Omitted results from Chapter 1

Rasmus Sylvester Bryder

January 11, 2018

Recall for C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ that we have defined a C^*-seminorm $\|x\|_\kappa = \sup\{\|\pi_{\varphi\odot\psi}(x)\| \mid \varphi \in S(\mathcal{A}),\ \psi \in S(\mathcal{B})\}$ for all $x \in \mathcal{A} \odot \mathcal{B}$.

Proposition 1.45. *The C^*-seminorm $\|\cdot\|_\kappa$ is a cross-norm, and all linear functionals in $\mathcal{A}^* \odot \mathcal{B}^*$ are bounded with respect to $\|\cdot\|_\kappa$.*

Before beginning the proof, we first consider the unique linear map $\Lambda\colon \mathcal{A} \odot \mathcal{B} \to B(\mathcal{A}^*, \mathcal{B})$ defined by $\Lambda(a \otimes b)(\varphi) = \varphi(a)b$. The map $x \mapsto \Lambda(x)$ is an injection, since by choosing $x = \sum_i a_i \otimes b_i$ with linearly independent b_i one finds that $\Lambda(x) = 0$ implies $\varphi(a_i) = 0$ for all $\varphi \in \mathcal{A}^*$ and i, meaning that $x = 0$. Now define a norm

$$\lambda(x) = \|\Lambda(x)\|.$$

Then for $x = \sum_i a_i \otimes b_i$ we have

$$\begin{aligned}\lambda(x) = \sup\{\|\Lambda(x)(\varphi)\| \mid \varphi \in (\mathcal{A}^*)_1\} &= \sup\left\{\left\|\sum_i \varphi(a_i)b_i\right\| : \varphi \in (\mathcal{A}^*)_1\right\} \\ &= \sup\left\{\left|\sum_i \varphi(a_i)\psi(b_i)\right| : \varphi \in (\mathcal{A}^*)_1,\ \psi \in (\mathcal{B}^*)_1\right\} \\ &= \sup\left\{|(\varphi \otimes \psi)(x)| : \varphi \in (\mathcal{A}^*)_1,\ \psi \in (\mathcal{B}^*)_1\right\}.\end{aligned}$$

For the same reason that the projective tensor norm is a cross-norm (Proposition 1.37), λ is a cross-norm.

Proof of Proposition 1.45. Let $f \in \mathcal{A}^* = (\mathcal{A}^{**})_*$ and $g \in \mathcal{B}^*$ of norm 1. Then by Proposition 2.45 there is a partial isometry $U \in \mathcal{A}^{**}$ and a state $F \in \mathcal{A}^*$ such that $f(a) = F(aU)$ for all $a \in \mathcal{A}$. By Kaplansky's density theorem, we may let (u_α) be a net in $(\mathcal{A})_1$ such that $u_\alpha \to U$ ultraweakly. Then

$$F(au_\alpha) \to f(a)$$

for all $a \in \mathcal{A}$. Similarly, we find a net (v_β) in $(\mathcal{B})_1$ and a state $G \in \mathcal{B}^*$ such that $G(bv_\beta) \to g(b)$ for all $b \in \mathcal{B}$. Now for all $x = \sum_{i=1}^n a_i \otimes b_i$ in $\mathcal{A} \odot \mathcal{B}$ and $\varepsilon > 0$, let α_0 and β_0 such that $|f(a_i)g(b_i) - F(a_i u_{\alpha_0})G(b_i v_{\beta_0})| < \frac{\varepsilon}{n}$ for all i. We have

$$\begin{aligned}|(f \otimes g)(x)| = \left|\sum_i f(a_i)g(b_i)\right| &\le \left|\sum_i F(a_i u_{\alpha_0})G(b_i v_{\beta_0})\right| + \varepsilon \\ &\le \sup\{|(F \otimes G)(x(a \otimes b))| \mid a \in (\mathcal{A})_1,\ b \in (\mathcal{B})_1,\ \varphi \in S(\mathcal{A}),\ \psi \in S(\mathcal{B})\} \\ &\le \sup\{\|x(a \otimes b)\|_\kappa \mid a \in (\mathcal{A})_1,\ b \in (\mathcal{B})_1\} \\ &\le \|x\|_\kappa.\end{aligned}$$

Since $f \in \mathcal{A}^*$ and $g \in \mathcal{B}^*$ were arbitrary, we have $\lambda(x) \le \|x\|_\kappa$. Since C^*-seminorms are sub--cross-norms and λ is a cross-norm, $\|\cdot\|_\kappa$ is a cross-norm. Moreover, any linear functional of the form $f \otimes g$ for $f \in \mathcal{A}^*$ and $g \in \mathcal{B}^*$ is evidently bounded with respect to λ, and therefore bounded with respect to $\|\cdot\|_\kappa$ as well. □

We next prove the following result:

Theorem 1.50. *For C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, let $\mathcal{A}_1$ resp. $\mathcal{B}_1$ be the C^*-algebras obtained from $\mathcal{A}$ resp. $\mathcal{B}$ by adjoining an identity if it is non-unital. Then there is a one-to-one correspondence between C^*-norms on $\mathcal{A} \odot \mathcal{B}$ and C^*-norms on $\mathcal{A}_1 \odot \mathcal{B}_1$.*

In the following, let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{A}_1$ and $\mathcal{B}_1$ be as above.

Lemma 1. *Let $\varphi\colon \mathcal{A}\odot\mathcal{B}\to\mathbb{C}$ be an algebraically positive linear functional. Then $\varphi=\|\varphi\|_{\mathrm{alg}}\psi$ for an algebraic state $\psi\colon \mathcal{A}\odot\mathcal{B}\to\mathbb{C}$, and if $\mathcal{A}$ and $\mathcal{B}$ are unital, then $\varphi(1\otimes 1)=\|\varphi\|_{\mathrm{alg}}$.*

Proof. We may assume that $\varphi\neq 0$. Notice that the linear functional $\varphi_b\colon a\mapsto\varphi(a\otimes b)$ is bounded for fixed $b\in\mathcal{B}$. Indeed, for $b\in\mathcal{B}_+$, the map φ_b is positive, hence bounded, and since $\mathcal{B}=\mathrm{span}\mathcal{B}_+$, the claim follows. Define bounded linear functionals $\psi_a\colon b\mapsto\varphi(a\otimes b)$ for $a\in\mathcal{A}$ as well. The collection of maps $\{\varphi_b\mid \|b\|\leq 1\}$ satisfy $|\varphi_b(a)|=|\psi_a(b)|\leq\|\psi_a\|$ for all $a\in\mathcal{A}$, so the Uniform Boundedness Principle yields $K>0$ such that

$$\|\varphi\|_{\mathrm{alg}}=\sup_{a\in(\mathcal{A})_1,\,b\in(\mathcal{B})_1}|\varphi(a\otimes b)|=\sup_{b\in(\mathcal{B})_1}\|\varphi_b\|<\infty.$$

Hence $\psi=\|\varphi\|_{\mathrm{alg}}^{-1}\varphi$ is an algebraic state. The second assertion follows from Proposition 1.39. $\square$

Lemma 2. *Let $\varphi\colon \mathcal{A}\odot\mathcal{B}\to\mathbb{C}$ be an algebraically positive linear functional. Then φ has a unique extension to an algebraically positive linear functional $\hat{\varphi}\colon\mathcal{A}_1\odot\mathcal{B}_1\to\mathbb{C}$ such that $\|\hat{\varphi}\|_{\mathrm{alg}}=\|\varphi\|_{\mathrm{alg}}$. Moreover, for bounded approximate identities (e_α) resp. (f_β) of $\mathcal{A}$ resp. $\mathcal{B}$, we have $\hat{\varphi}(1\otimes b)=\lim_\alpha\hat{\varphi}(e_\alpha\otimes b)$ for all $b\in\mathcal{B}_1$ and $\hat{\varphi}(a\otimes 1)=\lim_\beta\hat{\varphi}(a\otimes f_\beta)$ for all $a\in\mathcal{A}_1$, as well as*

$$\hat{\varphi}(1\otimes 1)=\lim_{\alpha,\beta}\varphi(e_\alpha\otimes f_\beta).$$

Proof. We may assume that $\|\varphi\|_{\mathrm{alg}}=1$. Let $\pi\colon\mathcal{A}\odot\mathcal{B}\to B(H)$ be the non-degenerate GNS representation of φ, and let $\pi_1\colon\mathcal{A}\to B(H)$ and $\pi_2\colon\mathcal{B}\to B(H)$ be commuting representations such that $\pi(a\otimes b)=\pi_1(a)\pi_2(b)$ for all $a\in\mathcal{A}$ and $b\in\mathcal{B}$ [Brown-Ozawa, Theorem 3.2.6]. Notice that these are uniquely determined. Indeed, for (e_α) and (f_β) as above, then we must have $1_H=\lim_\alpha\pi_1(e_\alpha)=\lim_\beta\pi_2(f_\beta)$ in the strong operator topology by π being non-degenerate (so that vectors of the form $\sum_i\pi(x_i\otimes y_i)\xi$ span a dense subspace of H). Hence we have

$$\pi_1(a)=\lim_\beta\pi(a\otimes f_\beta),\quad \pi_2(b)=\lim_\alpha(e_\alpha\otimes b),\quad a\in\mathcal{A},\ b\in\mathcal{B}.$$

We may then extend π_1 and π_2 to unital representations $\hat{\pi}_1\colon\mathcal{A}_1\to B(H)$ and $\hat{\pi}_2\colon\mathcal{B}_1\to B(H)$. As these commute, we may define an extension $\hat{\pi}=\hat{\pi}_1\times\hat{\pi}_2\colon\mathcal{A}_1\odot\mathcal{B}_1\to B(H)$ of π, and $\hat{\varphi}(x)=\langle\hat{\pi}(x)\xi,\xi\rangle$ for $x\in\mathcal{A}_1\odot\mathcal{B}_1$ where $\xi\in H$ is the unit vector from the GNS representation of φ. Then $\hat{\varphi}$ is clearly an algebraic state extending φ. Moreover, for (e_α) and (f_β) as above we have $\pi_1(e_\alpha)\to 1=\hat{\pi}_1(1)$ in the strong operator topology, so that

$$\hat{\varphi}(e_\alpha\otimes b)=\langle\pi_1(e_\alpha)\hat{\pi}_2(b)\xi,\xi\rangle\to\langle\hat{\pi}_1(1)\hat{\pi}_2(b)\xi,\xi\rangle=\hat{\varphi}(1\otimes b)$$

for all $b\in\mathcal{B}_1$. Similarly, one sees that $\hat{\varphi}(a\otimes f_\beta)\to\hat{\varphi}(a\otimes 1)$ for all $a\in\mathcal{A}_1$.

If $g\colon\mathcal{A}_1\odot\mathcal{B}_1\to\mathbb{C}$ extends φ and $\|g\|_{\mathrm{alg}}=\|\varphi\|_{\mathrm{alg}}$, let (v_β) be a contractive approximate identity of $\mathcal{B}$. For all $a\in\mathcal{A}_+$ we have

$$g(a\otimes 1)\geq g(a\otimes v_\beta)=\hat{\varphi}(a\otimes v_\beta)=\langle\hat{\pi}_1(a)\pi_2(v_\beta)\xi,\xi\rangle\to\langle\hat{\pi}_1(a)\hat{\pi}_2(1)\xi,\xi\rangle=\hat{\varphi}(a\otimes 1).$$

Hence $\hat{\varphi}(a\otimes 1)\leq g(a\otimes 1)$. If some $a'\in\mathcal{A}_+$ with $\|a'\|\leq 1$ satisfies $\delta:=g(a'\otimes 1)-\hat{\varphi}(a\otimes 1)>0$, then for all $a'\leq a''$ with $\|a''\|\leq 1$ we have

$$\|\varphi\|_{\mathrm{alg}}-\hat{\varphi}(a''\otimes 1)\geq g(a''\otimes 1)-\hat{\varphi}(a''\otimes 1)=g((a''-a')\otimes 1)-\hat{\varphi}((a''-a')\otimes 1)+\delta\geq\delta.$$

Letting $b_n=1-\frac{1}{n}(1-a)$ for $n\geq 1$, then $b_n\geq a'$ and $\|b_n\|\leq 1$, so that

$$\|\varphi\|_{\mathrm{alg}}=\lim_{n\to\infty}\hat{\varphi}(b_n)\leq\|\varphi\|_{\mathrm{alg}}-\delta,$$

a contradiction. Therefore $g(a\otimes 1)=\hat{\varphi}(a\otimes 1)$ for all $a\in\mathcal{A}_+$ and hence for all $a\in\mathcal{A}$. Similarly one sees that $g(1\otimes b)=\hat{\varphi}(1\otimes b)$ for all $b\in\mathcal{B}$. As $g|_{\mathcal{A}\odot\mathcal{B}}=\hat{\varphi}|_{\mathcal{A}\odot\mathcal{B}}$ by assumption, one concludes that $g=\hat{\varphi}$. $\square$

From here onward, we let $\hat{\varphi}$ denote the unique extension of an algebraically positive linear functional $\varphi\colon\mathcal{A}\odot\mathcal{B}\to\mathbb{C}$ to an algebraically positive linear functional $\mathcal{A}_1\odot\mathcal{B}_1\to\mathbb{C}$ with the same algebraic norm.

Lemma 3. *Let $x \in \mathcal{A}_1 \odot \mathcal{B}_1$. If $xy = 0$ for all $y \in \mathcal{A} \odot \mathcal{B}$, then $x = 0$.*

Proof. Let $\pi_1 \colon \mathcal{A} \to B(H_1)$ and $\pi_2 \colon \mathcal{B} \to B(H_2)$ be faithful non-degenerate representations; if $\mathcal{A}$ or $\mathcal{B}$ is unital, let the corresponding representation be unital. Now π_1 and π_2 extend to faithful unital representations of $\mathcal{A}_1$ and $\mathcal{B}_1$, the tensor product representation $\pi = \pi_1 \otimes \pi_2 \colon \mathcal{A}_1 \odot \mathcal{B}_1 \to B(H_1 \otimes H_2)$ is unital and faithful (Corollary 1.22), and $\pi|_{\mathcal{A}\odot\mathcal{B}}$ is non-degenerate. As $xy = 0$ for all $y \in \mathcal{A} \odot \mathcal{B}$ implies $\pi(x)\pi(y)\eta = 0$ for all $y \in \mathcal{A} \odot \mathcal{B}$, we have $\pi(x)\xi = 0$ for all $\xi \in H_1 \otimes H_2$ by $\pi|_{\mathcal{A}\odot\mathcal{B}}$ being non-degenerate, so $x = 0$. □

Lemma 4. *Let p be a C^*-norm on $\mathcal{A}\odot\mathcal{B}$. Then $\Gamma = \{\hat{\varphi} \in S(\mathcal{A}_1 \odot \mathcal{B}_1) \mid \varphi \in S_p(\mathcal{A}\odot\mathcal{B})\}$ is a separating subset of $S(\mathcal{A}_1 \odot \mathcal{B}_1)$, and thus defines a C^*-norm $\hat{p} = p_\Gamma$ on $\mathcal{A}_1 \odot \mathcal{B}_1$. Moreover, $\hat{p}|_{\mathcal{A}\odot\mathcal{B}} = p$.*

Proof. For $x \in \mathcal{A}_1 \odot \mathcal{B}_1$, $p_\Gamma(x) = 0$ implies

$$\|\pi_{\hat{\varphi}}(x^*x)\| = \|\pi_{\hat{\varphi}}(x)\|^2 \leq p_\Gamma(x)^2 = 0$$

and $\hat{\varphi}(y^*x^*xy) = \|\pi_{\hat{\varphi}}(x^*x)\| \|\pi_{\hat{\varphi}}(y)\xi_{\hat{\varphi}}\|^2 = 0$ for all $\varphi \in S_p(\mathcal{A}\otimes\mathcal{B})$ and $y \in \mathcal{A}_1 \odot \mathcal{B}_1$. In particular, $\varphi(y^*x^*xy) = 0$ for all $\varphi \in S_p(\mathcal{A}\otimes\mathcal{B})$ and $y \in \mathcal{A} \odot \mathcal{B}$, so $p(xy) = 0$ and $xy = 0$ for all $y \in \mathcal{A} \odot \mathcal{B}$ by Proposition 1.41. Since $x = 0$ by the preceding lemma, p_Γ is separating and is therefore a C^*-norm. Moreover, Γ is evidently convex.

Now for $\varphi \in S_p(\mathcal{A} \odot \mathcal{B})$ and $y \in \mathcal{A}_1 \odot \mathcal{B}_1$, then as $\hat{\varphi}(y^*x^*xy) \leq \|x\|^2\hat{\varphi}(y^*y)$ for all $x \in \mathcal{A}_1 \odot \mathcal{B}_1$, we may assume that $\hat{\varphi}(y^*y) > 0$ if we are to prove that the conditions of Lemma 1.42 hold. Defining $\psi(x) = \hat{\varphi}(y^*y)^{-1}\varphi(y^*xy)$ for $x \in \mathcal{A} \odot \mathcal{B}$, then ψ is an algebraic state on $\mathcal{A} \odot \mathcal{B}$. Since $\psi'(x) = \hat{\varphi}(y^*y)^{-1}\hat{\varphi}(y^*xy)$, $x \in \mathcal{A}_1 \odot \mathcal{B}_1$ defines an algebraic state on $\mathcal{A}_1 \odot \mathcal{B}_1$ extending ψ, we see that $\psi' = \hat{\psi}$ by uniqueness, and hence $\hat{\varphi}(y^*xy) = \hat{\varphi}(y^*y)\hat{\psi}(x)$ for all $x \in \mathcal{A}_1 \odot \mathcal{B}_1$. To see that $\hat{\psi} \in \Gamma$, simply note that φ is bounded with respect to p by assumption, so that ψ is bounded with respect to p as well, i.e., $\psi \in S_p(\mathcal{A} \odot \mathcal{B})$. Moreover, we see that

$$p_\Gamma(x) = \sup\{\hat{\varphi}(x^*x)^{1/2} : \varphi \in S_p(\mathcal{A} \odot \mathcal{B})\}$$

for all $x \in \mathcal{A}_1 \odot \mathcal{B}_1$, which completes the proof. □

For any positive linear functional $\sigma \colon \mathcal{A} \to \mathbb{C}$, let $\tilde{\sigma} = \sigma$ if $\mathcal{A}$ is unital, and let $\tilde{\sigma}(a+\alpha 1) = \rho(a)+\alpha\|\rho\|$ if $\mathcal{A}$ is non-unital. Then $\tilde{\sigma}$ is the unique positive linear functional on $\mathcal{A}_1$ extending σ and having the same norm as $\mathcal{A}$ [Master's thesis, Corollary 1.5.5]. Moreover, if $\mathcal{A}$ is non-unital, define a state $i_\mathcal{A} \colon \mathcal{A}_1 \to \mathbb{C}$ by $i_\mathcal{A}(a + \alpha 1) = \alpha$ for $a \in \mathcal{A}$ and $\alpha \in \mathbb{C}$.

Proof of Theorem 1.50. We are done once we prove that any C^*-norm p on $\mathcal{A}_1 \odot \mathcal{B}_1$ satisfies $p = \hat{p_0}$, where $p_0 = p|_{\mathcal{A}\odot\mathcal{B}}$.

With this notation, then let $\psi \in S_p(\mathcal{A}_1 \odot \mathcal{B}_1)$ and $\varphi = \psi|_{\mathcal{A}\odot\mathcal{B}}$. Suppose that $\mathcal{B}$ is non-unital. As in the proof of Lemma 2 we have $\hat{\varphi}(a \otimes 1) \leq \psi(a \otimes 1)$ for all $a \in \mathcal{A}_+$. Define the positive linear functional $\rho \colon \mathcal{A} \to \mathbb{C}$ by

$$\rho(a) = \psi(a \otimes 1) - \hat{\varphi}(a \otimes 1),$$

and let $\hat{\rho} = \tilde{\rho} \otimes i_\mathcal{B} \colon \mathcal{A}_1 \odot \mathcal{B}_1 \to \mathbb{C}$.

Let (e_α) be a bounded approximate identity for $\mathcal{A}$ (letting $e_\alpha = 1$ if $\mathcal{A}$ is unital). For $b' = b+\beta 1 \in \mathcal{B}_1$ positive, then by Proposition 0.3, we find that

$$\begin{aligned}
(\hat{\varphi} + \hat{\rho})(1 \otimes b') &= \lim_\alpha \hat{\varphi}(e_\alpha \otimes b') + \tilde{\rho}(1)\beta \\
&= \lim_\alpha(\hat{\varphi}(e_\alpha \otimes b) + \beta\hat{\varphi}(e_\alpha \otimes 1) + \beta\rho(e_\alpha)) \\
&= \lim_\alpha(\psi(e_\alpha \otimes b) + \beta\psi(e_\alpha \otimes 1)) \\
&= \lim_\alpha \psi(e_\alpha \otimes b') \leq \psi(1 \otimes b').
\end{aligned}$$

We may then define a positive linear functional $\sigma_1 \colon \mathcal{B}_1 \to \mathbb{C}$ by $\sigma_1(b) = \psi(1 \otimes b) - (\hat{\varphi} + \hat{\rho})(1 \otimes b)$ for $b \in \mathcal{B}_1$. If $\mathcal{A}$ is unital, let $\mu = 0$; if not, let $\mu = i_\mathcal{A} \otimes \sigma_1$. We then claim that

$$\psi = \hat{\varphi} + \hat{\rho} + \mu.$$

Note first that $i_\mathcal{B}|_\mathcal{B} = 0$, so that we always have $\hat{\rho}|_{\mathcal{A}_1\odot\mathcal{B}} = 0$. If $\mathcal{A}$ is unital, let $a \in \mathcal{A}$, $b \in \mathcal{B}$ and $\beta \in \mathbb{C}$, and note that

$$(\hat{\varphi} + \hat{\rho})(a \otimes (b + \beta 1)) = \psi(a \otimes b) + \beta(\hat{\varphi}(a \otimes 1) + \rho(a)) = \psi(a \otimes (b + \beta 1)).$$

If $\mathcal{A}$ is non-unital, then for $a' = a + \alpha 1 \in \mathcal{A}_1$ and $b' = b + \beta 1 \in \mathcal{B}_1$ we have

$$\begin{aligned}(\hat{\varphi} + \hat{\rho} + \mu)(a' \otimes b') &= \hat{\varphi}(a' \otimes b') + \beta\hat{\rho}(a' \otimes 1) + \alpha\sigma_1(b') \\ &= \hat{\varphi}(a' \otimes b') + \beta\hat{\rho}(a' \otimes 1) + \alpha\psi(1 \otimes b') - \alpha\hat{\varphi}(1 \otimes b') - \alpha\hat{\rho}(1 \otimes b') \\ &= \hat{\varphi}(a \otimes b') + \beta(\rho(a) + \alpha\|\rho\|) + \alpha\psi(1 \otimes b') - \|\rho\|\alpha\beta \\ &= \psi(a \otimes b) + \beta\hat{\varphi}(a \otimes 1) + \beta\rho(a) + \alpha\psi(1 \otimes b') \\ &= \psi(a \otimes b) + \beta\psi(a \otimes 1) + \alpha\psi(1 \otimes b') \\ &= \psi(a' \otimes b').\end{aligned}$$

With Γ as in the preceding lemma, then $\hat{\varphi}$ is continuous with respect to $(p_0)_\Gamma = \hat{p_0}$. For any positive linear functional $\delta\colon \mathcal{A} \to \mathbb{C}$, then the linear functional $\tilde{\delta}\colon \mathcal{A}_1 \to \mathbb{C}$ is continuous. Therefore $\tilde{\delta} \otimes i_{\mathcal{B}}$ is bounded with respect to $\|\cdot\|_\kappa$, due to Proposition 1.45. Since $\|\cdot\|_\kappa$ is the minimal C^*-norm, it follows that the previously defined algebraically positive linear functional $\hat{\rho}\colon \mathcal{A}_1 \odot \mathcal{B}_1 \to \mathbb{C}$ is bounded with respect to $\hat{p_0}$. In a similar manner one sees that μ is also bounded with respect to $\hat{p_0}$. Taken all together, it follows from our above considerations that $\psi = \hat{\varphi} + \hat{\rho} + \mu$ is bounded with respect to $\hat{p_0}$ as well, so that

$$S_p(\mathcal{A}_1 \odot \mathcal{B}_1) \subseteq S_{\hat{p_0}}(\mathcal{A}_1 \odot \mathcal{B}_1).$$

By Proposition 1.41, we have $p \leq \hat{p_0}$. (If $\mathcal{B}$ happens to be unital, the proof is completely analoguous.)

Conversely, if $\varphi \in S_{p_0}(\mathcal{A} \odot \mathcal{B})$, let us assume that both $\mathcal{A}$ and $\mathcal{B}$ are non-unital. For $a \in \mathcal{A}$, $b \in \mathcal{B}$ and $\gamma, \delta \in \mathbb{C}$, let (e_α) and (f_β) be bounded approximate identities for $\mathcal{A}$ resp. $\mathcal{B}$. Now for $a' = a + \gamma 1$ and $b' = b + \delta 1$, then $(a' \otimes b')(e_\alpha \otimes f_\beta) \in \mathcal{A} \odot \mathcal{B}$ and

$$\hat{\varphi}(a' \otimes b') = \lim_{\alpha,\beta}(\varphi(a \otimes b + \gamma e_\alpha \otimes b + a \otimes \delta f_\beta + \gamma\delta e_\alpha \otimes f_\beta) = \lim_{\alpha,\beta} \varphi((a' \otimes b')(e_\alpha \otimes f_\beta)).$$

This also holds if at least one of $\mathcal{A}$ and $\mathcal{B}$ is unital (in which case we replace the associated approximate identity by the identity element). For $x \in \mathcal{A}_1 \odot \mathcal{B}_1$ such that $p(x) \leq 1$, this means that $\hat{\varphi}(x^*x) = \lim_{\alpha,\beta} \varphi(x^*x(e_\alpha \otimes f_\beta))$, but we also have

$$|\varphi(x^*x(e_\alpha \otimes f_\beta))| \leq p_0(x^*x(e_\alpha \otimes f_\beta)) = p(x^*x(e_\alpha \otimes f_\beta)) \leq p(x^*x) \leq 1.$$

Hence $\hat{\varphi}(x^*x) \leq 1$. As found in the preceding lemma, we now have

$$\hat{p_0}(x) = \sup\{\hat{\varphi}(x^*x)^{1/2} : \varphi \in S_{p_0}(\mathcal{A} \odot \mathcal{B})\} \leq 1,$$

so that $\hat{p_0} \leq p$. This ends the proof. □