sum_{n=1}^{\infty} 2^{-n}\varphi_n(a).$$

φ is clearly well-defined, linear and satisfies $\|\varphi\| \leq \sum_{n=1}^{\infty} 2^{-n} = 1$. Moreover, as

$$\varphi(1_{\mathcal{A}}) = \sum_{n=1}^{\infty} 2^{-n} = 1,$$

we see φ is a state. Moreover, φ is faithful. Indeed, let $a \in \mathcal{A}$ be non-zero and positive and let $a' = \|a\|^{-1}a$. Then $a' \in \mathcal{B}$, so there exists $n \geq 1$ such that $\|a' - a_n\| < 1$. Then

$$|1 - \varphi_n(a')| = |\varphi_n(a_n - a')| \leq \|a_n - a'\| < 1,$$

so $\varphi_n(a') > 0$ and hence $\varphi(a') = \sum_{m=1}^{\infty} 2^{-m}\varphi_m(a') \geq \varphi_n(a') > 0$. Therefore $\varphi(a) > 0$, completing the proof. □

1.6 Effects of second adjoint maps on exact sequences

This section is devoted to one small lemma concerning what happens to inclusion and quotient maps on Banach spaces.

Lemma A.18. *Let $\mathfrak{X}$ be a Banach space and $\mathfrak{Y} \subseteq \mathfrak{X}$ a closed subspace, with the inclusion map $j\colon \mathfrak{Y} \hookrightarrow \mathfrak{X}$ and quotient map $\pi\colon \mathfrak{X} \to \mathfrak{X}/\mathfrak{Y}$. Then*

- (i) $j^{**}\colon \mathfrak{Y}^{**} \to \mathfrak{X}^{**}$ *is injective.*
- (ii) $\ker \pi^{**}$ *is equal to the weak* closures of $j^{**}(\iota_{\mathfrak{Y}}(\mathfrak{Y}))$ and $j^{**}(\mathfrak{Y}^{**})$ where $\iota_{\mathfrak{Y}}\colon \mathfrak{Y} \to \mathfrak{Y}^{**}$ is the canonical inclusion.*
- (iii) $(\mathfrak{X}/\mathfrak{Y})^{**}$ *is equal to the weak*-closures of $\pi^{**}(\iota_{\mathfrak{X}}(\mathfrak{X}))$ and $\pi^{**}(\mathfrak{X}^{**})$ where $\iota_{\mathfrak{X}}\colon \mathfrak{X} \to \mathfrak{X}^{**}$ is the canonical inclusion.*

Proof. Let $\iota_{\mathfrak{X}}$, $\iota_{\mathfrak{Y}}$ and $\iota_{\mathfrak{X}/\mathfrak{Y}}$ denote the inclusions into the biduals of $\mathfrak{X}$, $\mathfrak{Y}$ and $\mathfrak{X}/\mathfrak{Y}$ respectively, and recall that $\iota_{\mathfrak{X}} \circ j = j^{**} \circ \iota_{\mathfrak{Y}}$ and $\iota_{\mathfrak{X}/\mathfrak{Y}} \circ \pi = \pi^{**} \circ \iota_{\mathfrak{X}}$. As we will be using results about weak* convergence to no avail in this proof, we will abbreviate "in the weak* topology" as "w^*".

Assume that $j^{**}(\varphi) = 0$ for some $\varphi \in \mathfrak{Y}^{**}$. By Goldstine's theorem [29, Theorem II.A.13], there exists some net $(y_\alpha)_{\alpha\in A}$ in $\mathfrak{Y}$ such that $\iota_{\mathfrak{Y}}(y_\alpha) \to \varphi$ w^*. By w^*-continuity of the second adjoint maps, we then have $\iota_{\mathfrak{X}}(y_\alpha) \to 0$ w^*, or $\psi(y_\alpha) \to 0$ for all $\psi \in \mathfrak{X}^*$. The Hahn-Banach theorem [13, Theorem 5.7] now implies $\psi'(y_\alpha) \to 0$ for all $\psi' \in \mathfrak{Y}^*$, so $\iota_{\mathfrak{Y}}(y_\alpha) \to 0$ w^* and thus $\varphi = 0$. Hence j^{**} is injective, and moreover $j^{**}(\mathfrak{Y}^{**}) \subseteq \ker \pi^{**}$, as $\pi^{**} \circ j^{**} = (\pi \circ j)^{**} = 0$.

Assume now that $\varphi \in \ker \pi^{**}$. Then $\varphi \circ \pi^* = 0$ for all $\psi \in (\mathfrak{X}/\mathfrak{Y})^*$ or $\varphi(\psi) = 0$ for all $\psi \in \pi^*((\mathfrak{X}/\mathfrak{Y})^*)$. By [22, Theorem 4.9(b)],

$$\pi^*((\mathfrak{X}/\mathfrak{Y})^*) = \mathfrak{Y}^\perp := \{\psi \in \mathfrak{X}^* \mid \psi(y) = 0 \text{ for all } y \in \mathfrak{Y}\} = \{\psi \in \mathfrak{X}^* \mid \psi \circ j = 0\},$$

so $\varphi(\psi) = 0$ for all $\psi \in \mathfrak{Y}^\perp$. Assume now that $\varphi \notin \mathfrak{Z}$ where $\mathfrak{Z}$ denotes the w^*-closure of $j^{**}(\iota_{\mathfrak{Y}}(\mathfrak{Y})) \subseteq \mathfrak{X}^{**}$. By the Hahn-Banach theorem for locally convex topological vector spaces [14, Corollary 1.2.13], there is $\omega \in \mathfrak{X}^*$ such that $\psi(\omega) = 0$ for all $\psi \in \mathfrak{Z}$ and $\varphi(\omega) \neq 0$. Hence for all $y \in \mathfrak{Y}$, we have

$$0 = j^{**}(\iota_{\mathfrak{Y}}(y))(\omega) = \iota_{\mathfrak{Y}}(y)(j^*(\omega)) = \omega(j(y)) = \omega(y),$$

so $\omega \in \mathfrak{Y}^\perp$. Hence $\varphi(\omega) = 0$, a contradiction, so $\varphi \in \mathfrak{Z}$. Hence $\ker \pi^{**}$ is equal to the w^*-closure of $j^{**}(\iota_{\mathfrak{Y}}(\mathfrak{Y}))$, from which the last part follows by continuity of j^{**}.

Finally, note that $\iota_{\mathfrak{X}/\mathfrak{Y}}(\mathfrak{X}/\mathfrak{Y}) \subseteq \pi^{**}(\iota_{\mathfrak{X}}(\mathfrak{X}))$, so that the w^*-closure of $\pi^{**}(\iota_{\mathfrak{X}}(\mathfrak{X}))$ is $(\mathfrak{X}/\mathfrak{Y})^{**}$. The last part follows similarly. □

For the record, this result might not be very intriguing in itself, but watch what happens in Proposition 3.14 when we start working with second adjoints of *-homomorphisms. *That* result owes a lot to what we have just proved, but also proves that *-homomorphisms are slightly more "magical" than linear maps, for lack of a better term.

APPENDIX B

TRACE CLASS AND HILBERT-SCHMIDT OPERATORS

We will in this chapter discuss an important class of bounded operators on a Hilbert space. For the sake of completeness, we include some results concerning generalized sums. In the following, let $\mathcal{H}$ denote a Hilbert space and for any non-empty set I, let $\mathfrak{F}_I$ denote the directed set of finite subsets of I.

Proposition B.1. *Let $(x_i)_{i\in I}$ be a family of non-negative real numbers indexed by a non-empty set I. If $\sum_{i\in I} x_i$ converges in $\mathbb{R}$, then*

$$\sum_{i\in I} x_i = \sup_{G\in\mathfrak{F}_I} \sum_{i\in G} x_i.$$

Conversely, if the above supremum exists in $\mathbb{R}$, then the sum converges and is equal to the supremum.

Proof. The inequality $\leq$ is obvious. Let $x = \sum_{i\in I} x_i$. For $\varepsilon > 0$, let $F \in \mathfrak{F}_I$ be a finite subset such that for all $H \in \mathfrak{F}_I$ with $F \subseteq H$, we have $|x - \sum_{i\in H} x_i| < \varepsilon$. For any $G \in \mathfrak{F}_I$, we have

$$\sum_{i\in G} x_i \leq \sum_{i\in F\cup G} x_i < x + \varepsilon$$

since $F \subseteq F \cup G$. Hence the supremum over all $G \in \mathfrak{F}_I$ exists and

$$\sup_{G\in\mathfrak{F}_I} \sum_{i\in G} x_i \leq x + \varepsilon,$$

and since ε was arbitrary, the inequality $\geq$ follows.

Now assume that the supremum exists and denote it by X. Let $\varepsilon > 0$ and take $F \in \mathfrak{F}_I$ such that $\sum_{i\in F} x_i + \varepsilon > X$. Then for all $G \in \mathfrak{F}_I$ with $F \subseteq G$, we have

$$\left|\sum_{i\in G} x_i - X\right| = X - \sum_{i\in G} x_i \leq X - \sum_{i\in F} x_i \leq \varepsilon,$$

so that $\sum_{i\in I} x_i$ converges to X. □

Corollary B.2. *Let $(x_i)_{i\in I}$ and $(y_i)_{i\in I}$ be families of non-negative real numbers indexed by a non-empty set I such that $x_i \leq y_i$ for all $i \in I$, and assume that $\sum_{i\in I} y_i$ converges. Then $\sum_{i\in I} x_i$ converges too, and $\sum_{i\in I} x_i \leq \sum_{i\in I} y_i$.*

Proof. Let $y = \sum_{i\in I} y_i$. For $\varepsilon > 0$, let $F \in \mathfrak{F}_I$ such that $|\sum_{i\in G} y_i - y| < \varepsilon$ for all $G \in \mathfrak{F}_I$ such that $F \subseteq G$. For any $G \in \mathfrak{F}_I$, then

$$\sum_{i\in G} x_i \leq \sum_{i\in F\cup G} x_i \leq \sum_{i\in F\cup G} y_i < y + \varepsilon$$

since $F \subseteq F \cup G$. Since G was arbitrary, the supremum over all $G \in \mathfrak{F}_I$ exists and

$$\sup_{G\in\mathfrak{F}_I} \sum_{i\in G} x_i \leq y + \varepsilon.$$

Because ε was arbitrary, we can conclude that the supremum is less than or equal to y, and Theorem B.1 now yields that $\sum_{i\in I} x_i$ converges and that $\sum_{i\in I} x_i \leq y$. □

2.1 Potius sero quam numquam

We now turn our attention to Hilbert spaces.

Lemma B.3. *Let $(e_i)_{i\in I}$ and $(f_j)_{j\in J}$ be orthonormal bases of $\mathcal{H}$. For any positive operator $T \in B(\mathcal{H})$ then convergence of $\sum_{i\in I}\langle Te_i, e_i\rangle$ implies*

$$\sum_{j\in J}\langle Tf_j, f_j\rangle = \sum_{i\in I}\langle Te_i, e_i\rangle.$$

In particular, if $\sum_{i\in I}\langle Te_i, e_i\rangle$ does not converge, then $\sum_{i\in I}\langle Tf_i, f_i\rangle$ does not converge either.

Proof. First of all, $T = S^*S$ for some $S \in B(\mathcal{H})$. Note that $\sum_{j\in J}|\langle Se_i, f_j\rangle|^2 = \|Se_i\|^2$ for any $i \in I$, so

$$\sum_{i\in I}\sum_{j\in J}|\langle Se_i, f_j\rangle|^2 = \sum_{i\in I}\langle Te_i, e_i\rangle$$

converges. For any finite subset F of J, Corollary B.2 yields that

$$\sum_{i\in I}\sum_{j\in F}|\langle S^*e_i, f_j\rangle|^2 \leq \sum_{i\in I}\sum_{j\in J}|\langle S^*e_i, f_j\rangle|^2 = \sum_{i\in I}\langle Te_i, e_i\rangle.$$

Since

$$\sum_{j\in F}\langle Tf_j, f_j\rangle = \sum_{j\in F}\sum_{i\in I}|\langle Sf_j, e_i\rangle|^2 = \sum_{i\in I}\sum_{j\in F}|\langle S^*e_i, f_j\rangle|^2,$$

then $\sum_{j\in J}\langle Tf_i, f_i\rangle$ converges by Proposition B.1 with $\sum_{j\in J}\langle Tf_j, f_j\rangle \leq \sum_{i\in I}\langle Te_i, e_i\rangle$. Equality then follows. The second result also follows easily, since if $\sum_{j\in I}\langle Tf_j, f_j\rangle$ converged, then $\sum_{i\in I}\langle Te_i, e_i\rangle$ would converge as well. □

This particular result allows for a name for the above value in $[0,\infty]$ (since T was assumed to be positive), independent of the orthonormal basis chosen.

Definition B.1. Let $\mathcal{H}$ be a Hilbert space with orthonormal basis $(e_i)_{i\in I}$. For any positive operator $T \in B(\mathcal{H})$ the *trace* of T is given by

$$\mathrm{tr}\,T = \sum_{i\in I}\langle Te_i, e_i\rangle \in [0,\infty].$$

If an operator $T \in B(\mathcal{H})$ satisfies $\mathrm{tr}\,|T| = \mathrm{tr}\,(T^*T)^{1/2} < \infty$, T is called *a trace class operator* (we will oftentimes say that T is *trace class*), and the set of trace class operators on $\mathcal{H}$ is denoted by $\mathscr{T}(\mathcal{H})$.

Proposition B.4 (Properties of the trace). *For positive operators $S, T \in B(\mathcal{H})$ and $\lambda \geq 0$, we have*

(i) $\mathrm{tr}\,(S+T) = \mathrm{tr}\,S + \mathrm{tr}\,T$.
(ii) $\mathrm{tr}\,(\lambda S) = \lambda \cdot \mathrm{tr}\,S$.
(iii) $\mathrm{tr}\,(UTU^*) = \mathrm{tr}\,T$ *for any unitary operator* $U \in B(\mathcal{H})$.
(iv) *If* $S \leq T$, *then* $\mathrm{tr}\,S \leq \mathrm{tr}\,T$.

Proof. (i), (ii) and (iv) are clear. Finally, if $U \in B(\mathcal{H})$ is unitary, note that if $(e_i)_{i\in I}$ is an orthonormal basis of $\mathcal{H}$, then $(Ue_i)_{i\in I}$ is an orthonormal basis as well (indeed, if $\langle Ue_i, \xi\rangle = 0$ for all $i \in I$, then $U^*\xi = 0$, so $\xi = UU^*\xi = 0$). Hence

$$\mathrm{tr}\,(UTU^*) = \sum_{i\in I}\langle (UTU^*)Ue_i, Ue_i\rangle = \sum_{i\in I}\langle Te_i, e_i\rangle = \mathrm{tr}\,T,$$

completing the proof. □

Lemma B.5. *If $T \in B(\mathcal{H})$ is positive and $U \in B(\mathcal{H})$ is a partial isometry, then* $\mathrm{tr}\,(U^*TU) \leq \mathrm{tr}\,T$.

Proof. Let $(e_i)_{i\in I}$ be an orthonormal basis for $\mathcal{H}$ such that e_i either belongs to $\ker U$ or $(\ker U)^\perp$ for $i \in I$ and let J be the subset of I consisting of all i such that $e_i \in (\ker U)^\perp$. Then the set $(Ue_i)_{i\in J}$ is orthonormal and may be extended to a full orthonormal basis $(f_i)_{i\in I}$ for $\mathcal{H}$, yielding

$$\operatorname{tr}(U^*TU) = \sum_{i\in I}\langle Ue_i, TUe_i\rangle = \sum_{i\in J}\langle TUe_i, Ue_i\rangle \le \sum_{i\in I}\langle Tf_i, f_i\rangle = \operatorname{tr} T,$$

completing the proof. □

Proposition B.6. *Let $\mathcal{H}$ be a Hilbert space. Then*

(i) *$\mathscr{T}(\mathcal{H})$ is a vector space,*
(ii) *for all $T \in \mathscr{T}(\mathcal{H})$ and $S \in B(\mathcal{H})$, then $ST \in \mathscr{T}(\mathcal{H})$ and $TS \in \mathscr{T}(\mathcal{H})$,*
(iii) *if $T \in \mathscr{T}(\mathcal{H})$, then $T^* \in \mathscr{T}(\mathcal{H})$.*

Hence $\mathscr{T}(\mathcal{H})$ is a two-sided *-ideal of $B(\mathcal{H})$.

Proof. (i) Since $|\lambda T| = (|\lambda|^2 T^*T)^{1/2} = |\lambda||T|$ for all $T \in \mathscr{T}(\mathcal{H})$ and $\lambda \in \mathbb{C}$, we have $\operatorname{tr}|\lambda T| = |\lambda|\operatorname{tr}|T|$ by Proposition B.4 and thus $\mathscr{T}(\mathcal{H})$ is closed under scalar multiplication. Assume that S and T are in $\mathscr{T}(\mathcal{H})$. To show that $S + T \in \mathscr{T}(\mathcal{H})$, we will make use of the polar decomposition for bounded operators. Let U, V and W be partial isometries in $B(\mathcal{H})$ such that

$$S = U|S|, \quad T = V|T|, \quad S + T = W|S + T|.$$

Moreover, let $(e_i)_{i\in I}$ be an orthonormal basis of $\mathcal{H}$. For any finite subset F of I, note that

$$\sum_{i\in F}\langle |S+T|e_i, e_i\rangle = \sum_{i\in F}\langle W^*(S+T)e_i, e_i\rangle \le \sum_{i\in F}|\langle W^*U|S|e_i, e_i\rangle| + \sum_{i\in F}|\langle W^*V|T|e_i, e_i\rangle|.$$

By the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\sum_{i\in F}|\langle W^*U|S|e_i, e_i\rangle| &\le \sum_{i\in F}\left\||S|^{1/2}e_i\right\|\left\||S|^{1/2}U^*We_i\right\| \\
&\le \left[\sum_{i\in F}\left\||S|^{1/2}e_i\right\|^2\right]^{1/2}\left[\sum_{i\in F}\left\||S|^{1/2}U^*We_i\right\|^2\right]^{1/2} \\
&\le \left[\sum_{i\in I}\langle |S|e_i, e_i\rangle\right]^{1/2}\left[\sum_{i\in I}\langle W^*U|S|U^*We_i, e_i\rangle\right]^{1/2} \\
&\le \operatorname{tr}|S|^{1/2}\cdot\operatorname{tr}(W^*U|S|U^*W)^{1/2} \\
&\le \operatorname{tr}|S|^{1/2}\cdot\operatorname{tr}(U|S|U^*)^{1/2} \\
&\le \operatorname{tr}|S|,
\end{aligned}$$

using Lemma B.5 and the fact that W and U^* are partial isometries. Similarly one proves that $\sum_{i\in F}|\langle W^*V|T|e_i, e_i\rangle| \le \operatorname{tr}|T|$, yielding

$$\sum_{i\in F}\langle |S+T|e_i, e_i\rangle \le \operatorname{tr}|S| + \operatorname{tr}|T| < \infty$$

for all finite subsets F of I. Hence $\operatorname{tr}|S+T| \le \operatorname{tr}|S| + \operatorname{tr}|T| < \infty$, so $S + T \in \mathscr{T}(\mathcal{H})$, and we conclude that $\mathscr{T}(\mathcal{H})$ is a vector space.

(ii) Since any $S \in B(\mathcal{H})$ can be written as a finite linear combination of unitary operators [31, Theorem 10.6] and $\mathscr{T}(\mathcal{H})$ is a vector space, we need only show the result for unitary operators. For $U \in B(\mathcal{H})$ unitary and $T \in \mathscr{T}(\mathcal{H})$, we have $|UT| = (T^*U^*UT)^{1/2} = |T|$, yielding $UT \in \mathscr{T}(\mathcal{H})$; furthermore since

$$(U^*|T|U)^2 = U^*T^*TU = (TU)^*(TU),$$

we have $|TU| = U^*|T|U$ by uniqueness of the square root, so Proposition B.4(iv) yields that $TU \in \mathscr{T}(\mathcal{H})$.

(iii) Let U be a partial isometry in $B(\mathcal{H})$ such that $T = U|T|$ by polar decomposition. Then

$$TT^* = U|T|^2U^* = (U|T|U^*)^2,$$

so $|T^*| = U|T|U^*$ by uniqueness of the square root. Therefore $\operatorname{tr}|T^*| = \operatorname{tr}(U|T|U^*) \leq \operatorname{tr}|T| < \infty$ by Lemma B.5 since U^* is a partial isometry. Hence $T^* \in \mathscr{T}(\mathcal{H})$. □

Note that if $\operatorname{tr}|T| = 0$ for $T \in \mathscr{T}(\mathcal{H})$ and $(e_i)_{i\in\mathcal{H}}$ is an orthonormal basis for $\mathcal{H}$, then

$$\sum_{i\in I} \left\| |T|^{1/2} e_i \right\|^2 = \sum_{i\in I} \langle |T|e_i, e_i\rangle = 0,$$

so $|T|^{1/2} = 0$. Hence $T^*T = 0$ and thus $T = 0$. Since the proof of Proposition B.6 yielded that

$$\operatorname{tr}|\lambda T| = |\lambda| \operatorname{tr}|T|, \quad \operatorname{tr}|S+T| \leq \operatorname{tr}|S| + \operatorname{tr}|T|$$

for all $S, T \in \mathscr{T}(\mathcal{H})$ and $\lambda \in \mathbb{C}$, we obtain a norm on $\mathscr{T}(\mathcal{H})$.

Definition B.2. The *trace norm* $\|\cdot\|_1$ on $\mathscr{T}(\mathcal{H})$ is defined by

$$\|T\|_1 = \operatorname{tr}|T|, \quad T \in \mathscr{T}(\mathcal{H}).$$

In fact, $\mathscr{T}(\mathcal{H})$ is a Banach space with the trace norm, and we will prove this later. Now it is time to introduce a new class of operators, the definition of which will be expressed by the trace.

Definition B.3. For $T \in B(\mathcal{H})$ we say that T is a *Hilbert-Schmidt operator* if $|T|^2 = T^*T \in \mathscr{T}(\mathcal{H})$, i.e. if $\sum_{i\in I} \|Te_i\|^2 < \infty$ for some orthonormal basis $(e_i)_{i\in I}$. Note that the sum is (still) independent of the choice of basis. We denote the set of all Hilbert-Schmidt operators on $\mathcal{H}$ by $\mathscr{T}_2(\mathcal{H})$ and for $T \in \mathscr{T}_2(\mathcal{H})$ we define the *Hilbert-Schmidt norm* $\|T\|_2$ of T

$$\|T\|_2 = \||T|^2\|_1^{1/2},$$

i.e. the norm satisfies $\|T\|_2^2 = \||T|^2\|_1 = \sum_{i\in I} \|Te_i\|^2$ for any orthonormal basis $(e_i)_{i\in I}$.

Note that $\mathscr{T}(\mathcal{H}) \subseteq \mathscr{T}_2(\mathcal{H})$ by Proposition B.6. It is not immediately clear that the Hilbert-Schmidt norm is actually a norm, and we will establish this now, as well as a lot of other properties of $\mathscr{T}_2(\mathcal{H})$.

Proposition B.7. *Let $\mathcal{H}$ be a Hilbert space. Then*

(i) $(\mathscr{T}_2(\mathcal{H}), \|\cdot\|_2)$ *is a normed space.*
(ii) *If $T \in \mathscr{T}_2(\mathcal{H})$, then $T^* \in \mathscr{T}_2(\mathcal{H})$ and $\|T^*\|_2 = \|T\|_2$.*
(iii) *For $T \in \mathscr{T}_2(\mathcal{H})$, we have $\|T\| \leq \|T\|_2$.*
(iv) *For $S \in B(\mathcal{H})$ and $T \in \mathscr{T}_2(\mathcal{H})$, then $ST \in \mathscr{T}_2(\mathcal{H})$ and $TS \in \mathscr{T}_2(\mathcal{H})$ with $\|ST\|_2 \leq \|S\|\|T\|_2$ and $\|TS\|_2 \leq \|S\|\|T\|_2$.*
(v) *For $U \in B(\mathcal{H})$ unitary and $T \in \mathscr{T}_2(\mathcal{H})$, we have $\|UT\|_2 = \|TU\|_2 = \|T\|_2$.*

Hence $\mathscr{T}_2(\mathcal{H})$ is a two-sided *-ideal of $B(\mathcal{H})$.

Proof. Fix an orthonormal basis $(e_i)_{i\in I}$ for $\mathcal{H}$. For $S, T \in \mathscr{T}_2(\mathcal{H})$ and $\lambda \in \mathbb{C}$, note that for finite subsets F of I, the triangle inequality on the Hilbert space $\mathbb{C}^F$ yields

$$\begin{aligned}\left[\sum_{i\in F} \|(S+T)e_i\|^2\right]^{1/2} &\leq \left[\sum_{i\in F} (\|Se_i\| + \|Te_i\|)^2\right]^{1/2} \\ &\leq \left[\sum_{i\in F} \|Se_i\|^2\right]^{1/2} + \left[\sum_{i\in F} \|Te_i\|^2\right]^{1/2} = \|S\|_2 + \|T\|_2 < \infty.\end{aligned}$$

Hence $S + T \in \mathscr{T}_2(\mathcal{H})$ with $\|S+T\|_2 \leq \|S\|_2 + \|T\|_2$. It is clear that $\lambda T \in \mathscr{T}_2(\mathcal{H})$ as well with $\|\lambda T\|_2 = |\lambda|\|T\|_2$. Finally, if $\|T\|_2 = 0$ for some $T \in \mathscr{T}_2(\mathcal{H})$, then $\|Te_i\|^2 = 0$ for all $i \in I$, implying $T = 0$. Therefore (i) follows. Additionally

$$\sum_{i\in I} \|T^*e_i\|^2 = \sum_{i\in I}\sum_{i\in I} |\langle T^*e_i, e_i\rangle|^2 = \sum_{i\in I} |\langle Te_i, e_i\rangle|^2 = \sum_{i\in I} \|Te_i\|^2,$$

so (ii) holds as well. For (iii), let ξ be a unit vector of $\mathcal{H}$. By choosing an orthonormal basis $(e_i)_{i\in I}$ for $\mathcal{H}$ containing ξ, we have

$$\|T\xi\|^2 \le \sum_{i\in I} \|Te_i\|^2 = \|T\|_2^2.$$

Since ξ was arbitrary, we obtain (iii).

Finally, for an arbitrary orthonormal basis $(e_i)_{i\in I}$ for $\mathcal{H}$ and all $i \in I$, then $\|STe_i\|^2 = \|S\|^2\|Te_i\|^2$ for $S \in B(\mathcal{H})$ and $T \in \mathscr{T}_2(\mathcal{H})$, implying $ST \in \mathscr{T}_2(\mathcal{H})$ and $\|ST\|_2 \le \|S\|\|T\|_2$. Hence we can also infer $S^*T^* \in \mathscr{T}_2(\mathcal{H})$ for any $S \in B(\mathcal{H})$ and $T \in \mathscr{T}_2(\mathcal{H})$, so (ii) implies that $TS \in \mathscr{T}_2(\mathcal{H})$ as well, as $\|TS\|_2 = \|S^*T^*\|_2 \le \|S^*\|\|T^*\|_2 = \|S\|\|T\|_2$. For (v), note first that $UT \in \mathscr{T}_2(\mathcal{H})$ by (iv) and that $\sum_{i\in I} \|TUe_i\|^2 = \sum_{i\in I} \|Te_i\|^2$ because $(Ue_i)_{i\in I}$ is also an orthonormal basis for $\mathcal{H}$. Hence $\|TU\|_2 = \|T\|_2$. Therefore $\|UT\|_2 = \|T^*U^*\|_2 = \|T^*\|_2 = \|T\|_2$ as well. □

As a consequence of Proposition B.7(ii), we have

$$\|T\|_1 = \operatorname{tr}|T| = \||T|^{1/2}\|_2^2 \ge \||T|^{1/2}\|^2 = \||T|\| = \|T\|.$$

Already from the definition of Hilbert-Schmidt operators one might suspect that there is some deep connection between these and trace class operators, other than the inclusion. The following proposition sheds some light on this.

Proposition B.8. *For $T \in B(\mathcal{H})$, the following are equivalent:*

(i) $T \in \mathscr{T}(\mathcal{H})$.
(ii) $|T|^{1/2} \in \mathscr{T}_2(\mathcal{H})$.
(iii) *T is the product of two Hilbert-Schmidt operators.*
(iv) *$|T|$ is the product of two Hilbert-Schmidt operators.*

Proof. For (i) $\Rightarrow$ (ii), note that $\||T|^{1/2}\xi\|^2 = \langle \xi, |T|\xi\rangle$ for all $\xi \in \mathcal{H}$. In the following, let $T = U|T|$ be the polar decomposition of T. If (ii) holds, then $T = (U|T|^{1/2})(|T|^{1/2})$, and as $U|T|^{1/2} \in \mathscr{T}_2(\mathcal{H})$ by Proposition B.7, (iii) follows. If (iii) holds, then assume that $T = RS$ for $R, S \in \mathscr{T}_2(\mathcal{H})$. Then $(U^*R)S = U^*U|T| = |T|$ because U^*U is the projection onto the closure of the range of $|T|$, and $U^*R \in \mathscr{T}_2(\mathcal{H})$ by Proposition B.7. Finally, if (iv) holds, then suppose that $|T| = RS$ for $R, S \in \mathscr{T}_2(\mathcal{H})$. For any orthonormal basis $(e_i)_{i\in I}$ for $\mathcal{H}$ and $F \in \mathfrak{F}_I$, we have

$$\sum_{i\in F}\langle |T|e_i, e_i\rangle \le \sum_{i\in F} \|Se_i\|\|R^*e_i\| \le \left[\sum_{i\in I}\|Se_i\|^2\right]^{1/2}\left[\sum_{i\in I}\|R^*e_i\|^2\right]^{1/2} < \infty$$

by Proposition B.7, yielding $\operatorname{tr}|T| < \infty$. □

Our first big result (in the sense that it is immensely useful) concerning trace class operators is the following theorem that provides some structure not only for the aforementioned operators, but also the Hilbert-Schmidt operators.

Theorem B.9. *For $T \in \mathscr{T}(\mathcal{H})$ and any orthonormal basis $(e_i)_{i\in I}$ for $\mathcal{H}$, then $\sum_{i\in I}\langle Te_i, e_i\rangle$ converges absolutely in $\mathbb{C}$ and the limit is independent of the choice of basis.*

Proof. Since $T \in \mathscr{T}(\mathcal{H})$, we can write $T = S^*R$ for Hilbert-Schmidt operators R and S. For $\lambda \in \mathbb{C}$ and $i \in I$, note that since $\|(R - \lambda S)e_i\|^2 \ge 0$, we obtain

$$\|Re_i\|^2 + |\lambda|^2\|Se_i\|^2 \ge \langle Re_i, \lambda Se_i\rangle + \langle \lambda Se_i, Re_i\rangle = 2\mathrm{Re}\langle Re_i, \lambda Se_i\rangle = 2\mathrm{Re}\overline{\lambda}\langle Re_i, Se_i\rangle.$$

Hence if $|\lambda| = 1$ and $\overline{\lambda}\langle Re_i, Se_i\rangle = |\langle Re_i, Se_i\rangle|$, we obtain

$$|\langle Te_i, e_i\rangle| = |\langle Re_i, Se_i\rangle| \le \frac{1}{2}(\|Re_i\|^2 + \|Se_i\|^2)$$

for all $i \in I$. Therefore $\sum_{i\in I}|\langle Te_i, e_i\rangle| \le \sum_{i\in I}\|Re_i\|^2 + \sum_{i\in I}\|Se_i\|^2 < \infty$, so $\sum_{i\in I}\langle Te_i, e_i\rangle$ converges absolutely. To prove that the sum is independent of the choice of basis, observe that for any $j \in I$ we have

$$\|(R+S)e_j\|^2 - \|(R-S)e_j\|^2 = 2\langle Re_j, Se_j\rangle + 2\langle Se_j, Re_j\rangle = 4\mathrm{Re}\langle Re_j, Se_j\rangle = 4\mathrm{Re}\langle Te_j, e_j\rangle$$

and

$$\|(iR+S)e_j\|^2 - \|(iR-S)e_j\|^2 = 2\langle iRe_j, Se_j\rangle + 2\langle Se_j, iRe_j\rangle = 4\mathrm{Re}\langle iRe_j, Se_j\rangle = -4\mathrm{Im}\langle Te_j, e_j\rangle.$$

Recause $R+S$ and $R-S$ are also Hilbert-Schmidt operators, we see that

$$\begin{aligned}\mathrm{Re}\sum_{j\in I}\langle Te_j, e_j\rangle &= \sum_{j\in I}\mathrm{Re}\langle Te_j, e_j\rangle \\ &= \frac{1}{4}\left[\sum_{j\in I}\|(R+S)e_j\|^2 - \sum_{j\in I}\|(R-S)e_j\|^2\right] = \frac{1}{4}\left[\|R+S\|_2^2 - \|R-S\|_2^2\right],\end{aligned}$$

and likewise

$$\mathrm{Im}\sum_{j\in I}\langle Te_j, e_j\rangle = -\frac{1}{4}\left[\|iR+S\|_2^2 - \|iR-S\|_2^2\right].$$

Hence it follows that $\sum_{i\in I}\langle Te_i, e_i\rangle$ is independent of the choice of basis. □

This inspires the following definition.

Definition B.4. Let $\mathcal{H}$ be a Hilbert space. The map $\mathrm{tr}: \mathscr{T}(\mathcal{H}) \to \mathbb{C}$ given by

$$\mathrm{tr}\, T = \sum_{i\in I}\langle Te_i, e_i\rangle,$$

where $(e_i)_{i\in I}$ is any orthonormal basis, is called the *trace*.

Now the face of the reader might look like a question mark: is this the same trace as defined for positive operators? The answer would be yes and no. It is clear that we are dealing with two separate classes of operators, but it is also not hard to see that the traces of a bounded operator are equal if the operator is both positive *and* trace class. The reason that we "expand" our notion of the trace to the trace class operators is that it offers the operator algebraist a nice architectural background for considering $\mathscr{T}(\mathcal{H})$ as more than just a normed space. The first thing we shall look into is that it turns out that the trace defined above indeed lives up to its name.

Proposition B.10. tr *is a linear functional on* $\mathscr{T}(\mathcal{H})$ *satisfying*

$$\mathrm{tr}\,(T^*) = \overline{\mathrm{tr}\, T}, \quad \mathrm{tr}\,(TS) = \mathrm{tr}\,(ST), \quad |\mathrm{tr}\,(ST)| \leq \|S\|\|T\|_1 \quad T \in \mathscr{T}(\mathcal{H}),\ S \in B(\mathcal{H}).$$

Proof. Linearity and the first equality are clear from straightforward calculation. Fix an orthonormal basis $(e_i)_{i\in I}$ for $\mathcal{H}$. Since any $S \in B(\mathcal{H})$ is a finite linear combination of four unitary operators, it suffices to check that $\mathrm{tr}\,(TS) = \mathrm{tr}\,(ST)$ for $T \in \mathscr{T}(\mathcal{H})$ and a unitary operator $S \in B(\mathcal{H})$. In this case, defining $f_i = Se_i$ for $i \in I$, $(f_i)_{i\in I}$ is an orthonormal basis and $S^*f_i = e_i$ for all $i \in I$, yielding

$$\mathrm{tr}\,(TS) = \sum_{i\in I}\langle TSe_i, e_i\rangle = \sum_{i\in I}\langle Tf_i, S^*f_i\rangle = \sum_{i\in I}\langle STf_i, f_i\rangle = \mathrm{tr}\,(ST).$$

Finally, if $S \in B(\mathcal{H})$, $T \in \mathscr{T}(\mathcal{H})$ and $T = U|T|$ is the polar decomposition of T, note that the operators $|T|^{1/2}U^*S^*$ and $|T|^{1/2}$ are Hilbert-Schmidt by Proposition B.8. Hence for any finite subset F of I, the Cauchy-Schwarz inequality yields

$$\begin{aligned}\sum_{i\in F}|\langle TSe_i, e_i\rangle| &\leq \sum_{i\in F}|\langle |T|^{1/2}Se_i, |T|^{1/2}Ue_i\rangle| \\ &\leq \sum_{i\in F}\left\||T|^{1/2}Se_i\right\|\left\||T|^{1/2}Ue_i\right\| \\ &\leq \left[\sum_{i\in I}\left\||T|^{1/2}Se_i\right\|^2\right]^{1/2}\left[\sum_{i\in F}\left\||T|^{1/2}Ue_i\right\|^2\right]^{1/2} \\ &= \||T|^{1/2}S\|_2\||T|^{1/2}U\|_2.\end{aligned}$$

Hence

$$|\mathrm{tr}\,(TS)| \leq \||T|^{1/2}S\|_2\||T|^{1/2}U\|_2 \leq \||T|^{1/2}\|_2^2\|S\|\|U\| \leq \||T|\|_1\|S\| = \|T\|_1\|S\|,$$

completing the proof. □

From this it follows that for $S \in B(\mathcal{H})$, the map $T \mapsto \operatorname{tr}(ST)$ is a linear functional on $\mathscr{T}(\mathcal{H})$. This is not the whole story, though. A consequence of the trace being well-defined is that $\mathscr{T}_2(\mathcal{H})$ can in fact be equipped with an inner product.

Proposition B.11. *For any $S, T \in \mathscr{T}_2(\mathcal{H})$, define*

$$\langle S, T\rangle_2 = \operatorname{tr}(T^*S).$$

Then $\langle \cdot, \cdot\rangle_2$ is an inner product on $\mathscr{T}_2(\mathcal{H})$, inducing the Hilbert-Schmidt norm.

Proof. $\langle \cdot, \cdot\rangle_2$ is well-defined, as $T^*S \in \mathscr{T}(\mathcal{H})$ for any $S, T \in \mathscr{T}_2(\mathcal{H})$, from Proposition B.8. Linearity follows from the trace being linear. For any $S, T \in \mathscr{T}_2(\mathcal{H})$, we also have

$$\langle S, T\rangle_2 = \operatorname{tr}(T^*S) = \overline{\operatorname{tr}(S^*T)} = \langle T, S\rangle_2$$

from Proposition B.10. Finally, $\langle T, T\rangle_2 = \|T\|_2^2$ for any $T \in \mathscr{T}_2(\mathcal{H})$ so $\langle \cdot, \cdot\rangle_2$ is indeed an inner product on $\mathscr{T}_2(\mathcal{H})$ inducing the $\|\cdot\|_2$ norm. $\square$

We now expand our knowledge of finite rank operators a little. We will now race towards a great finish concerning the structure of $\mathscr{T}(\mathcal{H})$ and $\mathscr{T}_2(\mathcal{H})$ as normed spaces. It turns out that the norms turn the spaces into complete ones.

Proposition B.12. *Let $\mathcal{H}$ be a Hilbert space. Then $(\mathscr{T}(\mathcal{H}), \|\cdot\|_1)$ is a Banach space and $(\mathscr{T}_2(\mathcal{H}), \langle \cdot, \cdot\rangle_2)$ is a Hilbert space.*

Proof. Let $(e_i)_{i\in I}$ be an orthonormal basis for $\mathcal{H}$. If $(T_n)_{n\geq 1}$ is a Cauchy sequence of trace class operators in $\|\cdot\|_1$, it is in particular a Cauchy sequence in $\|\cdot\|$ and hence converges to some $T \in B(\mathcal{H})$ under the operator norm. This is our candidate for a limit under $\|\cdot\|_1$. For $\varepsilon > 0$ there exists $N \geq 1$ such that

$$\sum_{i\in G} \langle |T_n - T_m| e_i, e_i\rangle < \varepsilon$$

for all $n, m \geq N$ and an arbitrary finite subset $G \subseteq I$. If $S_n \to S$ in norm, then $S_n^* S_n \to S^*S$ in norm as well, and by approximating the square root function by polynomials on a non-negative interval, we obtain $|S_n| \to |S|$. Therefore $|T_n - T_m| \to |T_n - T|$ in norm for $m \to \infty$, so

$$\lim_{m\to\infty} \langle |T_n - T_m| e_i, e_i\rangle = \langle |T_n - T| e_i, e_i\rangle$$

for all $i \in I$. Hence

$$\sum_{i\in G} \langle |T_n - T| e_i, e_i\rangle \leq \varepsilon$$

for all $n \geq N$ and finite subsets $G \subseteq I$, and hence

$$\sum_{i\in I} \langle |T_n - T| e_i, e_i\rangle \leq \varepsilon$$

for $n \geq N$, so $T_N - T \in \mathscr{T}(\mathcal{H})$. Hence $T = T_N - (T_N - T) \in \mathscr{T}(\mathcal{H})$, and the above inequality yields convergence of $(T_n)_{n\geq 1}$ to T in $\|\cdot\|_1$. Therefore $\mathscr{T}(\mathcal{H})$ is a Banach space.

Similarly, if $(T_n)_{n\geq 1}$ is a Cauchy sequence of Hilbert-Schmidt operators in $\|\cdot\|_2$, then it is a Cauchy sequence in $\|\cdot\|$, converging to some $T \in B(\mathcal{H})$ in this norm. For $\varepsilon > 0$, take $N \in \mathbb{N}$ such that

$$\sum_{i\in G} \|(T_n - T_m)e_i\|^2 < \varepsilon^2$$

for $n, m \geq N$ and an arbitrary finite subset $G \subseteq I$. As before, this implies

$$\sum_{i\in I} \|(T_n - T)e_i\|^2 \leq \varepsilon^2$$

for all $n \geq N$, in turn yielding that T is Hilbert-Schmidt, and that $\|T_n - T\|_2 \to 0$. Hence $\mathscr{T}_2(\mathcal{H})$ is a Hilbert space. $\square$

The following results are useful facts concerning the relation of finite rank operators to our two new operator classes.

Proposition B.13. *Recall the* elementary operators $E_{\xi,\eta}\colon \mathcal{H} \to \mathcal{H}$ *given by* $E_{\xi,\eta}\omega = \langle \omega, \eta\rangle \xi$ *for* $\omega \in \mathcal{H}$. *We proved on page 31 that these span the set of finite rank operators on* $\mathcal{H}$. *It also holds that*

(i) $E^*_{\xi,\eta} = E_{\eta,\xi}$.
(ii) *For any* $T \in B(\mathcal{H})$, *we have* $TE_{\xi,\eta} = E_{T\xi,\eta}$ *and* $E_{\xi,\eta}T = E_{\xi,T^*\eta}$.
(iii) $E_{\xi,\eta} \in \mathscr{T}(\mathcal{H})$ *with* $\|E_{\xi,\eta}\|_1 = \|\xi\|\|\eta\|$ *and* $\operatorname{tr} E_{\xi,\eta} = \langle \xi, \eta\rangle$.

Proof. For $\omega_1, \omega_2 \in \mathcal{H}$

$$\langle \omega_1, E_{\eta,\xi}\omega_2\rangle = \overline{\langle \omega_2, \xi\rangle}\langle \omega_1, \eta\rangle = \langle \langle \omega_1, \eta\rangle \xi, \omega_2\rangle = \langle E_{\xi,\eta}\omega_1, \omega_2\rangle,$$

so (i) holds. (ii) is easily verified. Because $E_{\eta,\xi}E_{\xi,\eta}\omega_1 = \langle \omega_1, \eta\rangle\langle \xi, \xi\rangle \eta = \|\xi\|^2 E_{\eta,\eta}\omega_1$, we obtain $E^*_{\xi,\eta}E_{\xi,\eta} = \|\xi\|^2 E_{\eta,\eta}$. If $\eta = 0$, it is clear that $E_{\xi,\eta} \in \mathscr{T}(\mathcal{H})$ with $\|E_{\xi,\eta}\|_1 = \|\xi\|\|\eta\|$. Assuming from here onward that $\eta \neq 0$, we see that

$$\frac{\|\xi\|^2}{\|\eta\|^2} E^2_{\eta,\eta} = \|\xi\|^2 E_{\eta,\eta} = E^*_{\xi,\eta}E_{\xi,\eta},$$

so $|E_{\xi,\eta}| = {}^{\|\xi\|}/_{\|\eta\|} E_{\eta,\eta}$. Hence

$$\sum_{i\in I}\langle |E_{\xi,\eta}|e_i, e_i\rangle = \frac{\|\xi\|}{\|\eta\|}\sum_{i\in I}\langle E_{\eta,\eta}e_i, e_i\rangle = \frac{\|\xi\|}{\|\eta\|}\sum_{i\in I}\langle e_i, \eta\rangle\langle \eta, e_i\rangle = \frac{\|\xi\|}{\|\eta\|}\sum_{i\in I}|\langle \eta, e_i\rangle|^2 = \frac{\|\xi\|}{\|\eta\|}\|\eta\|^2 = \|\xi\|\|\eta\|$$

for any orthonormal basis $(e_i)_{i\in I}$, so $E_{\xi,\eta} \in \mathscr{T}(\mathcal{H})$ with $\|E_{\xi,\eta}\|_1 = \|\xi\|\|\eta\|$. Moreover,

$$\operatorname{tr} E_{\xi,\eta} = \sum_{i\in I}\langle E_{\xi,\eta}e_i, e_i\rangle = \sum_{i\in I}\langle \langle e_i, \eta\rangle \xi, e_i\rangle = \sum_{i\in I}\langle \langle \xi, e_i\rangle e_i, \eta\rangle = \left\langle \sum_{i\in I}\langle \xi, e_i\rangle e_i, \eta \right\rangle = \langle \xi, \eta\rangle.$$

Hence the proposition follows. □

Proposition B.14. *Let* $\mathcal{H}$ *be a Hilbert space. Then* $(\mathscr{T}_2(\mathcal{H}), \|\cdot\|_2)$ *contains the finite rank operators as a dense subspace, implying that all Hilbert-Schmidt operators are compact.*

Proof. Fix an orthonormal basis $(e_i)_{i\in I}$ for $\mathcal{H}$. Note that because

$$\sum_{i\in I}\|E_{\xi,\eta}e_i\|^2 \leq \|\xi\|^2 \sum_{i\in I}|\langle e_i, \eta\rangle|^2 = \|\xi\|^2\|\eta\|^2,$$

then all finite rank operators are Hilbert-Schmidt by Proposition B.7. For $T \in \mathscr{T}_2(\mathcal{H})$ and $\varepsilon > 0$, then since $\sum_{i\in I}\|Te_i\|^2$ converges, there exists a finite subset $F \subseteq I$ such that

$$\sum_{i\in I\setminus F}\|Te_i\|^2 < \varepsilon^2.$$

Defining $Se_i = Te_i$ for $i \in F$ and $Se_i = 0$ for $i \notin F$, we obtain a finite rank operator $S \in B(\mathcal{H})$. Then

$$\sum_{i\in I}\|(T-S)e_i\|^2 = \sum_{i\in I\setminus F}\|Te_i\|^2 < \varepsilon^2.$$

Hence the finite rank operators are dense in $\mathscr{T}_2(\mathcal{H})$. Because $\|\cdot\| \leq \|\cdot\|_2$, the inclusion into the compact operators follows. □

Corollary B.15. *For any* $S, T \in \mathscr{T}_2(\mathcal{H})$ *we have* $\operatorname{tr}(ST) = \operatorname{tr}(TS)$.

Proof. The equation makes sense because of Proposition B.8. As $|\operatorname{tr}(ST)| = |\langle T, S^*\rangle_2| \leq \|S\|_2\|T\|_2$, the map $\mathscr{T}_2(\mathcal{H}) \times \mathscr{T}_2(\mathcal{H}) \to \mathbb{C}$ given by $(S,T) \mapsto \operatorname{tr}(ST)$ is continuous in both variables, it suffices by Proposition B.14 to check the equality for T being finite rank, but this follows from Propositions B.13 and B.10. □

To the reader, all of these theorems may have seemed like a massive stroke of information overload. We have defined the trace, but have not used its properties yet. Why do we need to know about density of finite rank operators in the trace class operators? Is it essential that $\mathscr{T}(\mathcal{H})$ is a Banach space? One's confusion should be laid to rest immediately by looking at the next two theorems, concluding our past adventures with a colourful and surprising flourish.

Theorem B.16. *Let $\Phi\colon \mathscr{T}(\mathcal{H}) \to B(\mathcal{H})^*$ be the linear map defined by*

$$\Phi(S)(T) = \operatorname{tr}(ST), \quad S \in \mathscr{T}(\mathcal{H}),\ T \in B(\mathcal{H}).$$

Then for all $S \in \mathscr{T}(\mathcal{H})$, $\Phi(S)$ is ultraweakly continuous, and Ψ is an isometry, i.e. $\|S\|_1 = \|\Phi(S)\|$ for all $S \in \mathscr{T}(\mathcal{H})$. Conversely, if $\omega \in B(\mathcal{H})_*$, then there is a trace class operator $S \in \mathscr{T}(\mathcal{H})$ such that $\Phi(S) = \omega$, so Φ is in fact an isometric isomorphism $\mathscr{T}(\mathcal{H}) \to B(\mathcal{H})_*$. Moreover, $S \in \mathscr{T}(\mathcal{H})$ is positive if and only if $\Phi(S)$ is positive.

Proof. First of all, Proposition B.10 tells us that Φ is well-defined and linear. Fix an orthonormal basis $(e_i)_{i\in I}$ for $\mathcal{H}$ and assume that $T_\alpha \to T$ ultraweakly in $B(\mathcal{H})$ and that $S \in \mathscr{T}(\mathcal{H})$. By writing $S = R_1R_2^*$ for two Hilbert-Schmidt operators R_1 and R_2 by Proposition B.8, then there are only countably many $i \in I$ such that $R_1e_i \neq 0$ and $R_2e_i \neq 0$. Hence we can define a surjection Ω from $\mathbb{N}$ into $\{i \in I \mid R_1e_i \neq 0 \text{ or } R_2e_i \neq 0\}$ so that $(R_1e_{\Omega(n)})_{n\geq 1}$ and $(R_2e_{\Omega(n)})_{n\geq 1}$ are square-summable sequences in $\mathcal{H}$. Since for any $A \in B(\mathcal{H})$ we have $\operatorname{tr}(AR_1R_2^*) = \operatorname{tr}(R_2^*AR_1)$ by Proposition B.7 and Corollary B.15, we find that

$$\begin{aligned}\Phi(S)(T_\alpha) = \operatorname{tr}(ST_\alpha) = \operatorname{tr}(R_2^*T_\alpha R_1) &= \sum_{i\in I}\langle T_\alpha R_1e_i, R_2e_i\rangle = \sum_{n=1}^\infty \langle T_\alpha R_1e_{\Omega(n)}, R_2e_{\Omega(n)}\rangle \\ &\to \sum_{n=1}^\infty \langle TR_1e_{\Omega(n)}, R_2e_{\Omega(n)}\rangle = \sum_{i\in I}\langle TR_1e_i, R_2e_i\rangle = \operatorname{tr}(ST) = \Phi(S)(T).\end{aligned}$$

Hence $\Phi(S) \in B(\mathcal{H})_*$. Moreover, by Proposition B.10 we have $\|\Phi(S)\| \leq \|S\|_1$. If $S = U|S|$ denotes the polar decomposition of S, then because

$$\Phi(S)(U^*) = \operatorname{tr}(U^*S) = \operatorname{tr}|S| = \|S\|_1,$$

we have that Φ is an isometry.

For any $\omega \in B(\mathcal{H})_*$ then by Proposition 2.2 we have $\omega = \sum_{n=1}^n \omega_{\xi_n,\eta_n}$ for sequences $(\xi_n)_{n\geq 1}$ and $(\eta_n)_{n\geq 1}$ of $\mathcal{H}$ such that $\sum_{n=1}^\infty \|\xi_n\|^2 < \infty$ and $\sum_{n=1}^\infty \|\eta_n\|^2 < \infty$. Define

$$S = \sum_{n=1}^\infty E_{\xi_n,\eta_n},$$

which converges in $\mathscr{T}(\mathcal{H})$ in the trace norm, as $\mathscr{T}(\mathcal{H})$ is a Banach space and

$$\sum_{n=1}^\infty \|E_{\xi_n,\eta_n}\|_1 = \sum_{n=1}^\infty \|\xi_n\|\|\eta_n\| \leq \left(\sum_{n=1}^\infty \|\xi_n\|^2\right)^{1/2}\left(\sum_{n=1}^\infty \|\eta_n\|^2\right)^{1/2} < \infty$$

using Proposition B.13. Then because $S \mapsto \operatorname{tr}(TS)$, $S \in \mathscr{T}(\mathcal{H})$ is a bounded linear functional on $\mathscr{T}(\mathcal{H})$ for all $T \in B(\mathcal{H})$ by Proposition B.10, it follows that

$$\Phi(S)(T) = \operatorname{tr}(TS) = \sum_{n=1}^\infty \operatorname{tr}(TE_{\xi_n,\eta_n}) = \sum_{n=1}^\infty \langle T\xi_n, \eta_n\rangle = \omega(T), \quad T \in B(\mathcal{H}),$$

so that $\Phi(S) = \omega$.

Finally, let $S \in \mathscr{T}(\mathcal{H})$. If S is positive, then for all positive $T \in B(\mathcal{H})$, as $S^{1/2}$ is Hilbert-Schmidt by Proposition B.8, so

$$\operatorname{tr}(ST) = \operatorname{tr}(S^{1/2}TS^{1/2}) = \sum_{i\in I}\langle ST^{1/2}e_i, T^{1/2}e_i\rangle \geq 0$$

by Corollary B.15. If $\operatorname{tr}(ST) \geq 0$ for all positive $T \in B(\mathcal{H})$, then because $\|\xi\|^2 E_{\xi,\xi} = E_{\xi,\xi}^* E_{\xi,\xi}$ for all $\xi \in \mathcal{H}$, we see that $E_{\xi,\xi}$ is positive and hence

$$0 \leq \operatorname{tr}(SE_{\xi,\xi}) = \langle S\xi, \xi \rangle$$

for all $\xi \in \mathcal{H}$. Therefore S is positive. □

Just for closure (no pun intended), we include these two side effects, falling like dominoes.

Proposition B.17. *Let $\mathcal{H}$ be a Hilbert space. Any trace class operator is of the form*

$$\sum_{n=1}^{\infty} E_{\xi_n,\eta_n}$$

with the series converging in $\|\cdot\|_1$ and $(\xi_n)_{n\geq 1}$ and $(\eta_n)_{n\geq 1}$ being sequences such that $\sum_{n=1}^{\infty} \|\xi_n\|^2 < \infty$ and $\sum_{n=1}^{\infty} \|\eta_n\|^2 < \infty$. Hence $(\mathscr{T}(\mathcal{H}), \|\cdot\|_1)$ contains the finite rank operators as a dense subspace, implying that all trace class operators are compact.

Proof. Let $T \in \mathscr{T}(\mathcal{H})$. Using the isomorphism $\Phi\colon \mathscr{T}(\mathcal{H}) \to B(\mathcal{H})_*$ from Theorem B.16, then by Corollary 2.4 and Proposition 2.2, there are sequences $(\xi_n)_{n\geq 1}$ and $(\eta_n)_{n\geq 1}$ satisfying $\sum_{n=1}^{\infty} \|\xi_n\|^2 < \infty$ and $\sum_{n=1}^{\infty} \|\eta_n\|^2 < \infty$ such that $\Phi(T) = \sum_{n=1}^{\infty} \omega_{\xi_n,\eta_n}$ converging in norm. Since T is an isometry and $\Phi(E_{\xi_n,\eta_n}) = \omega_{\xi_n,\eta_n}$ for all $n \geq 1$, it follows that $T = \sum_{n=1}^{\infty} E_{\xi_n,\eta_n}$ with series converging under the trace norm. Since $\|\cdot\| \leq \|\cdot\|_1$, the rest of the statement follows. □

Theorem B.18. *Let $\mathscr{T}(\mathcal{H})$ be the Banach space of trace class operators equipped with the trace norm. The map $\Psi\colon B(\mathcal{H}) \to \mathscr{T}(\mathcal{H})^*$ given by*

$$\Psi(S)(T) = \operatorname{tr}(ST), \quad S \in B(\mathcal{H}),\ T \in \mathscr{T}(\mathcal{H})$$

is an isometric isomorphism. Moreover, Ψ is an ultraweak-to-weak homeomorphism, and $S \in B(\mathcal{H})$ is positive if and only if $\Psi(S) \in \mathscr{T}(\mathcal{H})^*$ is positive.*

Proof. Let $\Lambda\colon B(\mathcal{H}) \to (B(\mathcal{H})_*)^*$ be the canonical identification of Proposition 2.5 and let

$$\Phi^*\colon (B(\mathcal{H})_*)^* \to \mathscr{T}(\mathcal{H})^*$$

be the adjoint map of Φ. Φ^* is an isometric isomorphism with $(\Phi^*)^{-1} = (\Phi^{-1})^*$. Then

$$\Phi^*(\Lambda(S))(T) = \Lambda(S)(\Phi(T)) = \Phi(T)(S) = \operatorname{tr}(ST) = \Psi(S)(T),$$

so $\Psi = \Phi^* \circ \Lambda$. Hence Ψ is an isometric isomorphism. Moreover, since Λ is an ultraweakly-to-weak* homeomorphism by Corollary 2.8 and Φ^* is a weak*-to-weak* homeomorphism, it follows that Ψ is an ultraweak-to-weak* homeomorphism. The last statement can be proved the same way as it was in Theorem B.16. □

2.2 Further properties of the Hilbert-Schmidt norm

In this section, we will introduce the notion of a conjugate Hilbert space and find an interesting connection between the Hilbert-Schmidt operators over a Hilbert space and a Hilbert space tensor product related to the original Hilbert space.

Let $\mathcal{H}$ be a Hilbert space and define a map $F\colon \mathcal{H} \to \mathcal{H}^*$ by $F(\xi)(\eta) = \langle \eta, \xi \rangle$. F is conjugate linear, and by the Riesz representation theorem it is also a surjective isometry. Defining $\overline{\xi} = F(\xi)$ for $\xi \in \mathcal{H}$, we hence obtain a bijective map $\mathcal{H} \to \mathcal{H}^*$ given by $\xi \mapsto \overline{\xi}$ that satisfies

$$\overline{\xi + \eta} = \overline{\xi} + \overline{\eta}, \quad \overline{\lambda\xi} = \overline{\lambda}\,\overline{\xi}, \quad \xi, \eta \in \mathcal{H},\ \lambda \in \mathbb{C}. \tag{B.1}$$

We now define $\overline{\mathcal{H}} = \{\overline{\xi} \mid \xi \in \mathcal{H}\}$ as a set. The elements of $\overline{\mathcal{H}}$ are then in bijective correspondence with elements of $\mathcal{H}$, and we give $\overline{\mathcal{H}}$ a vector space structure by

$$\overline{\xi} + \overline{\eta} = \overline{\xi + \eta}, \quad \lambda\overline{\xi} = \overline{\overline{\lambda}\xi}, \quad \xi, \eta \in \mathcal{H},\ \lambda \in \mathbb{C}.$$

The vector space axioms can then be verified by using the equations of (B.1). Finally, $\overline{\mathcal{H}}$ can then be given an inner product by defining

$$\langle \overline{\xi}, \overline{\eta} \rangle_{\overline{\mathcal{H}}} = \langle \eta, \xi \rangle_{\mathcal{H}}, \quad \xi, \eta \in \mathcal{H},$$

so as $\|\overline{\xi}\|_{\overline{\mathcal{H}}} = \|\xi\|_{\mathcal{H}}$ for all $\xi \in \mathcal{H}$, $\overline{\mathcal{H}}$ becomes a Hilbert space, called the *conjugate Hilbert space* of $\mathcal{H}$. The map $\mathcal{H} \to \overline{\mathcal{H}}$ given by $\xi \mapsto \overline{\xi}$ is thereby a conjugate linear surjective isometry.

For any $T \in B(\mathcal{H})$, define $\overline{T}\colon \overline{\mathcal{H}} \to \overline{\mathcal{H}}$ by $\overline{T}\,\overline{\xi} = \overline{T\xi}$. $\overline{T}$ is then linear and bounded, as $\|\overline{T}\,\overline{\xi}\|_{\overline{\mathcal{H}}} = \|T\xi\|_{\mathcal{H}}$ for all $\xi \in \mathcal{H}$. Hence the map $B(\mathcal{H}) \to B(\overline{\mathcal{H}})$ given by $T \mapsto \overline{T}$ is an isometry. It is easily checked that the map is conjugate linear, and moreover, it is surjective, since for any $S \in B(\overline{\mathcal{H}})$, then by letting $G\colon \overline{\xi} \mapsto \xi$ denote the conjugate linear inverse of the map $\xi \mapsto \overline{\xi}$ and defining $T\xi = G(S\overline{\xi})$ for $\xi \in \mathcal{H}$, then T is linear and bounded, and $\overline{T\xi} = S\overline{\xi}$ for all $\xi \in \mathcal{H}$. Hence the map $T \mapsto \overline{T}$ is a conjugate linear isometric isomorphism $B(\mathcal{H}) \to B(\overline{\mathcal{H}})$. It is in fact also multiplicative and adjoint-preserving, as $\overline{T}\,\overline{S} = \overline{TS}$ and

$$\langle \overline{T}\,\overline{\xi}, \overline{\eta} \rangle = \langle \eta, T\xi \rangle = \langle T^*\eta, \xi \rangle = \langle \overline{\xi}, \overline{T^*}\,\overline{\eta} \rangle, \quad \xi, \eta \in \mathcal{H},$$

for all $S, T \in B(\mathcal{H})$.

Considering the map $B(\mathcal{H}) \to B(\overline{\mathcal{H}})$ given by $T \mapsto \overline{T^*}$, one can quickly check that it is linear, isometric and actually a bijection, since for $S \in B(\overline{\mathcal{H}})$, then by defining $T \in B(\mathcal{H})$ by $T\xi = G(S\overline{\xi})$ for $\xi \in \mathcal{H}$ we find that T^* maps to S, as

$$\langle \overline{(T^*)^*}\,\overline{\xi}, \overline{\eta} \rangle = \langle \overline{T\xi}, \overline{\eta} \rangle = \langle S\overline{\xi}, \overline{\eta} \rangle, \quad \xi, \eta \in \mathcal{H}.$$

Moreover, since $\overline{S^*}\,\overline{T^*} = \overline{S^*T^*} = \overline{(TS)^*}$ for all $S, T \in B(\mathcal{H})$, $T \mapsto \overline{T^*}$ is a linear isometric *anti-isomorphism*.

Since any finite rank operator on $\mathcal{H}$ is of the form $\sum_{i=1}^n E_{\xi_i,\eta_i}$ for $\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_n \in \mathcal{H}$, we can define a unique linear map from the finite rank operators on $\mathcal{H}$ to the inner product space $\mathcal{H} \odot \overline{\mathcal{H}}$ by

$$\sum_{i=1}^n E_{\xi_i,\eta_i} \mapsto \sum_{i=1}^n \xi_i \otimes \overline{\eta_i}.$$

This map is obviously linear and surjective. Because

$$E^*_{\xi_j,\eta_j} E_{\xi_i,\eta_i}\omega = E_{\eta_j,\xi_j} E_{\xi_i,\eta_i}\omega = \langle \omega, \eta_i \rangle \langle \xi_i, \xi_j \rangle \eta_j = E_{\langle \xi_i,\xi_j \rangle \eta_j,\eta_i}\omega$$

for all $\omega \in \mathcal{H}$ we have

$$\langle E_{\xi_i,\eta_i}, E_{\xi_j,\eta_j} \rangle_2 = \langle \xi_i, \xi_j \rangle \langle \eta_j, \eta_i \rangle.$$

We then see that

$$\left\| \sum_{i=1}^n E_{\xi_i,\eta_i} \right\|_2^2 = \sum_{i,j=1}^n \langle E_{\xi_i,\eta_i}, E_{\xi_j,\eta_j} \rangle_2 = \sum_{i,j=1}^n \langle \xi_i, \xi_j \rangle_{\mathcal{H}} \langle \overline{\eta_i}, \overline{\eta_j} \rangle_{\overline{\mathcal{H}}} = \left\| \sum_{i=1}^n \xi_i \otimes \overline{\eta_i} \right\|^2.$$

It follows from Proposition A.1 that the above map extends to a surjective isometric linear map $\mathscr{T}_2(\mathcal{H}) \to \mathcal{H} \otimes \overline{\mathcal{H}}$. We have hence proved the following result:

Proposition B.19. *The unique map $\rho\colon \mathscr{T}_2(\mathcal{H}) \to \mathcal{H} \otimes \overline{\mathcal{H}}$ satisfying $\rho(E_{\xi,\eta}) = \xi \otimes \eta$ is an isometric isomorphism.*

This isomorphism allows for a simple description of maps over $\mathscr{T}_2(\mathcal{H})$ in a tensor product language.

Proposition B.20. *Let $\mathcal{H}$ be a Hilbert space.*

(i) *The map $\mathscr{T}_2(\mathcal{H}) \to \mathscr{T}_2(\mathcal{H})$ given by $T \mapsto T^*$ corresponds to the unique isometry $\Phi\colon \mathcal{H} \otimes \overline{\mathcal{H}} \to \mathcal{H} \otimes \overline{\mathcal{H}}$ satisfying $\Phi(\xi \otimes \overline{\eta}) = \eta \otimes \overline{\xi}$.*

(ii) *Let $S \in B(\mathcal{H})$. The maps $\mathscr{T}_2(\mathcal{H}) \to \mathscr{T}_2(\mathcal{H})$ given by $T \mapsto ST$ and $T \mapsto TS$ correspond to the maps $S \otimes 1_{\overline{\mathcal{H}}} \subseteq B(\mathcal{H} \otimes \overline{\mathcal{H}})$ and $1_{\mathcal{H}} \otimes \overline{S^*} \subseteq B(\mathcal{H} \otimes \overline{\mathcal{H}})$.*

Proof. (i) It is clear from the definition of the tensor product and Proposition A.1 that Φ exists and is unique with the elementary tensor property. As the set of finite linear combinations of elementary operators is dense in $\mathscr{T}_2(\mathcal{H})$, we only need to check that the two maps correspond under the isomorphism for elementary operators, but this follows from from Proposition B.13(i).

(ii) Let $\xi, \eta \in \mathcal{H}$. As $SE_{\xi,\eta} = E_{S\xi,\eta}$ corresponds to the vector $S \otimes 1_{\overline{\mathcal{H}}}(\xi \otimes \overline{\eta})$ and $E_{\xi,\eta}S = E_{\xi,S^*\eta}$ corresponds to the vector $1_{\mathcal{H}} \otimes \overline{S^*}(\xi \otimes \overline{\eta})$, the statement again follows. We have used Proposition B.13(ii) for the above identities. □

Let $\mathscr{S}$ be any subset of $B(\mathcal{H})$ and let $\overline{\mathscr{S}} \subseteq B(\overline{\mathcal{H}})$ be the image of $\mathscr{S}$ under the map $T \mapsto \overline{T}$, and let ρ be the conjugate linear and multiplicative inverse of this map. For any given $T \in \mathscr{S}'$, then

$$\overline{T}\,\overline{S} = \overline{TS} = \overline{ST} = \overline{S}\,\overline{T}$$

for all $\overline{S} \in \overline{\mathscr{S}}$, so $\overline{\mathscr{S}'} \subseteq \overline{\mathscr{S}}'$. On the other hand, let $R \in \overline{\mathscr{S}}'$ and $S \in \mathscr{S}$. As $\overline{S}R = R\overline{S}$, we have $S\rho(R) = \rho(R)S$, so $\rho(R) \in \mathscr{S}'$. Hence $R \in \overline{\mathscr{S}'}$. Thus we have proved

$$\overline{\mathscr{S}'} = \overline{\mathscr{S}}', \quad \mathscr{S} \subseteq B(\mathcal{H}).$$

This implies that for any von Neumann algebra $\mathscr{M} \subseteq B(\mathcal{H})$, then $\overline{\mathscr{M}}$ is a von Neumann algebra, as

$$\overline{\mathscr{M}}'' = (\overline{\mathscr{M}'})' = \overline{\mathscr{M}''} = \overline{\mathscr{M}}.$$

$\overline{\mathscr{M}}$ is called *the conjugate von Neumann algebra of* $\mathscr{M}$.

We conclude this section with a proof of the Powers-Størmer inequality, needed in Chapter 5.

Proposition B.21 (Powers-Størmer inequality)**.** *Let S and T be positive operators in $B(\mathcal{H})$. If $S^{1/2}$ and $T^{1/2}$ are Hilbert-Schmidt operators, then $\|S^{1/2} - T^{1/2}\|_2 \leq \|S - T\|_1$.*

Proof. Define two self-adjoint operators $A, B \in B(\mathcal{H})$ by $A = S^{1/2} - T^{1/2}$ and $B = S^{1/2} + T^{1/2}$. By Proposition B.14, A is compact. This yields existence of an orthonormal basis $(e_i)_{i\in I}$ for $\mathcal{H}$ consisting of eigenvectors for A with corresponding real eigenvalues $(\lambda_i)_{i\in I}$ [32, Theorem 3.2.3]. It is clear that $B \geq A$ and $B \geq -A$, so $|\langle A\xi, \xi\rangle| \leq \langle B\xi, \xi\rangle$ for all $\xi \in \mathcal{H}$, and straight calculation yields that $\frac{1}{2}(AB + BA) = S - T$. Note also that for any self-adjoint operator $R \in B(\mathcal{H})$, we have $-|R| \leq R \leq |R|$ by the continuous functional calculus, and hence

$$-\langle |R|\xi, \xi\rangle \leq \langle R\xi, \xi\rangle \leq \langle |R|\xi, \xi\rangle$$

or $|\langle R\xi, \xi\rangle| \leq \langle |R|\xi, \xi\rangle$ for any $\xi \in \mathcal{H}$. As $AB + BA$ is self-adjoint, we see that

$$\begin{aligned}
\operatorname{tr}|S - T| &= \sum_{i\in I} \frac{1}{2}\langle |AB + BA|e_i, e_i\rangle \\
&\geq \sum_{i\in I} \frac{1}{2}|\langle (AB + BA)e_i, e_i\rangle| \\
&= \sum_{i\in I} \frac{1}{2}|\langle Be_i, Ae_i\rangle + \langle BAe_i, e_i\rangle| \\
&= \sum_{i\in I} \frac{1}{2}|\overline{\lambda_i}\langle Be_i, e_i\rangle + \lambda_i\langle Be_i, e_i\rangle| \\
&= \sum_{i\in I} |\lambda_i\langle Be_i, e_i\rangle| \\
&\geq \sum_{i\in I} |\lambda_i||\langle Ae_i, e_i\rangle| = \sum_{i\in I} |\lambda_i|^2 = \sum_{i\in I} |\langle A^2 e_i, e_i\rangle| = \operatorname{tr}\left[(S^{1/2} - T^{1/2})^2\right],
\end{aligned}$$

completing the proof. □

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BIBLIOGRAPHY

[1] B. Blackadar. *Operator Algebras: Theory of C*-Algebras and von Neumann Algebras*, volume 122 of *Encyclopaedia of Mathematical Sciences*. Springer, 2012.

[2] O. Bratteli and D. W. Robinson. *Operator Algebras and Quantum Statistical Mechanics 1*. Texts and Monographs in Physics. Springer-Verlag Berlin Heidelberg, second edition, 1987.

[3] N. P. Brown. The symbiosis of C^*- and W^*-algebras. `http://arxiv.org/pdf/0812.1763v1.pdf`, 2008.

[4] N. P. Brown and N. Ozawa. *C^*-algebras and Finite-Dimensional Approximations*, volume 88 of *Graduate Studies in Mathematics*. American Mathematical Society, 2008.

[5] R. S. Bryder. Cauchy completions. Sølvkorn 13, `http://math.ananas.nu/korn/013.pdf`, 2012.

[6] J. B. Conway. *A Course in Functional Analysis*, volume 96 of *Graduate Texts in Mathematics*. Springer-Verlag New York Inc., 1985.

[7] J. B. Conway. *A Course in Operator Theory*, volume 21 of *Graduate Studies in Mathematics*. American Mathematical Society, 1999.

[8] A. V. Daele. *Continuous crossed products and type III von Neumann algebras*. Cambridge University Press.

[9] J. Dixmier. *C^*-algebras*. North-Holland Publishing Company, 1977.

[10] J. Dixmier. *Von Neumann algebras*. North-Holland Publishing Company, 1981.

[11] R. G. Douglas. *Banach Algebra Techniques in Operator Theory*, volume 179 of *Graduate Texts in Mathematics*. Springer-Verlag Berlin Heidelberg New York, second edition, 1972.

[12] Ș. Strătilă and L. Zsidó. *Lectures on Von Neumann algebras*. Editura Academiei and Abacus Press, 1979.

[13] G. B. Folland. *Real Analysis: Modern Techniques and Their Applications*. Wiley Inter-Science. John Wiley & Sons, 1999.

[14] R. V. Kadison and J. R. Ringrose. *Fundamentals of the Theory of Operator Algebras. Volume I*. Pure and Applied Mathematics. Academic Press, Inc., 1983.

[15] R. V. Kadison and J. R. Ringrose. *Fundamentals of the Theory of Operator Algebras. Volume II*. Pure and Applied Mathematics. Academic Press, Inc., 1986.

[16] J. R. Munkres. *Topology*. Pearson Education International, Prentice Hall Incorporated, second edition, 1975.

[17] K. K. Olesen. *The Connes Embedding Problem – Sofic groups and the QWEP Conjecture*. Department of Mathematical Sciences, University of Copenhagen, 2012. Master's thesis.

[18] G. K. Pedersen. *C^*-algebras and their automorphism groups*, volume 14 of *L.M.S. Monographs*. Academic Press Inc. London, 1979.

[19] R. T. Powers and E. Størmer. *Free states of the canonical anticommutation relations*, volume 16(1) of *Communications in Mathematical Physics*. Springer-Verlag, 1970.

[20] M. Reed and B. Simon. *Methods of Modern Mathemathical Physics. Volume 1: Functional Analysis.* Academic Press Inc. London, 1980.

[21] M. Rørdam, F. Larsen, and N. J. Laustsen. *An Introduction to K-Theory for C^*-Algebras*, volume 49 of *London Mathematical Society Student Texts.* Cambridge University Press, 2000.

[22] W. Rudin. *Functional Analysis.* Tata McGraw-Hill Education Private Limited, second edition, 1971.

[23] V. Runde. *Lectures on Amenability*, volume 1774 of *Lecture notes in mathematics.* Springer-Verlag, 2002.

[24] M. Takesaki. *Theory of Operator Algebras I.* Studies in Advanced Mathematics. Springer-Verlag New York Inc., 1979.

[25] J. Tomiyama. *On the Projection of Norm One in W^*-algebras*, volume 33(10). Proceedings of the Japan Academy, 1959.

[26] J. Tomiyama. *Invitation to C^*-Algebras and Topological Dynamics*, volume 3 of *Advanced Series in Dynamic Systems.* World Scientific, 1987.

[27] D. M. Topping. *Lectures on Von Neumann algebras.* Van Nostrand Reinhold Mathematical Studies, 1971.

[28] A. M. Torpe. *Notes on nuclear C^*-algebras and injective von Neumann algebras.* Matematisk Institut, Odense Universitet, 1981.

[29] P. Wojtaszczyk. *Banach spaces for analysts*, volume 25 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, 1991.

[30] N. Young. *An introduction to Hilbert space.* Cambridge University Press, 1988.

[31] K. Zhu. *An Introduction to Operator Algebras.* Studies in Advanced Mathematics. CRC Press, 1993.

[32] R. J. Zimmer. *Essential Results of Functional Analysis.* Chicago Series in Mathematics. The University of Chicago Press, 1990.

[33] C. Zwarich. *Von Neumann Algebras for Abstract Harmonic Analysis.* University of Waterloo, 2008. Master's thesis.