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Master's thesis in mathematics

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Simplicity and uniqueness of trace for reduced group C^*-algebras

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Abstract

A locally compact group G is said to be C^*-simple (resp. has the unique trace property) if its reduced group C^*-algebra $C_r^*(G)$ is simple (resp. has a unique tracial state). After Powers' 1975 article containing a proof that the non-abelian free group of rank two was C^*-simple with unique trace (acting on a suggestion by Kadison), de la Harpe realized, among others, that part of Powers' proof could be applied to a variety of different groups to yield a large class of groups with the very same traits. In this thesis we will look into what effects C^*-simplicity or uniqueness of trace has on (mainly) discrete groups, the most intriguing being that the only normal, amenable subgroup is the trivial subgroup. It is still an open question whether any converse holds in general. We will also give several sufficient criteria for a discrete group to be C^*-simple with unique trace, by considering classes of groups with certain combinatorial properties, the first of which will be the class of Powers groups originally coined by de la Harpe in 1985. Since then, authors such as Boca, Niţică and Promislow have relaxed the defining property of a Powers group, resulting in new properties that retain C^*-simplicity and unique trace, and we consider the advantages of working with these weaker variants. Moreover, we provide an extensive overview of the ultraweak Powers groups of Bédos and their connection to reduced twisted crossed products, and we also establish some of the known permanence properties of C^*-simplicity and uniqueness of trace. Finally, we prove for $n \geq 2$ that subgroups of the projective special linear group $\mathrm{PSL}(n, \mathbb{R})$ containing $\mathrm{PSL}(n, \mathbb{Z})$ are C^*-simple with unique trace.

Resumé

En lokalkompakt gruppe G siges at være C^*-simpel (hhv. have entydigt spor) hvis dens reducerede gruppe-C^*-algebra $C_r^*(G)$ er simpel (hhv. har entydigt spor). Efter Powers' artikel fra 1975, hvori han viste at den frie gruppe af rang 2 var C^*-simpel og havde entydigt spor (ud fra et forslag fra Kadison), opdagede de la Harpe, blandt andre, at dele af Powers' bevis kunne anvendes på mange andre grupper til at konkludere C^*-simplicitet og entydighed af spor for disse. I dette speciale undersøger vi konsekvenserne af C^*-simplicitet og entydighed af spor på (hovedsageligt) diskrete grupper. En af de mest interessante af disse er, at den eneste normale, amenable undergruppe af en gruppe, der enten er C^*-simpel eller har entydigt spor, er den trivielle undergruppe. Det er stadig et uløst problem hvorvidt dette gælder den anden vej generelt. Vi giver også forskellige tilstrækkelige betingelser for at en diskret gruppe er C^*-simpel med entydigt spor, af hvilke hovedparten opnås ved at kigge på klasser af grupper med specielle kombinatoriske egenskaber. Den første af disse, der gennemgås, er den af de såkaldte Powers-grupper, oprindeligt defineret af de la Harpe i 1985. Siden da har andre, heriblandt Boca, Niţică og Promislow, taget de egenskaber, der er essentielle for at bibeholde C^*-simplicitet og entydighed af spor, for Powers-grupper ud for derpå at definere nye tilsvarende begreber, og vi undersøger fordelene ved at arbejde med disse varianter. Endvidere diskuterer vi ultrasvage Powers-grupper (defineret af Bédos) og deres forbindelse til reducerede krydsprodukter, og vi redegør også for visse stabilitetsegenskaber for C^*-simplicitet og entydighed af spor. Specialet rundes af med et bevis for, at der for $n \geq 2$ gælder, at undergrupper af den projektive specielle lineære gruppe $\mathrm{PSL}(n, \mathbb{R})$ som indeholder $\mathrm{PSL}(n, \mathbb{Z})$ er C^*-simple med entydigt spor.

Contents

Prologue

This thesis revolves around the topics of C^*-simplicity and the unique trace property, two properties of groups grounded in the theory of C^*-algebras that have steadily gained attention over the last four decades. If G is a locally compact group, we say that G is *C^*-simple* whenever its reduced group C^*-algebra $C_r^*(G)$ is a simple C^*-algebra, and that G has the *unique trace property* whenever $C_r^*(G)$ has a unique trace (i.e., tracial state). Since Powers proved that the non-abelian group on two generators satisfied both of these properties, a heap of other discrete groups have been shown to have these properties, with proofs of varying complexity but always grounded in the proof of Powers.

Seeing as the reduced group C^*-algebra of a discrete group is a much more manageable object than that of a non-discrete group, it is no surprise that all known C^*-simple groups are considered with the discrete topology. It does however call attention to the following question:

Question 1. *Does there exist a non-discrete locally compact group that is either C^*-simple or has the unique trace property?*

Even if one agrees to consider only discrete groups, there are still a number of questions that have not been answered. One of the most immediately gripping arises from the fact that so far, any discrete group that is C^*-simple has also been shown to have the unique trace property. It is therefore natural to ask the following.

Question 2. *Does there exist a discrete group that is C^*-simple but does not have the unique trace property, or vice versa?*

The study of C^*-simplicity and the unique trace property has of course also brought to light some interesting traits of groups with these properties. One of the most illuminating (if we are allowed to extend the metaphor) is the fact that all C^*-simple groups with unique trace have *trivial amenable radical*, i.e., the largest, normal, amenable subgroup is the subgroup containing only the identity element. Recently, it has been proved by Poznansky in [59] that if Γ is a countable, linear group, then the following three properties are equivalent:

(i) Γ is C^*-simple.
(ii) Γ has the unique trace property.
(iii) Γ has trivial amenable radical.

However, as of now the following question is still unanswered:

Question 3. *Does there exist a (countable) discrete group with trivial amenable radical that is not C^*-simple or does not have the unique trace property?*

This thesis does not attempt to answer any of the above three questions. Instead, we wish to give a general overview of the terrain, so to speak, by giving proofs of the most essential results related to the two main topics and provide interesting enough examples to motivate further investigation, mainly by means of combinatorial considerations, but also by using hyperbolic and algebraic geometry as a tool in our research. Our approach is inspired by two papers of de la Harpe ([32] and [31]), but we will also consider results by Boca and Niţică ([12]), Bédos ([5], [6], [7]) and Popa ([57]) in this regard.

Let us now give a short runthrough of what this thesis covers. Chapter 1 functions as an introduction to the reduced group C^*-algebra of a locally compact group and to the two central topics of this thesis, namely C^*-simplicity and uniqueness of trace. The main objective of the chapter is to prove that any C^*-simple locally compact group contains no non-trivial amenable, closed, normal subgroups, greatly affecting the possible traits of such groups. Along the way, we provide an alternate characterization of C^*-simplicity, expressed by means of weak containment of unitary representations. Finally, we provide some motivation for turning to discrete groups in order to obtain the greatest degree of positive results on both C^*-simplicity and uniqueness of trace.

In Chapter 2, we investigate the Dixmier property for unital C^*-algebras. The property itself was originally applied by Dixmier to von Neumann algebras to give a description of their ideals, and it is central to our studies because the reduced group C^*-algebra of a discrete group is simple and has unique trace *if and only if* it satisfies the Dixmier property; we give a proof for unital C^*-algebras in general. Seeing as the proof requires a bit of knowledge of finite, properly infinite and full projections in C^*-algebras, we have included an appendix in which we prove the most important facts about these.

Chapter 3 first and foremost concerns the class of Powers groups, yielding the first examples in the thesis of discrete groups with simple reduced group C^*-algebras with unique trace. We give a few easy examples and discuss permanence properties of Powers groups, before turning to the chief tool for finding Powers groups: examining group actions on Hausdorff spaces. We then give an introduction to the group $\mathrm{PSL}(2,\mathbb{R})$ of Möbius transformations on the extended upper half plane $\widetilde{\mathbb{H}}$ and prove that all non-elementary subgroups of $\mathrm{PSL}(2,\mathbb{R})$ are in fact Powers groups. Finally, we discuss how to relax the Powers property in order to obtain nicer permanence properties *and* retain both C^*-simplicity and unique trace, resulting in the notions of weak Powers groups and PH groups.

Chapter 4 provides an exposition of reduced twisted crossed products of C^*-algebras by discrete groups. The motivation for this is to look into ultraweak Powers groups, i.e., groups that contain a normal weak Powers subgroup with trivial centralizer. A central structure theorem for twisted crossed products gives us a means to show that ultraweak Powers groups are in fact also C^*-simple. We then discuss how ultraweak Powers groups can provide a lot of pretty permanence properties for C^*-simplicity in general. For the question of whether unique trace is preserved under the same circumstances, we turn to a von Neumann algebra variant of reduced twisted crossed products, namely regular extensions, for which we prove a lot of essential structure theorems. The final section of the chapter is devoted to showing that uniqueness of trace indeed has the same permanence properties as C^*-simplicity as found earlier.

Chapter 5 is an intermezzo of sorts, giving us the opportunity to investigate permanence properties of C^*-simplicity and uniqueness of trace in general. We consider direct products and inductive limits, before turning to finite index subgroups. Here we provide a gentle entrance to the study of indices of subfactors, our approach based on the work of Jones, Pimsner and Popa, and the notion of a subfactor having finite index turns out to be instrumental in proving that both C^*-simplicity and uniqueness of trace are preserved by passing to subgroups of finite index.

Finally, Chapter 6 is devoted to the topic of projective special linear groups and a proof of the fact that all subgroups of $\mathrm{PSL}(n,\mathbb{R})$ containing $\mathrm{PSL}(n,\mathbb{Z})$ are in fact C^*-simple with unique trace. To realize this, we first give an introduction to these groups and their action on the real projective space $\mathbb{P}^{n-1}(\mathbb{R})$. We then give a quick runthrough of the basic and not-so-basic facts about the Zariski topology, as the proof uses the properties of this particular topology to great effect. The main result of this section is due to Bekka, Cowling and de la Harpe from 1994 (see [8]). Finally, we wind up the thesis by using the techniques of this chapter to prove that certain subgroups of $\mathrm{PSL}(2,\mathbb{C})$ are in fact Powers groups.

Of course, as the reader may by now have gathered, there are important results aplenty related to the two chief topics that we have not included in this text. In particular there is a heap of examples of C^*-simple groups with unique trace that go entirely unmentioned. The reason is twofold: first, timely constraints are what they are, and second, most of these results require too far a deviation from the central objects of study in order for us to fully explain them. It is our hope that the reader is incited just a little bit to delve further into these topics.

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PREREQUISITES

The purpose of this unnumbered chapter is to briefly introduce the chief objects of study in the thesis. In order for the next six chapters to make sense, this is also the place where we put notation for these objects, along with an introduction to the properties that are the most relevant (for the thesis, at least). There will be no proofs; instead references will be given to literature where proofs for the non-well-known results can be found. Most of the concepts covered here are also described in greater detail in the chapter of prerequisites in [15].

For a set X and a subset $S \subseteq X$, we define the *characteristic function* $1_S \colon X \to \{0,1\}$ by

$$1_S(x) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{if } x \notin S, \end{cases}$$

If $\mathfrak{X}$ is a normed space and $r > 0$, the ball with radius $r > 0$ centered at 0 is denoted by

$$(\mathfrak{X})_r = \{x \in \mathfrak{X} \mid \|x\| \leq r\}.$$

Groups. If Γ is a group, the identity element of Γ will always be denoted by 1.

- If all conjugacy classes bar $\{1\}$ in a group Γ are infinite sets, we say that Γ is *icc* (short for *infinite conjugacy classes*).
- The *centralizer* of a subset $S \subseteq \Gamma$ is the set of all elements $x \in \Gamma$ that commute with all elements of S, i.e., $x \in \Gamma$ such that $xs = sx$ for all $s \in S$. The *center* of Γ is the centralizer of Γ in Γ.

Hilbert spaces. All Hilbert spaces in this thesis are usually denoted by $\mathcal{H}$ or $\mathcal{K}$, and the inner product on a Hilbert space $\mathcal{H}$ of two elements $\xi, \eta \in \mathcal{H}$ is denoted by $\langle \xi, \eta \rangle_{\mathcal{H}}$ or just $\langle \xi, \eta \rangle$ if the Hilbert space is clear from the context. The C^*-algebra of all bounded linear operators on the Hilbert space $\mathcal{H}$ is denoted by $B(\mathcal{H})$. If Γ is a group, we will often want to consider the Hilbert space $\ell^2(\Gamma)$ of all maps $\xi \colon \Gamma \to \mathbb{C}$ such that $\sum_{s \in \Gamma} |\xi(s)|^2 < \infty$.

- For all $s \in \Gamma$, we define the *Dirac point mass* (or *Dirac measure*) $\delta_s \colon \Gamma \to \mathbb{C}$ by

  $$\delta_s(t) = \begin{cases} 1 & \text{if } t = s \\ 0 & \text{if } t \neq s, \end{cases}$$

  and one can show that the family $(\delta_s)_{s \in \Gamma}$ constitutes an orthonormal basis of $\ell^2(\Gamma)$. Hence all elements in $\ell^2(\Gamma)$ are of the form $\sum_{s \in \Gamma} z_s \delta_s$ where $(z_s)_{s \in \Gamma}$ is a family of complex numbers such that $\sum_{s \in \Gamma} |z_s|^2 < \infty$.
- If $D \subseteq \Gamma$ is any subset, the closed subspace $\mathcal{M} \subseteq \ell^2(\Gamma)$ consisting of all functions $\xi \in \ell^2(\Gamma)$ such that $\xi(s) = 0$ for $s \notin D$ will be denoted by $\ell^2(D)$, so that we identify the Hilbert space $\ell^2(D)$ with a closed subspace of $\ell^2(\Gamma)$. The orthogonal complement of $\ell^2(D)$ in $\ell^2(\Gamma)$ is of course the closed subspace $\ell^2(\Gamma \setminus D)$.

C^*-algebras. The C^*-algebras in this thesis are usually named $\mathcal{A}$ or $\mathcal{B}$. We will often work with *unital* C^*-algebras, i.e., C^*-algebras with a multiplicative identity. If $\mathcal{A}$ is a unital C^*-algebra, the multiplicative identity will always be denoted by $1_{\mathcal{A}}$, or just 1 if there is no danger of misunderstanding.

- All ideals of C^*-algebras are assumed to be two-sided, unless otherwise stated. If $\mathcal{A}$ is a C^*-algebra, we say that $\mathcal{A}$ is ***simple*** if the only closed ideals of $\mathcal{A}$ are $\{0\}$ and $\mathcal{A}$ itself.

- The *spectrum* $\sigma_{\mathcal{A}}(a)$ of an element $a \in \mathcal{A}$ is the set of all $\lambda \in \mathbb{C}$ such that $\lambda 1_{\mathcal{A}} - a$ is not invertible (i.e., does not have a multiplicative inverse) in $\mathcal{A}$. We will write $\sigma(a)$ instead of $\sigma_{\mathcal{A}}(a)$ if the C^*-algebra is clear from the context.

- Let $\mathcal{A}$ be a unital C^*-algebra. An element $a \in \mathcal{A}$ is said to be
  - *self-adjoint* if $a = a^*$, and the set of all self-adjoint elements of $\mathcal{A}$ is denoted by $\mathcal{A}_{\text{sa}}$.
  - *positive* if there exists $x \in \mathcal{A}$ such that $a = x^*x$, and the cone of all positive elements of $\mathcal{A}$ is denoted by $\mathcal{A}_+$.
  - a *projection* if $a^2 = a = a^*$, and the set of projections of $\mathcal{A}$ is denoted by $\mathcal{P}(\mathcal{A})$.
  - *unitary* if $a^*a = aa^* = 1_{\mathcal{A}}$, and the group of unitary elements of $\mathcal{A}$ is denoted by $\mathcal{U}(\mathcal{A})$.

- If $a, b \in \mathcal{A}_{\text{sa}}$ and $b - a \in \mathcal{A}_+$, we will write $a \leq b$. The relation $\leq$ on $\mathcal{A}_{\text{sa}}$ defined in this manner is a partial ordering.

- A *-homomorphism of a Banach *-algebra into a C^*-algebra is always contractive [69, Proposition I.5.2].

- Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras. Then a map $\varphi\colon \mathcal{A} \to \mathcal{B}$ is said to be *positive* if $\varphi(\mathcal{A}_+) \subseteq \mathcal{B}_+$. A bounded linear functional $\varphi \in \mathcal{A}^*$ is therefore positive if it maps positive elements to positive numbers.
  - A positive bounded linear functional $\varphi \in \mathcal{A}^*$ is called a *state* if $\|\varphi\| = 1$, and the space of states on $\mathcal{A}$ is denoted by $S(\mathcal{A})$. If $\mathcal{A}$ is unital, then $\varphi \in \mathcal{A}^*$ is a state if and only if $\varphi(1_{\mathcal{A}}) = \|\varphi\| = 1$ [74, Theorem 13.5].
  - If $\varphi, \psi \in \mathcal{A}^*$ and $\psi - \varphi$ is positive, we say that ψ *majorizes* φ and write $\varphi \leq \psi$.
  - A positive linear functional $\varphi \in \mathcal{A}^*$ is said to be *faithful* if $\varphi(a^*a) > 0$ for all non-zero $a \in \mathcal{A}$.
  - If $\mathcal{A}$ is a C^*-algebra and $\varphi \in \mathcal{A}^*$, then φ is called a ***trace*** (or a *tracial state*) if it is positive, $\|\varphi\| = 1$ and $\varphi(ab) = \varphi(ba)$ for all $a, b \in \mathcal{A}$.

- Let $\mathcal{A}$ be a Banach *-algebra and let $\mathcal{H}$ be a Hilbert space. A *-homomorphism $\pi\colon \mathcal{A} \to B(\mathcal{H})$ is called a *representation* of $\mathcal{A}$ on $\mathcal{H}$. Letting $\mathscr{M} = \pi(\mathcal{A})$, then if the closed subspace of $\mathcal{H}$ generated by all vectors of the form $x\xi$ for $x \in \mathscr{M}$ and $\xi \in \mathcal{H}$ equals $\mathcal{H}$ itself, we say that π and $\mathscr{M}$ are both *non-degenerate*. Moreover, if $\xi \in \mathcal{H}$ and
  - $\mathcal{H}$ is the closure of the subspace $\mathscr{M}\xi$, then ξ is said to be *cyclic* for $\mathscr{M}$;
  - $x\xi = 0$ implies $x = 0$ for all $a \in \mathscr{M}$, then ξ is said to be *separating* for $\mathscr{M}$;
  - if $\langle xy\xi, \xi\rangle = \langle yx\xi, \xi\rangle$ for all $x, y \in \mathscr{M}$, ξ is called a *trace vector* for $\mathscr{M}$.

- For any positive linear functional φ on a C^*-algebra $\mathcal{A}$ there exists a Hilbert space $\mathcal{H}_\varphi$, a representation $\pi\colon \mathcal{A} \to B(\mathcal{H}_\varphi)$ and a unit vector $\xi_\varphi \in \mathcal{H}$ such that ξ_φ is cyclic for $\pi(\mathcal{A})$ (that is, the closed subspace generated by $\pi(\mathcal{A})\xi_\varphi$ is $\mathcal{H}$) and

  $$\varphi(x) = \langle \pi_\varphi(x)\xi_\varphi, \xi_\varphi\rangle, \quad x \in \mathcal{A}.$$

  The triple $(\pi_\varphi, \mathcal{H}_\varphi, \xi_\varphi)$ is called the *GNS triple* (or the *GNS representation*) for φ – it is one of those constructions where the proof that it holds is just as important as the construction itself. It is well-known that any C^*-algebra has a faithful (i.e., injective) representation on some Hilbert space, and it is typically proved by means of the GNS representation [15, p. x].

- If $\mathcal{A}$ is a C^*-algebra and $p, q \in \mathcal{P}(\mathcal{A})$, we say that p and q are *(Murray-von Neumann) equivalent* and write $p \sim q$ if there exists $v \in \mathcal{A}$ such that $p = vv^*$ and $q = v^*v$. We say that p and q are *subequivalent* and write $p \precsim q$ if there exists a projection $q_0 \in \mathcal{P}(\mathcal{A})$ such that $p \sim q_0 \leq q$. Finally, $p, q \in \mathcal{P}(\mathcal{A})$ are *orthogonal* if $pq = 0$.

- A projection p in a C^*-algebra $\mathcal{A}$ is said to be
  - *finite* if it holds for all $q \in \mathcal{P}(\mathcal{A})$ that $q \sim p$ and $q \leq p$ imply $q = p$.
  - *properly infinite* if there exist mutually orthogonal projections e and f such that $e \sim f \sim p$, $e \leq p$ and $f \leq p$.
  - *abelian* if the *-algebra $p\mathcal{A}p$ is commutative.

A unital C^*-algebra $\mathcal{A}$ is itself said to be *finite* (resp. *properly infinite*) if the identity $1_{\mathcal{A}}$ is a finite (resp. properly infinite) projection.

Von Neumann algebras. If $\mathcal{H}$ is a Hilbert space, a *von Neumann algebra* is a *-subalgebra $\mathscr{M}$ of $B(\mathcal{H})$ that is closed in the strong operator topology and contains the identity map $1_{\mathcal{H}}: \mathcal{H} \to \mathcal{H}$. All von Neumann algebras will usually be denoted by $\mathscr{M}$ or $\mathscr{N}$, and the identity element of a von Neumann algebra $\mathscr{M} \subseteq B(\mathcal{H})$ will either be denoted by $1_{\mathscr{M}}$ or $1_{\mathcal{H}}$, depending on the perspective.

- The *commutant* $\mathscr{S}'$ of any subset $\mathscr{S} \subseteq B(\mathcal{H})$ is the *-subalgebra of $B(\mathcal{H})$ of elements commuting with all elements of $\mathscr{S}$. If $\mathscr{S}$ is a self-adjoint subset, $\mathscr{S}'$ is a von Neumann algebra. If $\mathscr{M} \subseteq B(\mathcal{H})$ is a von Neumann algebra and $\mathscr{M} \cap \mathscr{M}' = \mathbb{C}1_{\mathscr{M}}$, then $\mathscr{M}$ is said to be a *factor*. The von Neumann algebra $\mathcal{Z}(\mathscr{M}) = \mathscr{M} \cap \mathscr{M}'$ is called the *center* of $\mathscr{M}$, and projections in $\mathcal{Z}(\mathscr{M})$ are called *central projections*.
- The *predual* $\mathscr{M}_*$ of a von Neumann algebra $\mathscr{M}$ is the Banach space of all *normal*, i.e., ultraweakly continuous, linear functionals on $\mathscr{M}$. It is a well-known result that $\mathscr{M}$ itself is isomorphic as a Banach space to the dual space $(\mathscr{M}_*)^*$.
- Any isomorphism of von Neumann algebras is automatically *normal*, i.e., ultraweakly-to-ultraweakly continuous [15, Proposition 2.49].
- Any C^*-algebra $\mathcal{A}$ has an *enveloping von Neumann algebra* $\mathcal{A}^{**}$, often called the *bidual* since it is identifiable with the double dual space of $\mathcal{A}$, in a way such that the canonical injection of $\mathcal{A}$ into the double dual space $\mathcal{A}^{**}$ becomes an injective *-homomorphism. It has the property that any non-degenerate representation $\pi: \mathcal{A} \to B(\mathcal{H})$ extends to a surjective normal *-homomorphism $\mathcal{A}^{**} \to \pi(\mathcal{A})''$ (if we consider $\mathcal{A}$ as a subalgebra of $\mathcal{A}^{**}$). Any bounded linear map $\varphi: \mathcal{A} \to \mathcal{B}$ of C^*-algebras extends to a normal linear map $\varphi^{**}: \mathcal{A}^{**} \to \mathcal{B}^{**}$ with $\|\varphi^{**}\| = \|\varphi\|$, and if φ is positive (resp. a *-homomorphism), then φ^{**} is also positive (resp. a *-homomorphism) [15, Proposition 3.13].
- If $\mathscr{M}$ is a von Neumann algebra, the *central support* $c(x)$ of an element $x \in \mathscr{M}$ is the smallest central projection p in $\mathscr{M}$ such that the range of x in $\mathcal{H}$ is contained in the range of p, and it satisfies $c(x)x = xc(x) = x$. For more information on central supports, consult [15, Section 2.3].
- A von Neumann algebra $\mathscr{M}$ with identity element 1 is said to be of
  - *type* I if it contains an abelian projection with central support 1, and more specifically of *type* I_n if 1 is the sum of n equivalent abelian projections where $n \in \mathbb{N} \cup \{\infty\}$;
  - *type* II if it contains no non-zero abelian projections but does contain a finite projection with central support 1 – we say that $\mathscr{M}$ is of *type* II_1 if 1 is finite and of *type* II_∞ if 1 is properly infinite;
  - *type* III if it contains no non-zero finite projections.

  In general, a von Neumann algebra $\mathscr{M}$ need not be one of the above types, but it can always be decomposed into a direct sum

  $$\mathscr{M} \cong \mathscr{M}_{\mathrm{I}} \oplus \mathscr{M}_{\mathrm{II}_1} \oplus \mathscr{M}_{\mathrm{II}_\infty} \oplus \mathscr{M}_{\mathrm{III}}$$

  where $\mathscr{M}_i$ is either a type i von Neumann algebra or $\{0\}$ for $i \in \{\mathrm{I}, \mathrm{II}_1, \mathrm{II}_\infty, \mathrm{III}\}$. A factor is always of *exactly* one of the these four types. A type I factor is even isomorphic to $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, so a *finite* factor $\mathscr{M}$ is always either of type I_n for $n < \infty$ or type II_1. Moreover, and this is absolutely essential knowledge, a finite factor $\mathscr{M}$ *always* has a unique trace $\tau: \mathscr{M} \to \mathbb{C}$ that is also faithful and normal. For proofs and more results of the same calibre, we refer to [40].

Chapter 1

C^*-SIMPLICITY OF LOCALLY COMPACT GROUPS

In this first chapter, we will mainly focus on the topic of locally compact groups and results on simplicity and uniqueness of trace of their reduced group C^*-algebras. This gives us an opportunity to construct the reduced group C^*-algebras from scratch, which we will take, as it gives us a great means to completely understand the conditions that simplicity imposes on them.

Throughout this next long overview, G will always denote a *locally compact group*, i.e., a topological group whose topology makes it into a locally compact Hausdorff space.

1.1 Recalling $L^1(G)$

Recall that if X is a locally compact Hausdorff space, then a Radon measure μ on X is a Borel measure satisfying the following properties:

(i) $\mu(K) < \infty$ for all compact $K \subseteq X$.
(ii) For all Borel sets E, $\mu(E) = \inf\{\mu(U) \mid U \text{ open}, E \subseteq U\}$ (outer regularity).
(iii) For all open sets U, $\mu(U) = \sup\{\mu(K) \mid K \text{ compact}, K \subseteq U\}$ (inner regularity).

Definition 1.1.1. A measure μ on a locally compact group G is said to be *left invariant* resp. *right invariant* if it holds that

$$\mu(sE) = \mu(E) \quad \text{resp.} \quad \mu(Es) = \mu(E)$$

for all Borel sets $E \subseteq G$ and $s \in G$. A *left (right) Haar measure* on G is a non-zero left (right) invariant Radon measure on G.

The following theorem is well-known:

Theorem 1.1.2 (Haar measure). *Any locally compact group G has a left Haar measure μ. Moreover, μ is unique in the sense that if ν is another left Haar measure on G, then there exists $c > 0$ such that $\nu = c\mu$.*

It holds that μ is a left Haar measure on G if and only if the measure

$$\tilde{\mu}(E) = \mu(E^{-1}), \quad E \subseteq G \text{ Borel}, \tag{1.1.1}$$

is a right Haar measure. Hence from the above theorem it follows that any locally compact group also possesses a right Haar measure, unique up to a scalar.

If μ is a left Haar measure on G, then for all $s \in G$ we can define a measure μ_s on G by $\mu_s(E) = \mu(Es)$ for all Borel sets E. This is a left Haar measure in itself, so by uniqueness of Haar measure there exists $\Delta(s) > 0$ such that $\mu_s = \Delta(s)\mu$. The map $\Delta\colon G \to \mathbb{R}_{>0}$ arising from this consideration is called the *modular function* of G. Important facts about the modular function include that it is independent of the choice of left Haar measure, that it is a continuous group homomorphism of G into the multiplicative group $\mathbb{R}_{>0}$, and that

$$\mathrm{d}\tilde{\mu}(s) = \Delta(s^{-1})\,\mathrm{d}\mu(s),$$

where $\tilde{\mu}$ is defined as in (1.1.1).

Unless otherwise mentioned we will always let μ denote a fixed left Haar measure on G and let Δ be the modular function of G. For any function $f\colon G \to \mathbb{C}$ and $s \in G$, we define functions $s.f, f.s\colon G \to \mathbb{C}$ by

$$(s.f)(t) = f(s^{-1}t), \quad (f.s)(t) = f(ts), \quad t \in G.$$

We will also make use of the function $\tilde{f}\colon G \to \mathbb{C}$ given by $\tilde{f}(x) = \overline{f(x^{-1})}$.

For $1 \leq p < \infty$, we consider the Banach spaces $L^p(G)$ of Borel-measurable functions on G that are p-integrable with respect to μ (identified modulo null sets), equipped with the usual norm:

$$\|f\|_p = \left(\int |f(t)|^p \,\mathrm{d}\mu(t)\right)^{1/p}.$$

For measurable functions $f, g\colon G \to \mathbb{C}$, we define the *convolution*

$$(f * g)(s) = \int f(t)g(t^{-1}s)\,\mathrm{d}\mu(t)$$

for all $s \in G$ such that the integral is well-defined. If $f \in L^1(G)$ and $g \in L^p(G)$, then the integral is well-defined for almost every $s \in G$, and $f * g \in L^p(G)$ with $\|f * g\|_p \leq \|f\|_1\|g\|_p$. Note that this also applies for $p = \infty$. Hence the convolution defines a product on $L^1(G)$ satisfying $\|f * g\|_1 \leq \|f\|_1\|g\|_1$ for all $f, g \in L^1(G)$. Moreover, if $f \in L^p(G)$ and a function $g \in L^1(G)$ has compact support, then $(f * g)(s)$ is also well-defined for almost all $s \in G$ and $f * g \in L^p(G)$. Finally, it is easy to see that

$$s.(f * g) = (s.f) * g, \quad s \in G,\ f, g \in L^1(G).$$

By defining

$$f^*(s) = \tilde{f}(s)\Delta(s^{-1}), \quad f \in L^1(G),\ s \in G,$$

one can check that $f^* \in L^1(G)$ for all $f \in L^1(G)$ and that $f \mapsto f^*$ is an isometric involution on $L^1(G)$, making it a Banach *-algebra. We call $L^1(G)$ the *group algebra of* G.

Before going on to establish important results about $L^1(G)$, we will first discuss some important properties of the function spaces $L^p(G)$, basically following from being able to work with the Haar measure.

Lemma 1.1.3. *If $U, V \subseteq G$ are neighbourhoods of 1, then UV is a neighbourhood of all $s \in U$ and all $t \in V$.*

Proof. For any $s \in U$ and $t \in V$, $sV \subseteq UV$ resp. $Ut \subseteq UV$ are neighbourhoods of s resp. t. □

Proposition 1.1.4. *Any locally compact group G has an open and closed σ-compact subgroup.*

Proof. Let V be a compact neighbourhood of 1 and define $U = V \cap V^{-1}$. If we now define compact subsets

$$U_n = \underbrace{UU\cdots U}_{n \text{ times}}$$

for all $n \geq 1$ and let $H = \bigcup_{n=1}^{\infty} U_n$, then H is clearly a σ-compact subgroup. Moreover, H is open, since $U_{n+1} = U_nU$ is a neighbourhood of all $s \in U_n$ by the above lemma. As the complement $G \setminus H = \bigcup_{x \notin H} xH$ is also open, H is closed as well. □

Let H be the open, closed and σ-compact subgroup of G arising from Proposition 1.1.4, and choose a *left transversal* T for H in G, i.e. a subset $T \subseteq G$ such that $G = \bigcup_{t \in T} tH$ and $t_1H = t_2H$ implies $t_1 = t_2$ for all $t_1, t_2 \in T$ (this is made possible by the axiom of choice). Then we have the following interesting result:

Proposition 1.1.5. *Let $E \subseteq G$ be a Borel set. Let $I = \{t \in T \mid E \cap tH \neq \emptyset\}$. Then $E \subseteq \bigcup_{t \in I} tH$. If I is countable, then $\mu(E) = \sum_{t \in I} \mu(E \cap tH)$. If I is uncountable, then $\mu(E) = \infty$.*

Proof. If I is countable, then since the sets tH are disjoint for all $t \in I$, then $\mu(E) = \sum_{t \in I} \mu(E \cap tH)$ by σ-additivity of μ. If I is uncountable, note that by outer regularity we can assume that E is open. Therefore, we have $\mu(E \cap tH) > 0$ for all $t \in I$ and thus

$$I = \bigcup_{n \geq 1} \left\{ t \in I \,\middle|\, \mu(E \cap tH) > \frac{1}{n} \right\}.$$

In particular there must exist some $n \geq 1$ such that $A = \{t \in I \mid \mu(E \cap tH) > \frac{1}{n}\}$ is uncountable and hence infinite. Therefore, if $N \geq 1$, then by taking $t_1, \ldots, t_N \in A$ we get

$$\mu(E) \geq \sum_{j=1}^{N} \mu(E \cap t_j H) > \frac{N}{n}.$$

Since N was arbitrary, we have $\mu(E) = \infty$. □

Corollary 1.1.6. *If $f \in L^p(G)$, then f vanishes outside a σ-compact subset of G.*

Proof. Define $F_n = \{s \in G \mid |f(s)|^p > \frac{1}{n}\}$ for all $n \geq 1$ and note that

$$\mu(F_n) = n \int \frac{1_{F_n}}{n} \,\mathrm{d}\mu \leq n \int |f|^p \,\mathrm{d}\mu < \infty.$$

Hence there exists a countable subset $I_n \subseteq T$ such that $F_n \subseteq \bigcup_{t \in I_n} tH$ by Proposition 1.1.5. Defining $I = \bigcup_{n \geq 1} I_n$, then f vanishes outside the σ-compact subset $\bigcup_{t \in I} tH$ of G. □

The above result has an important consequence regarding Fubini's theorem. Normally we have to take serious precautions, as Fubini's theorem only applies to products of σ-finite measure spaces. Therefore we cannot know for sure whether we can reverse the order of integration in the double integral $\int_G \int_G f(s,t) \,\mathrm{d}\mu(s) \,\mathrm{d}\mu(t)$, where $f \colon G \times G \to \mathbb{C}$ is some Borel-measurable function. There is an easy way to work around this, however. If f *vanishes outside some σ-compact subset E of $G \times G$*, then there exist σ-compact subsets E_1 and E_2 of G such that $E \subseteq E_1 \times E_2$. Therefore the measure spaces (E_1, μ) and (E_2, μ) (where μ is naturally restricted) are σ-finite, and as the domains of integration of the above integral can be changed to E_2 and E_1 respectively, Fubini's theorem applies. Most of the time we will indeed want to use Fubini's theorem on functions $(s,t) \mapsto f(s,t)$ with this property. A typical situation could be if

$$f(s,t) = g(s^{-1}t)h(t), \quad s,t \in G,$$

where $g \in L^p(G)$ and $h \in L^q(G)$ for numbers $p, q \geq 1$. In this case, Corollary 1.1.6 yields σ-compact subsets $A, B \subseteq G$ such that g vanishes outside A and h vanishes outside B, in which case f vanishes outside the σ-compact subset $BA^{-1} \times B$ of $G \times G$.

From here onward, we will use Fubini's theorem by implicitly restricting to σ-compact subsets. We define the *support* of a function $g \colon G \to \mathbb{C}$ to be the closed subset

$$\operatorname{supp} g = \overline{\{s \in G \mid g(s) \neq 0\}} \subseteq G.$$

The space of all continuous functions $G \to \mathbb{C}$ with compact support is denoted by $C_c(G)$, and it is a *-subalgebra of $L^1(G)$ with respect to the *-algebra structure defined earlier. Moreover, it is easy to show that if $g \colon G \to \mathbb{C}$ is some function and $A \subseteq G$ is a closed set, then $g(G \setminus A) = \{0\}$ implies $\operatorname{supp} g \subseteq A$.

Lemma 1.1.7. *If $1 \leq p < \infty$ and $f \in L^p(G)$, then $\|s.f - f\|_p \to 0$ and $\|f.s - f\|_p \to 0$ for $s \to 1$.*

Proof. Let U be a fixed compact neighbourhood of 1, let $f \in L^p(G)$ and let $\varepsilon > 0$. Note that $\|s.f\|_p = \|f\|_p$ and

$$\int |f(ts)|^p \,\mathrm{d}\mu(t) = \int |f(ts)|^p \Delta(t) \,\mathrm{d}\tilde{\mu}(t) = \int |f(t)|^p \Delta(t) \Delta(s^{-1}) \,\mathrm{d}\tilde{\mu}(t) = \Delta(s)^{-1} \|f\|_p^p,$$

so that $\|f.s\|_p = \Delta(s)^{-1/p} \|f\|_p$ for all $s \in G$. Since the function $s \mapsto \Delta(s)^{-1/p}$ is continuous and U is compact, there exists a $K > 0$ such that $\Delta(s)^{-1/p} \leq K$ for all $s \in U$, so that $\|f.s\|_p \leq K\|f\|_p$ for all $s \in U$. By [26, Proposition 7.9], $C_c(G)$ is dense in $L^p(G)$, so there exists $g \in C_c(G)$ such that

$$\|f - g\|_p < \frac{\varepsilon}{3(K+1)}.$$

Defining

$$A = (\operatorname{supp} g)U^{-1} \cup U(\operatorname{supp} g),$$

then A is compact, $\operatorname{supp} g \subseteq A$ and for all $s \in U$ we have $\operatorname{supp}(s.g) \subseteq A$ and $\operatorname{supp}(g.s) \subseteq A$. Since g is left and right uniformly continuous [26, Theorem 11.2], there is a neighbourhood V of 1 such that

$$\max\{\|s.g - g\|_\infty, \|g.s - g\|_\infty\} < \frac{\varepsilon}{3\mu(A)^{1/p}}$$

for all $s \in V$. Hence for all $s \in U \cap V$, we have

$$\|s.f - f\|_p \leq \|s.f - s.g\|_p + \|s.g - g\|_p + \|g - f\|_p \leq 2\|f - g\|_p + \mu(A)^{1/p}\|s.g - g\|_\infty < \varepsilon$$

and

$$\|f.s - f\|_p \leq \|(f - g).s\|_p + \|g.s - g\|_p + \|g - f\|_p \leq (K + 1)\|f - g\|_p + \mu(A)^{1/p}\|s.g - g\|_\infty < \varepsilon,$$

from which the wanted convergence follows. □

Proposition 1.1.8. *Let $\mathcal{U}$ be a neighbourhood base of the unit in G, ordered by reverse inclusion. For each $U \in \mathcal{U}$, let e_U be a measurable function on G such that $\operatorname{supp} e_U$ is compact and contained in U, $e_U(s^{-1}) = e_U(s)$ for all $s \in G$, $e_U \geq 0$ and $\int e_U \,\mathrm{d}\mu = 1$. Then $(e_U)_{U\in\mathcal{U}}$ is a net in $L^p(G)$ for all $1 \leq p < \infty$ such that $\|f * e_U - f\|_p \to 0$ and $\|e_U * f - f\|_p \to 0$.*

Proof. Note first that each e_U belongs to $L^p(G)$ for all $1 < p < \infty$ by Hölder's inequality, as

$$\int |e_U|^p \,\mathrm{d}\mu \leq \||e_U|^p\|_{1/p}\|\chi\|_{1/(1-p)} = \left(\int \chi \,\mathrm{d}\mu\right)^{p-1} = \mu(\operatorname{supp} e_U)^{p-1} < \infty,$$

where χ is the characteristic function for $\operatorname{supp} e_U$. For all $f \in L^p(G)$, then $f * e_U(s)$ is well-defined for almost all $s \in G$ and

$$\begin{aligned} f * e_U(s) - f(s) &= \int f(t)e_U(t^{-1}s) \,\mathrm{d}\mu(t) - f(s) \\ &= \int f(t)e_U(s^{-1}t) \,\mathrm{d}\mu(t) - \int f(s)e_U(t) \,\mathrm{d}\mu(t) \\ &= \int (f(st) - f(s))e_U(t) \,\mathrm{d}\mu(t). \end{aligned}$$

Note that $s \mapsto (f(st) - f(s))e_U(t)$ belongs to $L^p(G)$ for all $t \in G$ and

$$\int \|(f.t - f)e_U(t)\|_p \,\mathrm{d}\mu(t) = \int \|f.t - f\|_p e_U(t) \,\mathrm{d}\mu(t) \leq \sup\{\|f.t - f\|_p \,|\, t \in \operatorname{supp} e_U\} \leq K\|f\|_p < \infty$$

for some $K > 0$. Hence it follows from Minkowski's inequality for integrals [26, Theorem 6.19] that

$$\|f * e_U - f\|_p \leq \sup\{\|f.t - f\|_p \,|\, t \in \operatorname{supp} e_U\} \leq \sup\{\|f.t - f\|_p \,|\, t \in U\}.$$

Since $\operatorname{supp} e_U$ is compact, it follows that $f * e_U \in L^p(G)$ for all $U \in \mathcal{U}$. By Lemma 1.1.7, there exists a neighbourhood V of 1 such that $\|f.t - f\|_p < \varepsilon$ for all $t \in V$. Taking $U_0 \in \mathcal{U}$ such that $U_0 \subseteq V$, then for all $U \in \mathcal{U}$ with $U \subseteq U_0$, we have $\|f * e_U - f\|_p \leq \varepsilon$, proving that $\|f * e_U - f\|_p \to 0$. A similar argument applies to show that $\|e_U * f - f\|_p \to 0$. □

The above proposition yields approximate identities aplenty, as long as we choose them according to "the rules". For instance, recall that a neighbourhood U of 1 is *symmetric* if $U = U^{-1}$, and that all neighbourhoods U of 1 contain the symmetric neighbourhood $U \cap U^{-1}$. We can then let $\mathcal{U}$ be the directed system of compact, symmetric neighbourhoods U of 1 and define

$$e_U(s) = \frac{1}{\mu(U)^{1/p}} 1_U.$$

It is also possible to choose continuous e_U's: indeed, letting $\mathcal{U}$ be the system of symmetric neighbourhoods of 1, then for all $U \in \mathcal{U}$ the locally compact version of Urysohn's lemma [26, Lemma 4.32] gives us a function $g_U \colon G \to [0, 1]$ with $g_U = 1$ on a compact neighbourhood $K_U \subseteq U$ of 1 and $g_U = 0$ outside a compact subset of U. Defining $f_U \colon G \to [0, \infty)$ by

$$f_U(s) = g_U(s) + g_U(s^{-1}), \quad s \in G,$$

then each f_U is continuous with compact support and $\|f_U\|_p \geq \mu(K_U)^{1/p} > 0$. By normalizing in $L^p(G)$, we obtain a family with the wanted properties.

Before going any further, we need to get one problem out of the way: how $L^1(G)$ relates to $L^\infty(G)$. If we were only to work with locally compact groups G whose Haar measure makes the group into a σ-finite space, then it is a classic result of measure theory that the space $L^\infty(G)$ of essentially bounded measurable functions on G is in fact isomorphic to the dual space of $L^1(G)$ by means of the isomorphism

$$f \mapsto \left(g \mapsto \int fg \,\mathrm{d}\mu\right), \quad f \in L^\infty(G),\ g \in L^1(G). \tag{1.1.2}$$

However, when the group is not σ-finite, this is not necessarily true. We are therefore going to modify the usual definition of $L^\infty(G)$ in order to obtain this duality no matter which locally compact group we consider; for further discussion, see [25, Section 2.3]. We will say that a subset $E \subseteq G$ is

- *locally Borel* if $E \cap F$ is Borel for all Borel sets F with finite measure, and
- *locally null* if $E \cap F$ has measure zero for all Borel sets F with finite measure.

We then say that a function $f\colon G \to \mathbb{C}$ is *locally measurable* if the pre-image $f^{-1}(B)$ is locally Borel for all Borel sets $B \subseteq \mathbb{C}$. We now let $L^\infty(G)$ denote the space of locally measurable functions $G \to \mathbb{C}$ that are bounded except on a locally null set, in which functions are identified if they differ only on a locally null set. Defining a norm on $L^\infty(G)$ by

$$\|f\|_\infty = \inf\{c \geq 0 \mid \text{there exists } N \subseteq G \text{ locally null such that } |f(x)| \leq c \text{ for all } x \in G \setminus N\}$$

sure enough turns $L^\infty(G)$ into a Banach space, and it is in fact isomorphic to the dual space of $L^1(G)$ by means of the isomorphism described in (1.1.2). For a proof, the reader can consult the author's notes in [16]. Finally, note that if G is σ-finite with respect to its Haar measure, then our new definition of $L^\infty(G)$ coincides with the original one.

1.2 Unitary representations of locally compact groups

We must now take one step further back, in order to reveal the secret behind the representations of $L^1(G)$ on Hilbert spaces, bringing us closer to constructing C^*-algebras related to $L^1(G)$. It requires the following well-known notion:

Definition 1.2.1. Let G be a locally compact group and let $\mathcal{H}$ be a Hilbert space. If $\rho\colon G \to \mathcal{U}(\mathcal{H})$ is a strongly continuous group homomorphism, we say that $(\rho, \mathcal{H})$ is a *unitary representation* of G. By requiring ρ to be strongly continuous, we mean that the map $s \mapsto \rho(s)\xi$ should be continuous for all $\xi \in \mathcal{H}$. If $(\rho', \mathcal{H}')$ is another unitary representation of G and there exists a unitary operator $U\colon \mathcal{H} \to \mathcal{H}'$ such that $U\rho(s) = \rho'(s)U$ for all $s \in G$, we say that ρ and ρ' are *equivalent*.

It is then utterly crucial that we consider these next two important examples of unitary representations:

Example 1.2.2. (i) If $\mathcal{H} = \mathbb{C}$ and $1_G\colon G \to \mathcal{U}(\mathcal{H})$ maps all elements of G to the identity map, then 1_G is a unitary representation of G on $\mathbb{C}$, called the *trivial representation*.

(ii) Consider the Hilbert space $L^2(G)$ with the inner product

$$\langle f, g\rangle = \int f\overline{g}\,\mathrm{d}\mu, \quad f, g \in L^2(G).$$

By left invariance of μ, we have $s.f \in L^2(G)$ and $\langle s.f, s.g\rangle = \langle f, g\rangle$ for all $s \in G$ and $f, g \in L^2(G)$. Hence we can define a group homomorphism $\lambda_G\colon G \to \mathcal{U}(L^2(G))$ by

$$\lambda_G(s)f = s.f$$

for all $s \in G$ and $f \in L^2(G)$. Lemma 1.1.7 easily applies to show that λ_G is also strongly continuous, so λ_G is indeed a unitary representation of G on $L^2(G)$, called the *left-regular representation*.

To prove the next big theorem, we need to be acquainted with the concept of vector-valued integration, and more specifically, the notion of a *Bochner integral.* Bochner integration is in fact a logical extension of Lebesgue integration of complex-valued functions over a measure space, to integration of vector-valued functions. If $(X, \mathcal{A}, \nu)$ is a measure space and $\mathcal{Z}$ is a Banach space, we say that a function $s\colon X \to \mathcal{Z}$ is *simple* if it is of the form

$$s(x) = \sum_{i=1}^{n} 1_{A_i}(x) y_i, \quad x \in X,$$

where $A_1, \dots, A_n \in \mathcal{A}$ are pairwise disjoint with finite measure, and $y_1, \dots, y_n$ are vectors in $\mathcal{Z}$. The integral of the above function s is then defined as

$$\int^{B} s \, \mathrm{d}\nu := \sum_{i=1}^{n} \nu(A_i) y_i.$$

(We write $\int^B$ to distinguish Bochner integrals from other types of integrals.) One can check that the integral is independent of the representation of s, just as in the complex-valued case. Equipping $\mathcal{Z}$ with the Borel σ-algebra, we say that a measurable function $f\colon X \to \mathcal{Z}$ is *Bochner-integrable* with respect to ν if there exists a sequence $(s_n)_{n\geq 1}$ of simple functions such that $\int \|f - s_n\| \, \mathrm{d}\nu \to 0$. It then follows for any Bochner-integrable function $f\colon X \to \mathcal{Z}$ that the integrals of the sequence $(s_n)_{n\geq 1}$ approximating f converge to an element

$$\int^{B} f \, \mathrm{d}\nu \in \mathcal{Z},$$

called the Bochner integral. This element is in fact independent of the choice of sequence [20, Proposition B.6.1], and the Bochner integral of a complex-valued function is just the usual Lebesgue integral. Additionally, if $f\colon X \to \mathcal{Z}$ is Bochner-integrable and T is a bounded linear operator of $\mathcal{Z}$ into another Banach space $\mathcal{Y}$, then $T \circ f$ is Bochner-integrable and

$$T\left(\int^{B} f \, \mathrm{d}\nu\right) = \int^{B} T \circ f \, \mathrm{d}\nu. \tag{1.2.1}$$

Two other results related to Bochner-integrability need to be mentioned:

(i) If X is a locally compact Hausdorff space, then all continuous functions on X with compact support are Bochner-integrable with respect to any Radon measure.
(ii) If G is a locally compact group and μ is a left Haar measure, then for all $f \in C_c(G)$ and $g \in L^1(G)$, the function $G \to L^1(G)$ given by $s \mapsto f(s)s.g$ is Bochner-integrable and

$$f * g = \int^{B} f(s)s.g \, \mathrm{d}\mu.$$

We refer to [20, Corollary B.6.4 and Lemma B.6.5] for proofs of these statements.

Let us forget Bochner integrals for a moment. If G is our favourite locally compact group with fixed left Haar measure μ and $\mathcal{H}$ is a Hilbert space, we will let $L^1(G, B(\mathcal{H}))$ denote the set of maps $f\colon G \to B(\mathcal{H})$ satisfying the following two conditions:

(i) For all $\xi, \eta \in \mathcal{H}$, the function $G \to \mathbb{C}$ given by $s \mapsto \langle f(s)\xi, \eta\rangle$ is measurable.
(ii) The function $s \mapsto \|f(s)\|$ is contained in $L^1(G)$.

It is clear from the get-go that $L^1(G, B(\mathcal{H}))$ is a complex vector space. If $f \in L^1(G, B(\mathcal{H}))$, then the map

$$(\xi, \eta) \mapsto \int \langle f(s)\xi, \eta\rangle \, \mathrm{d}\mu(s)$$

is clearly a sesquilinear form on $\mathcal{H} \times \mathcal{H}$. Hence by the Riesz representation theorem, there exists a unique operator $T \in B(\mathcal{H})$ with $\|T\| \leq \int \|f(s)\| \, \mathrm{d}\mu$ such that

$$\langle T\xi, \eta\rangle = \int \langle f(s)\xi, \eta\rangle \, \mathrm{d}\mu(s), \quad \xi, \eta \in \mathcal{H}.$$

We shall write $T = \int f \, \mathrm{d}\mu$ in this case and call $\int f \, \mathrm{d}\mu$ the *operator integral of* f, and it is then easily verified that the map $f \mapsto \int f \, \mathrm{d}\mu$ is linear. Note moreover that if $f \in L^1(G, B(\mathcal{H}))$ and $S_1, S_2 \in B(\mathcal{H})$, then the map $s \mapsto S_1 f(s) S_2$ also belongs to $L^1(G, B(\mathcal{H}))$ and

$$\int S_1 f(s) S_2 \, \mathrm{d}\mu(s) = S_1 \left(\int f \, \mathrm{d}\mu \right) S_2. \tag{1.2.2}$$

Still, the question remains: do these operators obtained from functions in $L^1(G, B(\mathcal{H}))$ have anything to do with Bochner integrals? In fact they do:

Lemma 1.2.3. *Let $f \in C_c(G)$, $(\rho, \mathcal{H})$ be a unitary representation of G and $\xi \in \mathcal{H}$. Then the function $h \colon G \to \mathcal{H}$ given by $h(s) = f(s)\rho(s)\xi$ is Bochner-integrable and*

$$\int^B h \, \mathrm{d}\mu = \left(\int f(s)\rho(s) \, \mathrm{d}\mu(s) \right) \xi,$$

where the integral on the right hand side is the operator integral of the function $s \mapsto f(s)\rho(s)$.

Proof. Since ρ is strongly continuous and f is continuous with compact support, h is itself continuous with compact support and hence Bochner-integrable. On the other hand, $s \mapsto \langle f(s)\rho(s)\eta_1, \eta_2 \rangle$ is continuous and hence measurable for all $\eta_1, \eta_2 \in \mathcal{H}$, and $\|f(s)\rho(s)\| = |f(s)|$, so $s \mapsto f(s)\rho(s)$ belongs to $L^1(G, B(\mathcal{H}))$. For all $\eta \in \mathcal{H}$, (1.2.1) yields

$$\left\langle \int^B h \, \mathrm{d}\mu, \eta \right\rangle = \int^B \langle h(s), \eta \rangle \, \mathrm{d}\mu(s) = \left\langle \left(\int f(s)\rho(s) \, \mathrm{d}\mu(s) \right) \xi, \eta \right\rangle,$$

using the definition of the operator integral. Thus the proof is complete. □

We are now ready to prove a very beautiful theorem:

Theorem 1.2.4. *Let $(\rho, \mathcal{H})$ be a unitary representation of G. Then the map $\theta_\rho \colon L^1(G) \to B(\mathcal{H})$ given by*

$$\theta_\rho(f) = \int f(s)\rho(s) \, \mathrm{d}\mu(s)$$

is a nondegenerate representation of $L^1(G)$. Moreover, the map $\rho \mapsto \theta_\rho$ is a bijection from the set of unitary representations of G in $\mathcal{H}$ onto the set of nondegenerate representations of $L^1(G)$ in $\mathcal{H}$.

Proof. First and foremost, ρ is strongly continuous, implying that the map $s \mapsto \langle \rho(s)\xi, \eta \rangle$ is continuous and hence measurable. Given $f \in L^1(G)$, we therefore conclude that the map $s \mapsto \langle f(s)\rho(s)\xi, \eta \rangle$ is measurable, and furthermore, the map $s \mapsto \|f(s)\rho(s)\| = |f(s)|$ belongs to $L^1(G)$. Hence the map $s \mapsto f(s)\rho(s)$ belongs to $L^1(G, B(\mathcal{H}))$, so θ_ρ is well-defined. Moreover, θ_ρ is clearly linear, and since

$$\|\theta_\rho(f)\| \le \int |f(s)| \, \mathrm{d}\mu(s) = \|f\|_1$$

for all $f \in L^1(G)$, we see that θ_ρ is continuous.

To see that θ_ρ is a representation, it only remains to check that θ_ρ is adjoint-preserving and multiplicative. For all $f \in L^1(G)$ and $\xi, \eta \in \mathcal{H}$, the calculations

$$\begin{aligned}
\langle \xi, \theta_\rho(f^*)\eta \rangle &= \overline{\langle \theta_\rho(f^*)\eta, \xi \rangle} \\
&= \overline{\int \langle f^*(s)\rho(s)\eta, \xi \rangle \, \mathrm{d}\mu(s)} \\
&= \int f(s^{-1})\Delta(s^{-1})\langle \xi, \rho(s)\eta \rangle \, \mathrm{d}\mu(s) \\
&= \int f(s)\langle \xi, \rho(s^{-1})\eta \rangle \, \mathrm{d}\mu(s) \\
&= \int \langle f(s)\rho(s)\xi, \eta \rangle \, \mathrm{d}\mu(s) \\
&= \langle \theta_\rho(f)\xi, \eta \rangle
\end{aligned}$$

yield that $\theta_\rho(f^*) = \theta_\rho(f)^*$ for all $f \in L^1(G)$. Additionally, for all $f, g \in C_c(G)$ and $\xi, \eta \in \mathcal{H}$, note that (1.2.2) and Fubini's theorem tell us that

$$\begin{aligned}
\langle \theta_\rho(f * g)\xi, \eta\rangle &= \int\int \langle f(t)g(t^{-1}s)\rho(s)\xi, \eta\rangle \,\mathrm{d}\mu(t)\,\mathrm{d}\mu(s) \\
&= \int\int \langle f(t)g(t^{-1}s)\rho(s)\xi, \eta\rangle \,\mathrm{d}\mu(s)\,\mathrm{d}\mu(t) \\
&= \int\int \langle f(t)g(s)\rho(ts)\xi, \eta\rangle \,\mathrm{d}\mu(s)\,\mathrm{d}\mu(t) \\
&= \int f(t) \int \langle g(s)\rho(s)\xi, \rho(t)^*\eta\rangle \,\mathrm{d}\mu(s)\,\mathrm{d}\mu(t) \\
&= \int f(t)\langle \theta_\rho(g)\xi, \rho(t)^*\eta\rangle \,\mathrm{d}\mu(t) \\
&= \int \langle f(t)\rho(t)\theta_\rho(g)\xi, \eta\rangle \,\mathrm{d}\mu(t) \\
&= \langle \theta_\rho(f)\theta_\rho(g)\xi, \eta\rangle.
\end{aligned}$$

Since θ_ρ is continuous and $C_c(G)$ is dense in $L^1(G)$ with respect to $\|\cdot\|_1$, it follows that θ_ρ is multiplicative. Hence θ_ρ is a representation of $L^1(G)$.

We now let $(e_U)_{U\in\mathcal{U}}$ be any approximate identity for $L^1(G)$ obtained by means of Proposition 1.1.8. Fix $s \in G$, and observe that for any $\xi \in \mathcal{H}$ and $\varepsilon > 0$, strong continuity yields a neighbourhood V_0 of s such that $\|\rho(t)\xi - \rho(s)\xi\| < \varepsilon$ for all $t \in V_0$. Taking $U_0 \in \mathcal{U}$ such that $U_0 \subseteq s^{-1}V_0$, then for all $U \in \mathcal{U}$ with $U \subseteq U_0$ it is evident that e_U has support inside U by construction. Hence $s.e_U$ has support inside $sU \subseteq V_0$. Letting $\eta \in \mathcal{H}$ such that $\|\eta\| = 1$ and

$$\langle(\theta_\rho(s.e_U) - \rho(s))\xi, \eta\rangle = \|(\theta_\rho(s.e_U) - \rho(s))\xi\|,$$

we then get from (1.2.2)

$$\begin{aligned}
\|\theta_\rho(s.e_U)\xi - \rho(s)\xi\| &= \left\langle \left(\int (s.e_U)(t)\rho(t)\,\mathrm{d}\mu(t)\right)\xi - \rho(s)\xi, \eta\right\rangle \\
&= \left\langle \left(\int (s.e_U)(t)(\rho(t) - \rho(s))\,\mathrm{d}\mu(t)\right)\xi, \eta\right\rangle \\
&= \int (s.e_U)(t)\langle(\rho(t) - \rho(s))\xi, \eta\rangle \,\mathrm{d}\mu(t) \\
&\le \int_{V_0} (s.e_U)(t)\|\rho(t)\xi - \rho(s)\xi\|\|\eta\| \,\mathrm{d}\mu(t) \\
&\le \int (s.e_U)(t)\varepsilon \,\mathrm{d}\mu(s) = \varepsilon.
\end{aligned}$$

Since $U \subseteq U_0$ and $s \in G$ were arbitrary, it follows that $\theta_\rho(s.e_U) \to \rho(s)$ strongly for all $s \in G$. The case $s = 1$ yields that θ_ρ is nondegenerate. Moreover, if $(\rho', \mathcal{H})$ is a unitary representation of G such that $\theta_\rho = \theta_{\rho'}$, then $\theta_\rho(s.e_U) = \theta_{\rho'}(s.e_U) \to \rho'(s)$ for all $s \in G$, implying that $\rho = \rho'$. Hence the map $\rho \mapsto \theta_\rho$ is injective.

It only remains to show that the map $\rho \mapsto \theta_\rho$ is surjective. Let $\theta\colon L^1(G) \to B(\mathcal{H})$ be a non-degenerate representation, and let $\mathcal{H}_0$ be the linear span in $\mathcal{H}$ of all vectors of the form $\theta(f)\xi$, where $f \in L^1(G)$ and $\xi \in \mathcal{H}$. Then by the assumption of non-degeneracy, $\mathcal{H}_0$ is dense in $\mathcal{H}$. If $s \in G$, then we have

$$\|(s.e_U) * f - s.f\|_1 = \|e_U * f - f\|_1 \to 0$$

and hence

$$\|\theta(s.e_U)\theta(f) - \theta(s.f)\| \to 0$$

by continuity of θ. This observation allows us to define a linear operator $\rho(s)\colon \mathcal{H}_0 \to \mathcal{H}_0$ given by

$$\rho(s)\left(\sum_{i=1}^n \theta(f_i)\xi_i\right) = \sum_{i=1}^n \theta(s.f_i)\xi_i.$$

This operator is well-defined: indeed, if $\sum_{i=1}^n \theta(f_i)\xi_i = 0$ for functions $f_1, \dots, f_n \in L^1(G)$ and vectors $\xi_1, \dots, \xi_n \in \mathcal{H}$, then

$$\sum_{i=1}^n \theta(s.f_i)\xi_i = \lim_{U\in\mathcal{U}} \sum_{i=1}^n \theta(s.e_U)\theta(f_i)\xi_i = \lim_{U\in\mathcal{U}} \theta(s.e_U)\left(\sum_{i=1}^n \theta(f_i)\xi_i\right) = 0.$$

For any $\eta \in \mathcal{H}$ of the form $\sum_{i=1}^n \theta(f_i)\xi_i$ with $f_1, \dots, f_n \in L^1(G)$ and $\xi_1, \dots, \xi_n \in \mathcal{H}$, observe that

$$\|\theta(s.e_U)\eta\| \leq \|s.e_U\|_1\|\eta\| = \|\eta\|.$$

By continuity of the norm, this implies that $\|\rho(s)\| \leq 1$, allowing us to extend $\rho(s)$ to a bounded linear operator on $\mathcal{H}$ of norm less than or equal to 1, which we will also denote by $\rho(s)$. Letting $s, t \in G$ and $f \in L^1(G)$, we now have

$$\rho(st)\theta(f) = \theta((st).f) = \theta(s.(t.f)) = \rho(s)\theta(t.f) = \rho(s)\rho(t)\theta(f)$$

from which we deduce that $\rho(st)\eta = \rho(s)\rho(t)\eta$ for all $\eta \in \mathcal{H}_0$, and thus for all $\eta \in \mathcal{H}$ by continuity. Since $\rho(1)\theta(f) = \theta(f)$, we similarly deduce that $\rho(1) = 1_{\mathcal{H}}$. Finally, if $(s_\beta)_{\beta\in B}$ is a net in G converging to s, then

$$\rho(s_\beta)\theta(f)\xi = \theta(s_\beta.f)\xi \to \theta(s.f)\xi = \rho(s)\theta(f)\xi, \quad f \in L^1(G),\ \xi \in \mathcal{H},$$

by continuity of the map $s \mapsto s.f$ (Lemma 1.1.7). Consequently, $\rho(s_\beta)\xi \to \rho(s)\xi$ for all $\xi \in \mathcal{H}$. Finally, since $\|\rho(s)\| \leq 1$, $\rho(s)^{-1} = \rho(s^{-1})$ and $\|\xi\| = \|\rho(s^{-1})\rho(s)\xi\| \leq \|\rho(s)\xi\|$ for all $\xi \in \mathcal{H}$, we conclude that $\rho(s)$ is a linear surjective isometry $\mathcal{H} \to \mathcal{H}$ and hence a unitary for all $s \in G$. This proves that ρ is a unitary representation of G in $\mathcal{H}$.

We now claim that $\theta_\rho = \theta$, and this is where the Bochner integrals finally enter the picture. Let $f \in C_c(G)$, $g \in L^1(G)$ and $\xi \in \mathcal{H}$. Then the map $L^1(G) \to \mathcal{H}$ given by $h \mapsto \theta(h)\xi$ is a bounded linear operator, so by (1.2.1) and Lemma 1.2.3 we now see that

$$\begin{aligned}
\theta(f)(\theta(g)\xi) &= \theta(f * g)\xi \\
&= \theta\left(\int^B f(s)s.g \,\mathrm{d}\mu(s)\right)\xi \\
&= \int^B \theta(f(s)s.g)\xi \,\mathrm{d}\mu(s) \\
&= \int^B f(s)\theta(s.g)\xi \,\mathrm{d}\mu(s) \\
&= \int^B (f(s)\rho(s))\theta(g)\xi \,\mathrm{d}\mu(s) \\
&= \theta_\rho(f)\theta(g)\xi.
\end{aligned}$$

This implies that $\theta(f) = \theta_\rho(f)$ on $\mathcal{H}_0$ and thus on $\mathcal{H}$ by continuity. Since $C_c(G)$ is dense in $L^1(G)$ with respect to $\|\cdot\|_1$, continuity finally yields $\theta = \theta_\rho$, and hence the map $\rho \mapsto \theta_\rho$ is surjective. This completes the proof. □

When passing from a unitary representation $(\rho, \mathcal{H})$ of G to its associated representation on $L^1(G)$, we will usually denote the two by the same symbol, i.e., the representation $L^1(G) \to B(\mathcal{H})$ obtained by means of Theorem 1.2.4 will also be called ρ.

Example 1.2.5. We now examine the unitary representations of Example 1.2.2 in light of Theorem 1.2.4.

(i) Consider the trivial representation $1_G\colon G \to \mathbb{C}$ given by $1_G(s) = 1$ for all $s \in G$. The associated representation $1_G\colon L^1(G) \to \mathbb{C}$ is a character satisfying $1_G(f) = \int f(s)\,\mathrm{d}\mu(s)$ for all $f \in L^1(G)$. Conversely, if $\varphi\colon L^1(G) \to \mathbb{C}$ is a character, then there exists a unitary representation $\chi\colon G \to \mathbb{C}$, which is in this case a continuous homomorphism of G into the unit circle $\mathbb{T}$, such that

$$\varphi(f) = \int \chi(s)f(s)\,\mathrm{d}\mu(s), \quad f \in L^1(G).$$

Not coincidentally, continuous homomorphisms G into $\mathbb{T}$ are also called *characters*.

(ii) The left-regular representation $\lambda_G\colon G \to B(L^2(G))$ was defined by $\lambda(s)(g) = s.g$ for all $s \in G$ and $g \in L^2(G)$. The associated representation $\lambda_G\colon L^1(G) \to B(L^2(G))$ then satisfies

$$\begin{aligned}\langle \lambda_G(f)g, h\rangle &= \int \langle f(s)s.g, h\rangle \,\mathrm{d}\mu(s) \\ &= \iint \langle f(s)g(s^{-1}t)\overline{h(t)} \,\mathrm{d}\mu(s)\,\mathrm{d}\mu(t) \\ &= \int (f * g)(t)\overline{h(t)}\,\mathrm{d}\mu(t) \\ &= \langle f * g, h\rangle\end{aligned}$$

for all $f \in L^1(G)$ and $g, h \in L^2(G)$ by Fubini's theorem, so that for any $f \in L^1(G)$ the operator $\lambda_G(f)$ acts on $L^2(G)$ by left convolution.

Moreover, the representation $\lambda_G\colon L^1(G) \to B(L^2(G))$ is actually *faithful* as a *-homomorphism of Banach *-algebras. Indeed, if $f \in L^1(G)$ satisfies $f * g = 0$ for all $g \in L^2(G)$, then for any approximate identity $(e_U)_{U\in\mathcal{U}}$ obtained by means of Proposition 1.1.8 we have $\|f * e_U - f\|_1 \to 0$, so that f must be the zero element.

With what we now know, we are in fact able to construct a C^*-algebra from $L^1(G)$ and determine some of its most important properties almost right away.

1.3 The group C^*-algebras of a locally compact group

We now employ the fact that $L^1(G)$ is a Banach *-algebra along with Theorem 1.2.4 from the previous section and general C^*-algebraic results to obtain the central objects of study in this thesis.

Let $\mathcal{A}$ be a fixed Banach *-algebra with a contractive approximate identity and define $\|\cdot\|'\colon \mathcal{A} \to [0,\infty)$ by

$$\|x\|' = \sup\{\|\pi(x)\| \mid \pi \text{ is a representation of } \mathcal{A}\}, \quad x \in \mathcal{A}.$$

Then $x \mapsto \|x\|'$ is a seminorm on $\mathcal{A}$ such that

$$\|xy\|' \le \|x\|'\|y\|', \quad \|x^*\|' = \|x\|', \quad \|x^*x\|' = (\|x\|')^2$$

for all $x, y \in \mathcal{A}$ [22, Proposition 2.7.1]. Letting $\mathfrak{I} \subseteq \mathcal{A}$ be the set of $x \in \mathcal{A}$ such that $\|x\|' = 0$, we see that $\mathfrak{I}$ is a closed, two-sided and self-adjoint ideal of $\mathcal{A}$. Then the map $x + \mathfrak{I} \mapsto \|x\|'$ is a well-defined norm on the *-algebra $\mathcal{A}/\mathfrak{I}$. Letting $\mathcal{B}$ denote the completion of $\mathcal{A}/\mathfrak{I}$ with respect to this norm, then $\mathcal{B}$ is a C^*-algebra, called the *enveloping C^*-algebra of* $\mathcal{A}$. The map $j\colon (\mathcal{A}, \|\cdot\|) \to (\mathcal{B}, \|\cdot\|')$ given by $j(x) = x + \mathfrak{I}$ is then a contractive homomorphism with dense image. If $\mathcal{A}$ is a C^*-algebra, then $\|\cdot\| = \|\cdot\|'$, so that $\mathcal{B} = \mathcal{A}$.

Definition 1.3.1. The enveloping C^*-algebra of $L^1(G)$ is called the *full group C^*-algebra of G* and is denoted by $C^*(G)$.

Note that because $L^1(G)$ admits the faithful representation λ_G as seen in Example 1.2.5, $f \mapsto \|f\|'$ is in fact a norm on $L^1(G)$, and $C^*(G)$ is just the completion of $L^1(G)$ with respect to this norm. Moreover, $C_c(G)$ is dense in $C^*(G)$ with respect to the norm on $C^*(G)$, since $\|f\|' \le \|f\|$ for all $f \in L^1(G)$.

It will be useful now to know how representations of $C^*(G)$ arise. To see this, we have the following result.

Proposition 1.3.2. *Let $\mathcal{A}$, $\mathcal{B}$ and j be as above. For any representation π of $\mathcal{A}$, there exists exactly one representation ρ of $\mathcal{B}$ such that $\pi = \rho \circ j$, in which case $\rho(\mathcal{B})$ is the C^*-algebra generated by $\pi(\mathcal{A})$. The map $\pi \mapsto \rho$ is a bijection of the set of representations of $\mathcal{A}$ onto the set of representations of $\mathcal{B}$. Moreover, π is nondegenerate if and only if ρ is nondegenerate.*

Proof. Let π be a representation of $\mathcal{A}$. Then π vanishes on $\mathfrak{I}$ and hence induces a representation π' of $\mathcal{A}/\mathfrak{I}$ given by $\pi'(x + \mathfrak{I}) = \pi(x)$, so that $\|\pi'(x + \mathfrak{I})\| = \|\pi(x)\| \le \|x\|' = \|x + \mathfrak{I}\|$. Since $\mathcal{A}/\mathfrak{I}$ is dense in $\mathcal{B}$, π' extends to a unique representation ρ on $\mathcal{B}$ such that $\pi = \rho \circ j$. This immediately implies that $\pi(\mathcal{A})$ is dense in $\rho(\mathcal{B})$, and since $\rho(\mathcal{B})$ is a C^*-algebra [74, Theorem 11.1], it is the C^*-algebra generated by $\pi(\mathcal{A})$. The map $\pi \mapsto \rho$ is clearly injective, and the map $\rho \mapsto \rho \circ j$ is its inverse.

Supposing now that π represents $\mathcal{A}$ on the Hilbert space $\mathcal{H}$, then π (resp. ρ) is nondegenerate if and only if $T\xi = 0$ for all $T \in \pi(\mathcal{A})$ (resp. $T \in \rho(\mathcal{B})$) implies $\xi = 0$ for all $\xi \in \mathcal{H}$. By a standard density argument, it is then clear that nondegeneracy of π is equivalent to nondegeneracy of ρ. □

Remark 1.3.3. As an additional fact, note that

$$\|x\|' = \sup\{\|\pi(x)\| \mid \pi \text{ is a cyclic representation of } \mathcal{A}\}, \quad x \in \mathcal{A}.$$

Indeed, fix $x \in \mathcal{A}$ and let $\pi\colon \mathcal{A} \to B(\mathcal{H})$ be some represenation of $\mathcal{A}$. Letting $\mathcal{A}_\pi$ be the closure of $\pi(\mathcal{A})$ in $B(\mathcal{H})$, then $\mathcal{A}_\pi$ is a C^*-algebra. Then there exists a state φ on $\mathcal{A}_\pi$ such that

$$|\varphi(\pi(x^*x))| = \|\pi(x^*x)\|.$$

If $(\pi_\varphi, \mathcal{H}_\varphi, \xi_\varphi)$ is the GNS triple associated to φ, we then have

$$\|\pi(x)\|^2 = \|\pi(x^*x)\| = |\varphi(\pi(x^*x))| = |\langle \pi_\varphi(\pi(x^*x))\xi_\varphi, \xi_\varphi\rangle| \leq \|\pi_\varphi(\pi(x^*x))\| = \|\pi_\varphi \circ \pi(x)\|^2.$$

Since the representation $\pi_\varphi \circ \pi$ has a cyclic vector ξ_φ, the equality clearly follows. ✺

Corollary 1.3.4. *Any unitary representation $(\rho, \mathcal{H})$ of a locally compact group G induces a unique nondegenerate representation $\rho\colon C^*(G) \to B(\mathcal{H})$, such that*

$$\rho(f) = \int f(s)\rho(s)\,\mathrm{d}\mu(s)$$

for all $f \in L^1(G)$. Conversely, any nondegenerate representation of $C^*(G)$ arises from a unitary representation in this way.

Proof. This follows from Theorem 1.2.4 and Proposition 1.3.2. □

Remark 1.3.5. If a unitary representation $(\rho, \mathcal{H})$ of G has a cyclic vector $\xi_0 \in \mathcal{H}$, i.e., $\mathcal{H}$ equals the closed subspace generated by vectors of the form $\rho(s)\xi_0$ for $s \in G$, then ξ_0 is also a cyclic vector for the corresponding representation $\rho\colon C^*(G) \to B(\mathcal{H})$, as we found in the proof of Theorem 1.2.4 that $\theta_\rho(s.e_U) \to \rho(s)$ strongly for certain bounded approximate identities $(e_U)_{U\in\mathcal{U}}$ for $L^1(G)$. ✺

Another imposing question is how the positive linear functionals on $C^*(G)$ arise, and this can also be answered almost right away.

Lemma 1.3.6. *Let $\mathcal{A}$ be a Banach *-algebra with an approximate identity. Then*

$$\|x\|' = \sup_{\varphi\in\mathfrak{X}} \varphi(x^*x)^{1/2}, \quad x \in \mathcal{A},$$

where $\mathfrak{X}$ is the set of continuous positive linear functionals of norm less than or equal to 1.

Proof. If $\pi\colon \mathcal{A} \to B(\mathcal{H})$ is a representation of $\mathcal{A}$, then

$$\|\pi(x)\| = \sup_{\xi\in(\mathcal{H})_1} \|\pi(x)\xi\| = \sup_{\xi\in(\mathcal{H})_1} \langle \pi(x^*x)\xi, \xi\rangle^{1/2}$$

for all $x \in \mathcal{A}$, so we obtain "$\leq$" since $x \mapsto \langle \pi(x)\xi, \xi\rangle$ is a continuous positive linear functional with norm ≤ 1. Conversely, if φ is a continuous positive linear functional with norm less than or equal to 1, then the GNS construction yields a nondegenerate representation $\pi_\varphi\colon \mathcal{A} \to B(\mathcal{H}_\varphi)$ and a vector $\xi \in \mathcal{H}$ such that $\varphi(x) = \langle \pi_\varphi(x)\xi_\varphi, \xi_\varphi\rangle$, in which case $\pi_\varphi(e_\alpha) \to 1_\mathcal{H}$ and $\varphi(e_\alpha) \to \|\xi_\varphi\|^2$ for any approximate identity $(e_\alpha)_{\alpha\in A}$ in $\mathcal{A}$. Therefore $\|\xi_\varphi\|^2 = \|\varphi\| \leq 1$, so

$$\varphi(x^*x)^{1/2} = \|\pi_\varphi(x)\xi_\varphi\| \leq \|\pi_\varphi(x)\| \leq \|x\|'$$

for all $x \in \mathcal{A}$, from which the reverse inequality follows. □

Proposition 1.3.7. *Let $\mathcal{A}$, $\mathcal{B}$ and j be as in Proposition 1.3.2. If φ is a continuous positive linear functional on $\mathcal{A}$, then there is a unique positive linear functional $\tilde{\varphi}$ on $\mathcal{B}$ such that $\varphi = \tilde{\varphi} \circ j$ and $\|\tilde{\varphi}\| = \|\varphi\|$. The map $\pi \mapsto \rho$ is a bijection of the continuous positive linear functionals on $\mathcal{A}$ onto the set of positive linear functionals on $\mathcal{B}$, and when restricted to bounded subsets it is a weak*-to-weak* homeomorphism onto its image.*

Proof. Let φ be a continuous positive linear functional on $\mathcal{A}$. For all $x \in \mathcal{A}$ we then have

$$|\varphi(x)| \leq \|\varphi\|^{1/2}\varphi(x^*x)^{1/2} \leq \|\varphi\| \left(\frac{\varphi(x^*x)}{\|\varphi\|}\right)^{1/2} \leq \|\varphi\|\|x\|'$$

by the previous lemma, so φ extends to a bounded linear functional $\tilde{\varphi}$ on $\mathcal{B}$ with $\|\tilde{\varphi}\| \leq \|\varphi\|$. If $y \in \mathcal{B}$, then there exists a sequence $(x_n)_{n\geq 1}$ in $\mathcal{A}$ such that $j(x_n) \to y$, so that

$$\tilde{\varphi}(y^*y) = \lim_{n\to\infty} \varphi(x_n^* x_n) \geq 0,$$

yielding that $\tilde{\varphi}$ is positive. Finally, for all $x \in \mathcal{A}$, we have

$$|\varphi(x)| = |\tilde{\varphi}(j(x))| \leq \|\tilde{\varphi}\|\|j(x)\| \leq \|\tilde{\varphi}\|\|x\|,$$

so $\|\tilde{\varphi}\| = \|\varphi\|$. By continuity, the extension of φ to $\mathcal{B}$ is unique, and it is clear that the resultant map $\varphi \mapsto \tilde{\varphi}$ is bijective with inverse $\sigma \mapsto \sigma \circ j$.

Finally, let $\mathscr{S}$ be a bounded subset of the continuous positive linear functionals on $\mathcal{A}$ and let $\mathscr{S}_\mathcal{B}$ be its image under the map $\varphi \mapsto \tilde{\varphi}$. Then $\mathscr{S}_\mathcal{B}$ is bounded. If $(\varphi_i)_{i\in I}$ is a net in $\mathscr{S}$ and $\varphi \in \mathscr{S}$, then we have $\tilde{\varphi}_i \circ j = \varphi_i \to \varphi = \tilde{\varphi} \circ j$ in the weak* topology on $\mathscr{S}$ if and only if $\tilde{\varphi}_i \to \tilde{\varphi}$ in the weak* topology on $\mathscr{S}_\mathcal{B}$. Indeed, one implication is clear and the other follows from a standard "$3 \cdot \frac{\varepsilon}{3}$" argument using that $j(\mathcal{A})$ is dense in $\mathcal{B}$ and that $\mathscr{S}_\mathcal{B}$ is bounded. □

Remark 1.3.8. Suppose that $\pi\colon \mathcal{A} \to B(\mathcal{H})$ is a representation of the Banach *-algebra $\mathcal{A}$. Then for all $\xi \in \mathcal{H}$, the linear functional $\varphi_\xi\colon x \mapsto \langle \pi(x)\xi, \xi\rangle$, $x \in \mathcal{A}$, is bounded and positive. Letting $\tilde{\pi}\colon \mathcal{B} \to B(\mathcal{H})$ be the representation of the enveloping C^*-algebra $\mathcal{B}$ of $\mathcal{A}$ such that $\tilde{\pi} \circ j = \pi$ by Proposition 1.3.2, note now that by defining

$$\tilde{\varphi_\xi}(y) = \langle \tilde{\pi}(y)\xi, \xi\rangle, \quad y \in \mathcal{B},$$

we have $\tilde{\varphi_\xi} \circ j = \varphi_\xi$. Conversely, It now follows from Proposition 1.3.7 that there is a bijection between positive linear functionals associated with π and the positive linear functionals associated with $\tilde{\pi}$. ✺

For any unitary representation $(\rho, \mathcal{H})$ of G, then the proof of Theorem 1.2.4 yields that the subset

$$\left\{ \int f(s)\rho(s)\,\mathrm{d}\mu(s) \,\middle|\, f \in L^1(G) \right\} \subseteq B(\mathcal{H})$$

is a *-subalgebra of $B(\mathcal{H})$. Taking the norm closure, we obtain the *group C^*-algebra associated to ρ*, denoted by $C^*_\rho(G)$. In fact, we also have

$$C^*_\rho(G) = \overline{\left\{ \int f(s)\rho(s)\,\mathrm{d}\mu(s) \,\middle|\, f \in C_c(G) \right\}}. \tag{1.3.1}$$

By Corollary 1.3.4, $(\rho, \mathcal{H})$ also yields a non-degenerate representation $\rho\colon C^*(G) \to B(\mathcal{H})$ with image $C^*_\rho(G)$. Denoting the kernel of the surjective *-homomorphism $\rho\colon C^*(G) \to C^*_\rho(G)$ by $C^*\ker\rho$, we then have a *-isomorphism

$$C^*(G)/C^*\ker\rho \to C^*_\rho(G)$$

given by $x + C^*\ker\rho \mapsto \rho(x)$ for $x \in C^*(G)$.

Before going any further, we will define two central structures in the study of operator algebras.

Definition 1.3.9. For any locally compact group G, the C^*-algebra associated to the left-regular representation λ_G of Example 1.2.2 (ii) is called the *reduced group C^*-algebra of G* and is denoted by $C^*_r(G)$, i.e.,

$$C^*_r(G) = C^*_{\lambda_G}(G).$$

The *group von Neumann algebra* $L(G)$ is the von Neumann algebra $C^*_r(G)'' \subseteq B(L^2(G))$.

Having finally defined the reduced group C^*-algebra, we can define the two notions that the thesis is all about:

Definition 1.3.10. Let G be a locally compact group. We say that G is *C^*-simple* if the reduced group C^*-algebra $C_r^*(G)$ is a simple C^*-algebra, and we say that G has the *unique trace property*, or simply that G has *unique trace*, if $C_r^*(G)$ has a unique trace.

It is immediate that the full group C^*-algebra $C^*(G)$ is only simple whenever G is the trivial group, as $C^*(G)$ always has a one-dimensional representation by Corollary 1.3.4 applied to the trivial representation. However, whether $C_r^*(G)$ is simple or not is another matter entirely. To shed some light on this problem, we will from here onward attempt to find necessary and sufficient criteria for a locally compact group G to be C^*-simple.

It is appropriate at this point to discuss functoriality of the reduced group C^*-algebra. It is *not* true that any continuous group homomorphism φ of locally compact groups induces a homomorphism of their reduced group C^*-algebras; if we even assume that φ is injective, it need not be true. However, we do have the following result:

Proposition 1.3.11 (Eymard, 1964). *Let G be a locally compact group and let H be an* open *subgroup of G. Then there exists an isometric embedding $J\colon C_r^*(H) \to C_r^*(G)$ such that $J(\lambda_H(f)) = \lambda_G(\bar{f})$ for all $f \in L^1(H)$, where $\bar{f}$ is the natural extension of $f \in L^1(H)$ to G obtained by defining $\bar{f}(s) = 0$ for all $s \in G \setminus H$.*

Proof. Note first for any locally compact group G that $C_r^*(G)$ is the C^*-algebra generated by $\lambda_G(L^1(G))$. Since λ_G is an faithful representation of $L^1(G)$ on $B(L^2(G))$, it follows that $C_r^*(G)$ is the completion of $L^1(G)$ with respect to the norm $\|f\|_G = \|\lambda_G(f)\|$. Now, if we restrict the left Haar measure μ of G to H, we obtain a left Haar measure on H (which we also denote by μ), in part because $\mu(H) > 0$. In fact, if we were only to assume that H was a Borel measurable subgroup with $\mu(H) > 0$, then H would automatically be open (cf. [36, Corollary 20.17]). Consider the embedding $L^1(H) \to L^1(G)$ given by $f \mapsto \bar{f}$. If we can prove that $\|f\|_H = \|\bar{f}\|_G$ for all $f \in L^1(H)$, we then obtain the desired *-homomorphism by passing to the completions of $L^1(H)$ and $L^1(G)$ with respect to these norms.

Letting T be a right transversal for H in G, we will naturally consider $L^2(Ht)$ as a closed subspace of $L^2(G)$ for all $t \in T$, so that $L^2(G) = \bigoplus_{t\in T} L^2(Ht)$. Let $t \in T$. Given $f \in L^1(H)$, $g \in L^2(Ht)$ and $w \in G$, then if $g(s^{-1}w) \neq 0$ for some $s \in H$ we must have $w \in Ht$, so that $w \notin Ht$ implies $g(s^{-1}w) = 0$ for all $s \in H$. Hence for all $w \notin Ht$ we find that

$$(\lambda_G(\bar{f})g)(w) = (\bar{f} * g)(w) = \int_H f(s)g(s^{-1}w)\,\mathrm{d}\mu(s) = 0.$$

This means that $\lambda_G(\bar{f})g \in L^2(Ht)$, so that $L^2(Ht)$ is invariant under $\lambda_G(\bar{f})$. We then define unitary maps $U_t\colon L^2(Ht) \to L^2(H)$ by

$$(U_t g)(s) = \Delta(t)^{-1/2} g(st^{-1}), \quad s \in G.$$

Now fix $f \in L^1(H)$. For all $g \in L^2(Ht)$ and $w \in H$ we then have

$$U_t(\bar{f} * g)(w) = \Delta(t)^{-1/2} \int_H f(s)g(s^{-1}wt^{-1})\,\mathrm{d}\mu(s) = (f * U_t g)(w).$$

Therefore, if $g \in L^2(G)$, then by letting P_t denote the projection of $L^2(G)$ onto $L^2(Ht)$ for $t \in T$ we see that

$$\|\lambda_G(\bar{f})g\|_2^2 = \sum_{t\in T} \|P_t\lambda_G(\bar{f})P_t g\|_2^2 = \sum_{t\in T} \|U_t\lambda_G(\bar{f})P_t g\|_2^2 = \sum_{t\in T} \|\lambda_H(f)U_t P_t g\|_2^2 \leq \|\lambda_H(f)\|^2\|g\|_2^2.$$

Conversely, for $h \in L^2(H)$, take some $t \in T$ and let $g \in L^2(Ht)$ such that $U_t g = h$. Then

$$\|\lambda_H(f)h\| = \|U_t(\bar{f} * h)\| \leq \|\lambda_G(\bar{f})\|\|h\|_2.$$

Hence $\|f\|_H = \|\bar{f}\|_G$, as wanted. ☐

In fact, as long as we assume that H is an open subgroup of G, we obtain similar results for the full group C^*-algebras and the group von Neumann algebras; we refer to [45] for details.

1.4 Continuous positive definite functions

In order to fully describe the conditions that C^*-simplicity imposes on a locally compact group G, we need to study the notion of a continuous positive definite function on G.

Definition 1.4.1. A continuous function $\varphi\colon G \to \mathbb{C}$ is said to be *positive definite* if the complex matrix

$$[\varphi(s_i^{-1}s_j)]_{i,j=1}^n$$

is positive in $M_n(\mathbb{C})$ for all $s_1, \dots, s_n \in G$. The space of all continuous positive definite functions φ on G with $\varphi(1) = 1$ is denoted by $P_1(G)$.

Rephrasing the definition by virtue of the inner product on $\mathbb{C}^n$, a continuous function $\varphi\colon G \to \mathbb{C}$ is positive definite if and only if

$$\sum_{i,j=1}^n \varphi(s_i^{-1}s_j)\overline{\lambda_i}\lambda_j \geq 0 \tag{1.4.1}$$

for all $n \geq 1$, $s_1, \dots, s_n \in G$ and $\lambda_1, \dots, \lambda_n \in \mathbb{C}$.

Lemma 1.4.2. *A continuous positive definite function $\varphi\colon G \to \mathbb{C}$ is bounded with $\|\varphi\|_\infty = \varphi(1)$ and satisfies $\overline{\varphi(s)} = \varphi(s^{-1})$ for all $s \in G$.*

Proof. Let $s \in G$. If φ is positive definite, then the matrix

$$\begin{pmatrix} \varphi(1) & \varphi(s) \\ \varphi(s^{-1}) & \varphi(1) \end{pmatrix}$$

is positive and in particular self-adjoint, yielding $\overline{\varphi(s)} = \varphi(s^{-1})$. By (1.4.1) we then get

$$|\lambda_1|^2\varphi(1) + 2\mathrm{Re}\,\overline{\lambda_1}\lambda_2\varphi(s) + |\lambda_2|^2\varphi(1) \geq 0$$

for all $\lambda_1, \lambda_2 \in \mathbb{C}$. Therefore $\varphi(1) \geq 0$, and by choosing

$$(\lambda_1, \lambda_2) = \left(\frac{\varphi(s)}{|\varphi(s)|}, -1\right)$$

we immediately see that $|\varphi(s)| \leq \varphi(1)$, completing the proof. □

Example 1.4.3. If $(\rho, \mathcal{H})$ is a unitary representation of G and $\xi \in \mathcal{H}$, then $s \mapsto \langle \pi(s)\xi, \xi\rangle$ is a continuous positive definite function on G. Indeed, for all $s_1, \dots, s_n \in G$ and $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ we have

$$\sum_{i,j=1}^n \langle \pi(s_i^{-1}s_j)\xi, \xi\rangle\overline{\lambda_i}\lambda_j = \sum_{i,j=1}^n \langle \pi(s_j)\xi, \pi(s_i)\xi\rangle\lambda_j\overline{\lambda_i} = \left\|\sum_{j=1}^n \lambda_j\pi(s_j)\xi\right\|^2 \geq 0.$$

We say that a positive definite function of this form is *associated with* ρ.

As we shall see now, all continuous positive definite functions arise from unitary representations:

Proposition 1.4.4. *Let $\varphi\colon G \to \mathbb{C}$ be a continuous, bounded function. Then the following are equivalent:*

(i) *φ is positive definite.*
(ii) *There exists a unitary representation $(\pi, \mathcal{H})$ of G such that*

$$\varphi(s) = \langle \pi(s)\xi, \xi\rangle, \quad s \in G$$

for some vector $\xi \in \mathcal{H}$ with $\|\xi\|^2 = \varphi(1)$.
(iii) *The linear functional $\psi\colon L^1(G) \to \mathbb{C}$ given by*

$$\psi(f) = \int \varphi(s)f(s)\,\mathrm{d}\mu(s), \quad f \in L^1(G). \tag{1.4.2}$$

is bounded with norm $\|\varphi\|_\infty$ and positive in the sense that $\psi(f^* * f) \geq 0$ for all $f \in L^1(G)$.

(iv) *For all $f \in C_c(G)$, we have*

$$\int \varphi(s)(f^* * f)(s)\,\mathrm{d}\mu(s) \geq 0. \tag{1.4.3}$$

Proof. We already have (ii) ⇒ (i) from Example 1.4.3. We will prove that (i) ⇒ (iv) ⇒ (iii) ⇒ (ii).

Suppose that φ is positive definite and let $f \in C_c(G)$ with $K = \operatorname{supp} f$. By defining

$$F(s,t) = \varphi(s^{-1}t)\overline{f(s)}f(t), \quad s,t \in G,$$

F is continuous and has compact support inside $K \times K$. Considering F as a function on the topological group $G \times G$, we then know that F is left and right uniformly continuous [26, Proposition 11.2]. Letting $\varepsilon > 0$, there exists a neighbourhood $V_0 \subseteq G \times G$ of $(1,1)$ such that $|F(s^{-1}s', t^{-1}t') - F(s',t')| < \frac{\varepsilon}{2}$ for all $(s,t) \in V_0$ and $(s',t') \in G \times G$. Taking an open neighbourhood V of 1 in G such that $V \times V \subseteq V_0$, then if $(s_0,t_0) \in G \times G$ we have $|F(s,t) - F(s',t')| < \varepsilon$ for all (s,t) and (s',t') inside $Vs_0 \times Vt_0$. Since K is compact, we can cover K by finitely many right translates of V, i.e., there exist distinct $s_1,\ldots,s_n \in G$ such that $K \subseteq \bigcup_{i=1}^n Vs_i$. Defining $E_1 = K \cap Vs_1$ and

$$E_i = K \cap \left(\bigcup_{j=1}^i Vs_j \setminus \bigcup_{j=1}^{i-1} Vs_j\right)$$

for all $i = 2,\ldots,n$, we obtain a finite partition $\{E_i\}_{i=1}^n$ of K consisting of Borel sets, such that $E_i \subseteq Vs_i$ for all i. We can safely assume that each E_i is non-empty. Fixing $s_i \in E_i$ for all $i = 1,\ldots,n$, then with the aid of Fubini's theorem we see that

$$\begin{aligned}
\int \varphi(t)(f^* * f)(t)\,\mathrm{d}\mu(t) &= \iint \varphi(t)\overline{f(s^{-1})}\Delta(s)^{-1}f(s^{-1}t)\,\mathrm{d}\mu(s)\mu(t)\\
&= \iint \varphi(t)\overline{f(s)}f(st)\,\mathrm{d}\mu(s)\mu(t)\\
&= \iint \varphi(s^{-1}t)\overline{f(s)}f(t)\,\mathrm{d}\mu(s)\mu(t)\\
&= \sum_{i,j=1}^n \int_{E_i}\int_{E_j} F(s,t)\,\mathrm{d}\mu(s)\mu(t)\\
&= \sum_{i,j=1}^n \mu(E_i)\mu(E_j)\varphi(s_i^{-1}s_j)\overline{f(s_i)}f(s_j) + R,
\end{aligned}$$

where

$$R = \sum_{i,j=1}^n \int_{E_i}\int_{E_j} (F(s,t) - F(s_i,s_j))\,\mathrm{d}\mu(s)\mu(t).$$

Since $E_i \times E_j \subseteq Vs_i \times Vs_j$ for all $i,j = 1,\ldots,n$, it follows that

$$|R| \leq \sum_{i,j=1}^n \int_{E_i}\int_{E_j} |F(s,t) - F(s_i,s_j)|\,\mathrm{d}\mu(s)\mu(t) \leq \varepsilon\mu(K)^2.$$

Hence by the assumption that φ is positive definite, we have

$$\int \varphi(s)(f^* * f)(s)\,\mathrm{d}\mu(s) = \sum_{i,j=1}^n \varphi(s_j^{-1}s_i)\overline{\mu(E_i)f(s_i)}\mu(E_j)f(s_j) + R \geq -\varepsilon\mu(K)^2.$$

Because $\varepsilon > 0$ was arbitrary, we obtain (iv).

Assuming now that (iv) holds, then because φ is continuous and bounded by assumption we know that $\varphi \in L^\infty(G)$. Hence (1.4.2) defines a bounded linear functional ψ on $L^1(G)$ with norm $\|\varphi\|_\infty$. Since $\psi(f^* * f) \geq 0$ for all $f \in C_c(G)$ by (1.4.3), continuity yields that this inequality also holds for all $f \in L^1(G)$, making ψ positive with respect to the *-algebra structure of $L^1(G)$. Therefore (iv) implies (iii).

Finally assuming that (iii) holds, let $(\mathcal{H}, \pi, \xi)$ be the GNS triple associated to ψ. Since π is nondegenerate, Theorem 1.2.4 yields a unitary representation $(\rho, \mathcal{H})$ of G such that

$$\pi(f) = \int f(s)\rho(s)\,\mathrm{d}\mu(s), \quad f \in L^1(G).$$

Hence for all $f \in L^1(G)$ we have

$$\int \varphi(s)f(s)\,\mathrm{d}\mu(s) = \psi(f) = \langle \pi(f)\xi, \xi\rangle = \int \langle \rho(s)\xi, \xi\rangle f(s)\,\mathrm{d}\mu(s).$$

For any compact set $K \subseteq G$, we then deduce that $\varphi(s) = \langle \rho(s)\xi, \xi\rangle$ for almost all $s \in K$ and hence for all $s \in K$ by continuity. This proves that (iii) implies (ii), and the proof is complete. □

Remark 1.4.5. Fix a unitary representation $(\rho, \mathcal{H})$ of G and for all $\xi \in \mathcal{H}$, define

$$\rho_\xi(s) = \langle \rho(s)\xi, \xi\rangle, \quad s \in G.$$

We have already seen that each ρ_ξ is continuous and positive definite. Letting $\rho\colon L^1(G) \to B(\mathcal{H})$ be the nondegenerate representation by means of Theorem 1.2.4, note that Proposition 1.4.4 for any $\xi \in \mathcal{H}$ yields that the linear functional

$$\tilde{\rho}_\xi\colon f \mapsto \int f(s)\rho_\xi(s)\,\mathrm{d}\mu(s) = \langle \rho(f)\xi, \xi\rangle, \quad f \in L^1(G),$$

is bounded with norm $\|\xi\|^2$ and positive. It is clear that any continuous positive linear functional on $L^1(G)$ associated with ρ is of this form, so we obtain a norm-preserving bijection $\rho_\xi \mapsto \tilde{\rho}_\xi$ of the set of continuous positive definite functions on G associated with ρ onto the set of continuous positive linear functionals on $L^1(G)$ associated with ρ.

The continuous positive definite functions on G constitute a convex cone in $L^\infty(G)$, and the same goes for the continuous positive linear functionals on $L^1(G)$ in $(L^1(G))^*$. With the above notation, this allows us to extend the above bijection from the conic hull (i.e., sets of positive linear combinations) of

$$\{\rho_\xi \mid (\rho, \mathcal{H}) \text{ unitary representation of } G,\ \xi \in \mathcal{H}\} \subseteq L^\infty(G)$$

to the conic hull of

$$\{\tilde{\rho}_\xi \mid (\rho, \mathcal{H}) \text{ unitary representation of } G,\ \xi \in \mathcal{H}\} \subseteq (L^1(G))^*,$$

simply by letting $\sum_{i=1}^n \lambda_i(\rho_i)_{\xi_i}$ map to $\sum_{i=1}^n \lambda_i(\tilde{\rho}_i)_{\xi_i}$ for all $n \geq 1$, positive numbers $\lambda_1, \dots, \lambda_n$, unitary representations $(\rho_i, \mathcal{H}_i)$ and vectors $\xi_i \in \mathcal{H}_i$ for $1 \leq i \leq n$. This map is evidently well-defined and it is a weak*-to-weak* homeomorphism by construction. ✺

Lemma 1.4.6. *For any continuous positive definite function $\varphi\colon G \to \mathbb{C}$, we have*

$$|\varphi(s) - \varphi(t)|^2 \leq 2\varphi(1)(\varphi(1) - \operatorname{Re}\varphi(s^{-1}t)), \quad s, t \in G.$$

Proof. By Proposition 1.4.4, there exists a unitary representation $(\pi, \mathcal{H})$ of G and $\xi \in \mathcal{H}$ such that $\varphi(s) = \langle \pi(s)\xi, \xi\rangle$ and $\|\xi\|^2 = \varphi(1)$. Hence

$$\begin{aligned}
|\varphi(s) - \varphi(t)|^2 &= |\langle \pi(s)\xi - \pi(t)\xi, \xi\rangle|^2 \\
&\leq \|\xi\|^2\|\pi(s)\xi - \pi(t)\xi\|^2 \\
&= \varphi(1)(\|\pi(s)\xi\|^2 + \|\pi(t)\xi\|^2 - 2\operatorname{Re}\langle \pi(t)\xi, \pi(s)\xi\rangle) \\
&= 2\varphi(1)(\|\xi\|^2 - \operatorname{Re}\langle \pi(s^{-1}t)\xi, \xi\rangle) \\
&= 2\varphi(1)(\varphi(1) - \operatorname{Re}\varphi(s^{-1}t)),
\end{aligned}$$

completing the proof. □

Letting $C(G)$ denote the space of complex-valued continuous functions on G, then for any compact subset $F \subseteq G$ we define

$$p_F(f) = \sup_{s \in F} |f(s)|, \quad f \in C(G).$$

If we let $\mathcal{F}$ be the collection of all compact subsets of G, then $(p_F)_{F \in \mathcal{F}}$ is a separating family of seminorms on $C(G)$, and it generates a locally convex Hausdorff topology called the *topology of compact convergence.*

When we speak of the *weak* topology* on $L^\infty(G)$, we mean the topology that $L^\infty(G)$ inherits by being isometrically isomorphic to the dual space $L^1(G)^*$. Hence a net $(g_i)_{i \in I}$ in $L^\infty(G)$ converges to $g \in L^\infty(G)$ in the weak* topology if and only if

$$\int g_i(s) f(s) \,\mathrm{d}\mu(s) \to \int g(s) f(s) \,\mathrm{d}\mu(s)$$

for all $f \in L^1(G)$. We will write

$$\langle g, f \rangle = \int g(s) f(s) \,\mathrm{d}\mu(s)$$

for all $g \in L^\infty(G)$ and $f \in L^1(G)$.

Now comes the central result of this section, originally put forth by Raĭkov in 1947.

Theorem 1.4.7. *The subspace $P_1(G)$ is closed in $C(G)$ when the latter is equipped with the topology of compact convergence. If we view $P_1(G)$ as a subspace of $L^\infty(G)$ equipped with the weak* topology, then the weak* topology coincides with the topology of compact convergence on $P_1(G)$.*

First recall the following. If X is a space equipped with two topologies τ_1 and τ_2 and all convergent nets in (X, τ_1) also converge in (X, τ_2) with the same limit, then τ_1 is finer than τ_2.

Proof. Assume first that $(\varphi_i)_{i \in I}$ is a net in $P_1(G)$ converging to $\varphi \in C(G)$ in the topology of compact convergence. Then φ_i converges pointwise to φ, so that $\varphi(1) = 1$ and $\|\varphi\|_\infty \leq 1$, since $\varphi_i(1) = 1$ for all $i \in I$. If $f \in C_c(G)$ and $K = \operatorname{supp} f$, then $\operatorname{supp}(f^* * f) \subseteq K^{-1}K$, and so

$$\left| \int (f^* * f)(s)(\varphi_i(s) - \varphi(s)) \,\mathrm{d}\mu(s) \right| \leq \sup_{s \in K^{-1}K} |\varphi_i(s) - \varphi(s)| \|f^* * f\|_1 \to 0.$$

Thus $\varphi \in P_1(G)$ by Proposition 1.4.4, so that $P_1(G)$ is closed in the topology of compact convergence. Note also that uniform boundedness of the net $(\varphi_i)_{i \in I}$ ensures that $\int \varphi_i(s) f(s) \,\mathrm{d}\mu(s) \to \int \varphi(s) f(s) \mu(s)$ for all $f \in L^1(G)$ by Lebesgue's dominated convergence theorem, so that $\varphi_i \to \varphi$ in the weak* topology on $L^\infty(G)$. Hence the weak* topology is coarser than the topology of compact convergence on $P_1(G)$.

To show that the topologies are in fact equal, let $\varphi_0 \in P_1(G)$, $F \subseteq G$ be compact and $\varepsilon > 0$. We will show that there exists a weak*-open neighbourhood N of φ_0 in $P_1(G)$ such that $\varphi \in N$ implies $|\varphi(s) - \varphi_0(s)| < 7\varepsilon$ for all $s \in F$. This will imply that the weak* topology is finer than the topology of compact convergence.

By continuity of φ_0 and G being locally compact, there exists a compact neighbourhood U of 1 such that

$$|1 - \varphi_0(s)| = |\varphi_0(1) - \varphi_0(s)| < \varepsilon^2$$

for all $s \in U$. Set $\lambda = \mu(U) > 0$ and define $f = \lambda^{-1} 1_U \in L^1(G)$ and a weak*-open neighbourhood N_1 in $P_1(G)$ of φ_0 by

$$N_1 = \{\varphi \in P_1(G) \,|\, |\langle \varphi, 1_U \rangle - \langle \varphi_0, 1_U \rangle| < \lambda \varepsilon^2\} = \{\varphi \in P_1(G) \,|\, |\langle \varphi, f \rangle - \langle \varphi_0, f \rangle| < \varepsilon^2\}.$$

Note now that for all $\varphi \in N_1$ we then have

$$\left| \int_U (1 - \varphi(s)) \,\mathrm{d}\mu(s) \right| \leq \left| \int_U (1 - \varphi_0(s)) \,\mathrm{d}\mu(s) \right| + \left| \int_U (\varphi_0(s) - \varphi(s)) \,\mathrm{d}\mu(s) \right| \leq 2\lambda\varepsilon^2. \tag{1.4.4}$$

If we also let $s \in G$, we obtain

$$\begin{aligned}|(f * \varphi)(s) - \varphi(s)| &= \left|\lambda^{-1} \int_U \varphi(t^{-1}s)\,\mathrm{d}\mu(t) - \varphi(s)\right| \\ &= \left|\lambda^{-1} \int_U (\varphi(t^{-1}s) - \varphi(s))\,\mathrm{d}\mu(t)\right| \\ &\leq \lambda^{-1} \int_U |\varphi(t^{-1}s) - \varphi(s)|\,\mathrm{d}\mu(t).\end{aligned}$$

Because φ is positive definite and $\varphi(1) = 1$, it follows from Lemmas 1.4.2 and 1.4.6 that

$$|\varphi(t^{-1}s) - \varphi(s)| = |\varphi(s^{-1}t) - \varphi(s^{-1})| \leq \left(2 - 2\mathrm{Re}\,\varphi(t^{-1})\right)^{1/2} = (2 - 2\mathrm{Re}\,\varphi(t))^{1/2}.$$

This last expression almost begs us to use Hölder's inequality when integrating it, and so we have

$$\begin{aligned}|(f * \varphi)(s) - \varphi(s)| &\leq \lambda^{-1} \int_U (2 - 2\mathrm{Re}\,\varphi(t))^{1/2}\,\mathrm{d}\mu(t) \\ &\leq \sqrt{2}\lambda^{-1} \left(\int_U (1 - \mathrm{Re}\,\varphi(t))\,\mathrm{d}\mu(t)\right)^{1/2} \left(\int 1_U\,\mathrm{d}\mu(t)\right)^{1/2} \\ &= \sqrt{2}\lambda^{-1/2} \left(\mathrm{Re} \int_U (1 - \varphi(t))\,\mathrm{d}\mu(t)\right)^{1/2} \\ &\leq \sqrt{2}\lambda^{-1/2} \left|\int_U (1 - \varphi(s))\,\mathrm{d}\mu(s)\right|^{1/2} \\ &< 2\varepsilon,\end{aligned}$$

using (1.4.4) at the last inequality.

By continuity of the map $G \mapsto L^1(G)$ given by $s \mapsto s^{-1}.f$ (following from Lemma 1.1.7), the set $F' = \{s^{-1}.f \mid s \in F\} \subseteq L^1(G)$ is compact. Hence there exist $g_1, \ldots, g_n \in F'$ such that

$$F' \subseteq \bigcup_{i=1}^{n} \{g \in L^1(G) \mid \|g - g_i\|_1 < \varepsilon\}.$$

We now define another weak*-open neighbourhood N_2 of φ_0 in $P_1(G)$ by

$$N_2 = \{\varphi \in P_1(G) \mid |\langle \varphi, \overline{g_i^*}\rangle - \langle \varphi_0, \overline{g_i^*}\rangle| < \varepsilon \text{ for all } i = 1, \ldots, n\}.$$

If we define $\hat{g}(s) = g(s^{-1})$ for all $g \in L^\infty(G)$, then note that

$$\langle g, h^*\rangle = \int g(t)\overline{h(t^{-1})}\Delta(t)^{-1}\,\mathrm{d}\mu(t) = \int g(t)\overline{h(t^{-1})}\,\mathrm{d}\tilde{\mu}(t) = \int g(t^{-1})\overline{h(t)}\,\mathrm{d}\tilde{\mu}(t) = \langle \hat{g}, \overline{h}\rangle$$

and that

$$(h * g)(s) = \int h(t)g(t^{-1}s)\,\mathrm{d}\mu(t) = \int h(st)g(t^{-1})\,\mathrm{d}\mu(t) = \langle \hat{g}, s^{-1}.h\rangle$$

for all $g \in L^\infty(G)$ and $h \in L^1(G)$. Hence if $\varphi \in N_2$ and $s \in F$, there exists $1 \leq i \leq n$ such that $\|s^{-1}.f - g_i\|_1 < \varepsilon$ and thus

$$\begin{aligned}|(f * \varphi)(s) - (f * \varphi_0)(s)| &= |\langle \hat{\varphi}, s^{-1}.f\rangle - \langle \hat{\varphi}_0, s^{-1}.f\rangle| \\ &\leq |\langle \hat{\varphi}, s^{-1}.f - g_i\rangle| + |\langle \hat{\varphi}, g_i\rangle - \langle \hat{\varphi}_0, g_i\rangle| + |\langle \hat{\varphi}_0, g_i - s^{-1}.f\rangle| \\ &\leq 2\|s^{-1}.f - g_i\|_1 + |\langle \varphi, \overline{g_i^*}\rangle - \langle \varphi_0, \overline{g_i^*}\rangle| \\ &< 3\varepsilon.\end{aligned}$$

Now $N = N_1 \cap N_2$ is a weak*-open neighbourhood of φ_0 satisfying the wanted properties. Indeed, for all $\varphi \in N$ and $s \in F$ we have

$$\begin{aligned}|\varphi(s) - \varphi_0(s)| &\leq |\varphi(s) - (f * \varphi)(s)| + |(f * \varphi)(s) - (f * \varphi_0)(s)| + |(f * \varphi_0)(s) - \varphi_0(s)| \\ &< 2\varepsilon + 3\varepsilon + 2\varepsilon = 7\varepsilon,\end{aligned}$$

completing the proof. □

1.5 Weak containment of unitary representations

As the previous section helped us establish a connection between unitary representations of a locally compact groups and continuous positive definite functions, we now formulate this connection in a more C^*-algebraic manner, in the description of which the next notion becomes an important tool.

Definition 1.5.1. Let $\rho\colon G \to \mathcal{U}(\mathcal{H})$ and $\sigma\colon G \to \mathcal{U}(\mathcal{K})$ be unitary representations of G in Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$. We say that ρ is *weakly contained* in σ and write $\rho \prec \sigma$ if for any $\varepsilon > 0$, compact subset $F \subseteq G$ and $\xi \in \mathcal{H}$, there exist vectors $\eta_1, \dots, \eta_n \in \mathcal{K}$ such that

$$\left|\langle \rho(s)\xi, \xi\rangle - \sum_{i=1}^{n} \langle \sigma(s)\eta_i, \eta_i\rangle\right| < \varepsilon$$

for all $s \in F$. If ρ and σ are weakly contained in one another, we say that ρ and σ are *weakly equivalent* and write $\rho \sim \sigma$.

The first thing we might note is that the above definition might be simplified in order to see ties with Theorem 1.4.7:

Lemma 1.5.2. *Let G be a locally compact group and let $(\rho, \mathcal{H})$ and $(\sigma, \mathcal{K})$ be unitary representations of G. Then the following are equivalent:*

(i) $\rho \prec \sigma$.

(ii) *For any unit vector $\xi \in \mathcal{H}$, compact subset $F \subseteq G$ and $\varepsilon > 0$, there exist vectors $\eta_1, \dots, \eta_n \in \mathcal{K}$ such that $s \mapsto \sum_{i=1}^{n} \langle \sigma(s)\eta_i, \eta_i\rangle$ belongs to $P_1(G)$ and*

$$\left|\langle \rho(s)\xi, \xi\rangle - \sum_{i=1}^{n} \langle \sigma(s)\eta_i, \eta_i\rangle\right| < \varepsilon$$

for all $s \in F$.

Proof. First let $\xi \in \mathcal{H}$ be a fixed unit vector and $F \subseteq G$ be a fixed compact subset. If $\rho \prec \sigma$, then for all $k \geq 1$ there exist an $m_k \geq 1$ and vectors $\tilde{\eta}_1^n, \dots, \tilde{\eta}_{m_k}^k \in \mathcal{K}$ such that

$$\left|\langle \rho(s)\xi, \xi\rangle - \sum_{i=1}^{m_k} \langle \sigma(s)\tilde{\eta}_i^k, \tilde{\eta}_i^k\rangle\right| < \frac{1}{k}$$

for all $s \in F \cup \{1\}$. Defining $S_k\colon G \to \mathbb{C}$ by $S_k(s) = \sum_{i=1}^{m_k} \langle \sigma(s)\tilde{\eta}_i^k, \tilde{\eta}_i^k\rangle$ for $s \in G$, it is then clear that $S_k(1) \to 1$ and that

$$1 - \frac{1}{k} < S_k(1) < 1 + \frac{1}{k} \leq 2$$

for all $k \geq 1$. Letting $\varepsilon > 0$, we can take an $N \geq 1$ such that

$$\frac{1}{N} < \frac{\varepsilon}{2} \quad \text{and} \quad |1 - S_N(1)^{-1}| < \frac{\varepsilon}{4}.$$

Let $n = m_N$ and define $\eta_i = S_N(1)^{-1/2}\tilde{\eta}_i^N$ for $i = 1, \dots, n$. Then $s \mapsto \sum_{i=1}^{n} \langle \sigma(s)\eta_i, \eta_i\rangle = S_N(1)^{-1}S_N(s)$ belongs to $P_1(G)$ and for all $s \in F$, we have

$$\begin{aligned}\left|\langle \rho(s)\xi, \xi\rangle - \sum_{i=1}^{n} \langle \sigma(s)\eta_i, \eta_i\rangle\right| &< \frac{\varepsilon}{2} + \left|S_N(s) - \sum_{i=1}^{n} \langle \sigma(s)\eta_i, \eta_i\rangle\right| \\ &= \frac{\varepsilon}{2} + |1 - S_N(1)^{-1}||S_N(s)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{4}|S_N(1)| \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \leq \varepsilon,\end{aligned}$$

applying Lemma 1.4.2 along the way. Hence (i) implies (ii).

Assuming that (ii) holds, then if $\xi \in \mathcal{H}$ is a non-zero vector, F is a compact subset and $\varepsilon > 0$, there exist $\tilde{\eta}_1, \dots, \tilde{\eta}_n \in \mathcal{K}$ such that

$$\left| \frac{1}{\|\xi\|^2} \langle \rho(s)\xi, \xi\rangle - \sum_{i=1}^{n} \langle \sigma(s)\tilde{\eta}_i, \tilde{\eta}_i\rangle \right| < \frac{\varepsilon}{\|\xi\|^2}$$

for all $s \in F$. Defining $\eta_i = \|\xi\|\tilde{\eta}_i$ for all $1 \le i \le n$ yields that $\rho \prec \sigma$. □

The curious reader may wonder why we don't just use condition (ii) of the above lemma as our definition of weak containment, as it is that condition we will check for later in this section, but as is, our original definition is often more convenient to work with.

Let $\mathcal{A}$ be a C^*-algebra and let $\pi\colon \mathcal{A} \to B(\mathcal{H})$ be a representation of $\mathcal{A}$. If we define $\omega_\xi\colon B(\mathcal{H}) \to \mathbb{C}$ by

$$\omega_\xi(T) = \langle T\xi, \xi\rangle, \quad T \in B(\mathcal{H}),$$

then any positive linear functional on $\mathcal{A}$ of the form $\omega_\xi \circ \pi$ is said to be *associated with* π. In order to relate weak containment of unitary representations to the reduced group C^*-algebras associated with them, we will need to prove the following proposition:

Proposition 1.5.3. *Let $\mathcal{A}$ be a C^*-algebra, let $\pi\colon \mathcal{A} \to B(\mathcal{H})$ and $\rho\colon \mathcal{A} \to B(\mathcal{K})$ be two representations of $\mathcal{A}$. Then the following are equivalent:*

(i) $\ker\rho \subseteq \ker\pi$.
(ii) *Any positive linear functional associated with π on $\mathcal{A}$ is a weak*-limit of finite sums of positive linear functionals associated with ρ.*
(iii) *Any state associated with π on $\mathcal{A}$ is a weak*-limit of states which are finite sums of positive linear functionals associated with ρ.*

If $\xi_0 \in \mathcal{H}$ is a cyclic vector for π, then all of the above conditions are equivalent to:

(iv) *The positive linear functional $x \mapsto \langle \pi(x)\xi_0, \xi_0\rangle$ on $\mathcal{A}$ is a weak*-limit of finite sums of positive linear functionals associated with ρ.*

Note that if $(e_i)_{i\in I}$ is a contractive approximate identity of $\mathcal{A}$, then $\varphi(e_i) \to \|\varphi\|$ for all positive linear functionals φ on $\mathcal{A}$ (cf. [15, Proposition 0.3]), in which case $\|\xi\|^2 \ge \|\varphi\|$. If π is non-degenerate, then we in fact have $\pi(e_i) \to 1_{\mathcal{H}}$, and therefore $\|\xi\|^2 = \|\omega_\xi \circ \pi\|$.

Our first goal is to determine what positive linear functionals on the unitization $\tilde{\mathcal{A}}$ of a non-unital C^*-algebra $\mathcal{A}$ look like.

Proposition 1.5.4. *Let $\mathcal{A}$ be a non-unital C^*-algebra.*

(i) *For any positive linear functional φ on $\mathcal{A}$ and any fixed $\mu \ge \|\varphi\|$, define a functional $\tilde{\varphi}\colon \tilde{\mathcal{A}} \to \mathbb{C}$ by*
$$\tilde{\varphi}(x + \lambda 1) = \varphi(x) + \lambda\mu, \quad x \in \mathcal{A},\ \lambda \in \mathbb{C}.$$
Then $\tilde{\varphi}$ is a positive linear functional on $\tilde{\mathcal{A}}$ satisfying $\tilde{\varphi}|_{\mathcal{A}} = \varphi$.
(ii) *If σ is a positive linear functional on $\tilde{\mathcal{A}}$, then there exists a positive linear functional φ on $\mathcal{A}$ and $\mu \ge \|\varphi\|$ such that*
$$\sigma(x + \lambda 1) = \varphi(x) + \lambda\mu, \quad x \in \mathcal{A},\ \lambda \in \mathbb{C}.$$

Proof. (i) It is clear that $\tilde{\varphi}$ is a well-defined linear functional on $\tilde{\mathcal{A}}$ extending φ. Let $x \in \mathcal{A}$ be self-adjoint and $\lambda \ge 0$. By defining $y = (x + \lambda 1)^2$, we then have

$$\begin{aligned}\tilde{\varphi}(y) &= \tilde{\varphi}(x^2 + \lambda x + \lambda^2 1)\\ &= \varphi(x^2) + \lambda\varphi(x) + \lambda^2\mu\\ &= (\varphi(x^2)^{1/2} - \lambda\mu^{1/2})^2 + 2\lambda\mu^{1/2}\varphi(x^2)^{1/2} + \lambda\varphi(x)\\ &\ge 0,\end{aligned}$$

since $-\lambda\varphi(x) \le |\lambda\varphi(x)| \le \lambda\|\varphi\|^{1/2}\varphi(x^2)^{1/2} \le \lambda\mu^{1/2}\varphi(x^2)^{1/2}$. If $x \in \mathcal{A}$ is self-adjoint and $\lambda < 0$, then

$$\tilde{\varphi}((x-\lambda 1)^2) = (\varphi(x^2)^{1/2} + \lambda\mu^{1/2})^2 - 2\lambda\mu^{1/2}\varphi(x^2)^{1/2} - \lambda\varphi(x) \ge 0,$$

since $\lambda\varphi(x) \le |\lambda\varphi(x)| \le -\lambda\mu^{1/2}\varphi(x^2)^{1/2}$.

Now, if $y \in \tilde{\mathcal{A}}$ is positive, then we can write $\sqrt{y} = x + \lambda 1$ for some $x \in \mathcal{A}$ and $\lambda \in \mathbb{C}$. Since $\sqrt{y}$ is self-adjoint, it follows that x is self-adjoint and $\lambda \in \mathbb{R}$. By what we have already shown, it follows that

$$\tilde{\varphi}(y) = \tilde{\varphi}(\sqrt{y}^2) \ge 0,$$

so that $\tilde{\varphi}$ is positive.

(ii) Let σ be a positive linear functional on $\tilde{\mathcal{A}}$. Denote the restriction of σ to $\mathcal{A}$ by φ and let $\mu = \|\sigma\| \ge \|\varphi\|$. Since $\|\sigma\| = \sigma(1)$ we have $\sigma(x + \lambda 1) = \varphi(x) + \lambda\mu$ for all $x \in \mathcal{A}$ and $\lambda \in \mathbb{C}$. □

Corollary 1.5.5. *Let φ be a positive linear functional on a non-unital C^*-algebra. Then φ has a unique extension to a positive linear functional on $\tilde{\mathcal{A}}$ with the same norm, called the* canonical extension *of φ to $\tilde{\mathcal{A}}$.*

Proof. Define $\tilde{\varphi}(x+\lambda 1) = \varphi(x)+\lambda\|\varphi\|$ for $x \in \mathcal{A}$ and $\lambda \in \mathbb{C}$. By Proposition 1.5.4, $\tilde{\varphi}$ is a positive linear functional extending φ, and $\|\tilde{\varphi}\| = \tilde{\varphi}(1) = \|\varphi\|$. If σ is a positive linear functional on $\tilde{\mathcal{A}}$ extending φ and $\|\sigma\| = \|\varphi\|$, then $\|\varphi\| = \|\sigma\| = \sigma(1)$, in which case

$$\sigma(x + \lambda 1) = \varphi(x) + \lambda\|\varphi\| = \tilde{\varphi}(x + \lambda 1), \quad x \in \mathcal{A},\ \lambda \in \mathbb{C}.$$

Hence $\tilde{\varphi}$ is unique with this property. □

If $\pi\colon \mathcal{A} \to B(\mathcal{H})$ is a representation of a non-unital C^*-algebra $\mathcal{A}$, then we define the *canonical extension of* π to the unitization $\tilde{\mathcal{A}}$ by

$$\tilde{\pi}(x + \lambda 1) = \pi(x) + \lambda 1_{\mathcal{H}}, \quad x \in \mathcal{A},\ \lambda \in \mathbb{C}.$$

Clearly $\tilde{\pi}\colon \tilde{\mathcal{A}} \to B(\mathcal{H})$ is also a representation, and if π is faithful then $\tilde{\pi}$ is also faithful.

Corollary 1.5.6. *Let $\mathcal{A}$ be a non-unital C^*-algebra and let $\pi\colon \mathcal{A} \to B(\mathcal{H})$ be a nondegenerate representation. If $\xi \in \mathcal{H}$ and φ is the positive linear functional on $\mathcal{A}$ given by $\varphi(x) = \langle \pi(x)\xi, \xi\rangle$, then the canonical extension $\tilde{\varphi}$ of φ to $\tilde{\mathcal{A}}$ is given by*

$$\tilde{\varphi}(x) = \langle \tilde{\pi}(x)\xi, \xi\rangle, \quad x \in \tilde{\mathcal{A}}.$$

Proof. It is clear that $\tilde{\varphi}$ is a positive linear functional on $\tilde{\mathcal{A}}$ extending φ. Since π is nondegenerate, we have

$$\|\varphi\| = \|\xi\|^2 = \tilde{\varphi}(1) = \|\tilde{\varphi}\|,$$

so that $\tilde{\varphi}$ is indeed the canonical extension of φ. □

Before we state the next lemma, let us introduce some notation. For a subset $\mathcal{H}_0$ of a Hilbert space $\mathcal{H}$ and a *-subalgebra $\mathscr{M} \subseteq B(\mathcal{H})$, $[\mathscr{M}\mathcal{H}_0]$ denotes the closed subspace generated by all vectors of the form $x\xi$ for $x \in \mathscr{M}$ and $\xi \in \mathcal{H}_0$. A representation $\pi\colon \mathcal{A} \to B(\mathcal{H})$ is then non-degenerate if $[\pi(\mathcal{A})\mathcal{H}] = \mathcal{H}$.

Proposition 1.5.7. *Let $\mathcal{A}$ be a C^*-algebra and let $\pi\colon \mathcal{A} \to B(\mathcal{H})$ be a representation. Then $[\pi(\mathcal{A})\mathcal{H}]$ is invariant under π and $[\pi(\mathcal{A})\mathcal{H}]^\perp$ consists of all $\xi \in \mathcal{H}$ such that $\pi(a)\xi = 0$ for all $a \in \mathcal{A}$.*

Proof. Easy. □

Lemma 1.5.8. *Let $\mathcal{A}$ be a unital C^*-algebra and let $\mathfrak{X} \subseteq S(\mathcal{A})$. Suppose that it holds for all $a \in \mathcal{A}_{\mathrm{sa}}$ that $\varphi(a) \ge 0$ for all $\varphi \in \mathfrak{X}$ implies $a \in \mathcal{A}_+$. Then the convex hull of $\mathfrak{X}$ is weak*-dense in $S(\mathcal{A})$.*

Proof. See [15, Lemma A.4] or [22, Lemma 3.4.1]. □

With the necessary preliminary results established, we can now safely embark upon a proof of Proposition 1.5.3.

Proof of Proposition 1.5.3. If (ii) holds and $a \in \ker \rho$, then all positive linear functionals of the form $x \mapsto \langle \rho(x)\eta, \eta \rangle$ for $\eta \in \mathcal{K}$ vanish at a^*a. Hence for all $\xi \in \mathcal{H}$ we have $\|\pi(a)\xi\|^2 = \langle \pi(a^*a)\xi, \xi\rangle = 0$, so $\pi(a) = 0$ and $a \in \ker \pi$, hence proving that (ii) $\Rightarrow$ (i). The implication (iii) $\Rightarrow$ (ii) is clear.

We therefore assume that (i) holds and want to prove (iii). Let $\mathcal{K}_0 = [\rho(\mathcal{A})\mathcal{K}]$, let $P \in B(\mathcal{K})$ be the projection onto $\mathcal{K}_0$ and define $\Omega \colon B(\mathcal{K}) \to B(\mathcal{K}_0)$ by $\Omega(T) = PT|_{P(\mathcal{K})}$. Then Ω is a *-homomorphism and $\rho_1 = \Omega \circ \rho$ is a nondegenerate representation. Let $\mathfrak{I} = \ker \rho_1$ and $\mathcal{B} = \mathcal{A}/\ker \rho_1$. Then ρ_1 induces a nondegenerate faithful representation $\rho_2 \colon \mathcal{B} \to B(\mathcal{K}_0)$ given by $\rho_2(x + \mathfrak{I}) = \rho_1(x)$.

We now split into two cases:

- If $\mathcal{B}$ is unital, then let $\mathfrak{X}$ be the set of states of the form $\omega_\xi \circ \rho_2$, where $\xi \in \mathcal{K}_0$ is a unit vector. If $x \in \mathcal{B}$ is self-adjoint and $\omega_\xi(\rho_2(x)) \geq 0$ for all unit vectors $\xi \in \mathcal{K}_0$, then clearly $\rho_2(x) \geq 0$ is positive and therefore $x \geq 0$ by faithfulness of ρ_2. Therefore the weak*-closed convex hull of $\mathfrak{X}$ is $S(\mathcal{B})$ by Lemma 1.5.8.
- If $\mathcal{B}$ is non-unital, we instead consider the unitization $\tilde{\mathcal{B}}$ and let $\mathfrak{X}$ be the set of states on $\tilde{\mathcal{B}}$ of the form $\omega_\xi \circ \tilde{\rho}_2$, where $\xi \in \mathcal{K}_0$ is a unit vector. Since $\tilde{\rho}_2$ is faithful, the same argument as above yields that the weak*-closed convex hull of $\mathfrak{X}$ is $S(\tilde{\mathcal{B}})$. If we let $\varphi \in S(\mathcal{B})$, then the canonical extension $\tilde{\varphi}$ of φ is also a state. Hence $\tilde{\varphi}$ is the weak*-limit of a net of convex combinations of states in $\mathfrak{X}$, so by restriction it follows that φ is a weak*-limit of a net of convex combinations of states of the form $\omega_\xi \circ \rho_2$, where $\xi \in \mathcal{K}_0$ is a unit vector.

Assume now that φ is a state on $\mathcal{A}$ given by $\varphi(x) = \langle \pi(x)\xi, \xi\rangle$ where $\xi \in \mathcal{H}$. Then φ vanishes on $\ker \rho \subseteq \ker \rho_1$ and hence induces a positive linear functional φ' on $\mathcal{B}$ given by $\varphi'(x + \mathfrak{I}) = \langle \pi(x)\xi, \xi\rangle$. After scaling by the norm, then by virtue of what we have shown above, φ' is a weak*-limit of convex combinations of states of the form $\omega_\xi \circ \rho_2$ for $\xi \in \mathcal{K}_0$, so φ is a weak*-limit of convex combinations of the states of the form $\omega_\xi \circ \rho_1 = \omega_\xi \circ \rho$ for $\xi \in \mathcal{K}_0$, as wanted.

Finally, it is clear that (ii) implies (iv). To see that (iv) implies (ii), define $\varphi(x) = \langle \pi(x)\xi_0, \xi_0\rangle$ for all $x \in \mathcal{A}$. By assumption, φ is the weak*-limit of a net $(\varphi_i)_{i \in I}$ of positive linear functionals on $\mathcal{A}$ associated with ρ. Now let $\xi \in \mathcal{H}$. Given an $\varepsilon > 0$ there exists $x_0 \in \mathcal{A}$ such that $\|\pi(x_0)\xi_0 - \xi\| < \varepsilon$, in turn implying

$$\begin{aligned} |\langle \pi(x)\xi, \xi\rangle - \varphi(x_0^* x x_0)| &= |\langle \pi(x)\xi, \xi\rangle - \langle \pi(x)\pi(x_0)\xi_0, \pi(x_0)\xi\rangle| \\ &\leq \|\pi(x)\xi\| \|\pi(x_0)\xi_0 - \xi\| + \|\pi(x)\pi(x_0)\xi_0 - \pi(x)\xi\| \|\pi(x_0)\xi_0\| \\ &\leq \varepsilon_0 \|x\| (2\|\xi\| + \varepsilon_0) \end{aligned}$$

for all $x \in \mathcal{A}$. Note that the latter expression can be made arbitrarily small if x is fixed and ε_0 is chosen small enough. By defining $\psi_i(x) = \varphi_i(x_0^* x x_0)$ for all $i \in I$, then $(\psi_i)_{i \in I}$ is also a net of positive linear functionals on $\mathcal{A}$ associated with ρ, converging in the weak* topology to the linear functional $x \mapsto \varphi(x_0^* x x_0)$. It is now clear that $\psi_i(x) \to \langle \pi(x)\xi, \xi\rangle$ for all $x \in \mathcal{A}$, so that (ii) holds. ☐

Theorem 1.5.3 now allows for the following characterization of weak containment.

Theorem 1.5.9. *Let G be a locally compact group and let $(\rho, \mathcal{H})$ and $(\sigma, \mathcal{K})$ be unitary representations of G. Then the following are equivalent:*

(i) $\rho \prec \sigma$.
(ii) $C^* \ker \sigma \subseteq C^* \ker \rho$.
(iii) $\|\rho(x)\| \leq \|\sigma(x)\|$ *for all* $x \in C^*(G)$.

If $\xi_0 \in \mathcal{H}$ is a cyclic vector for ρ, then all of the above conditions are equivalent to:

(iv) *The function $s \mapsto \langle \rho(s)\xi_0, \xi_0\rangle$ of G is the limit in the topology of compact convergence of finite sums of continuous positive definite functions of G associated with σ.*

Proof. If (ii) holds, then the inclusion induces a *-homomorphism

$$C^*(G)/C^* \ker \sigma \to C^*(G)/C^* \ker \rho.$$

As the former is *-isomorphic to $C^*_\sigma(G)$ and the latter to $C^*_\rho(G)$, (iii) follows from *-homomorphisms being contractions. Conversely, (iii) immediately implies (ii).

We now prove that (i) $\Leftrightarrow$ (ii). We know that (ii) holds if and only if every state on $C^*(G)$ associated with ρ is the weak*-limit of a net of states which are finite sums of positive linear functionals associated with σ. Since states are bounded, it follows from Proposition 1.3.7 and Remark 1.3.8 that the latter condition occurs if and only if every state on $L^1(G)$ associated with ρ is the weak*-limit of a net of states on $L^1(G)$ related to σ in the same way as above. By Remark 1.4.5, Lemma 1.4.2 and Theorem 1.4.7, this happens if and only if any $\varphi \in P_1(G)$ associated with ρ is the limit in the topology of compact convergence of a net $(\varphi_i)_{i\in I}$ in $P_1(G)$, where each φ_i is a finite sum of continuous positive definite functions associated with σ, but this is equivalent to (i) by virtue of Lemma 1.5.2. In the same manner, one can prove that (ii) and (iv) are equivalent if ρ has a cyclic vector, again invoking Theorem 1.5.3. □

Having proved the above theorem, we obtain the following characterization of C^*-simplicity:

Corollary 1.5.10. *A locally compact group G is C^*-simple if and only if the conditions $\rho \sim \lambda_G$ and $\rho \prec \lambda_G$ are equivalent for any unitary representation ρ of G.*

Proof. Assume first that G is C^*-simple. If ρ is a unitary representation of G such that $\rho \prec \lambda_G$, then we have a *-homomorphism

$$C^*_r(G) \cong {}^{C^*(G)}/_{C^* \ker \lambda_G} \to {}^{C^*(G)}/_{C^* \ker \rho} \cong C^*_\rho(G)$$

given by $\lambda_G(x) \mapsto \rho(x)$ for $x \in C^*(G)$. By assumption this map is injective and hence isometric, so Theorem 1.5.9 yields that $\rho \sim \lambda_G$. Conversely, if $\mathfrak{I}$ is a two-sided, closed, proper ideal of $C^*_r(G)$, then we can take a faithful nondegenerate representation $\varphi\colon C^*_r(G)/\mathfrak{I} \to B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, yielding a nondegenerate representation $\rho\colon C^*(G) \to B(\mathcal{H})$ given by $\rho(x) = \varphi(\lambda_G(x) + \mathfrak{I})$. By Corollary 1.3.4, ρ arises from a unitary representation $\rho\colon G \to \mathcal{U}(\mathcal{H})$. Since *-homomorphisms are contractions, we have

$$\|\rho(x)\| \leq \|\lambda_G(x) + \mathfrak{I}\| \leq \|\lambda_G(x)\|, \quad x \in C^*(G),$$

and hence $\rho \prec \lambda_G$ by Theorem 1.5.9. If this entails that $\rho \sim \lambda_G$, then $\|x + \mathfrak{I}\| = \|x\|$ for all $x \in C^*_r(G)$, so $\mathfrak{I} = \{0\}$. Hence G is C^*-simple. □

1.6 The induced representation

Let G be a locally compact group and let H be a closed subgroup of G. Supposing that $(\sigma, \mathcal{H})$ is a unitary representation of H, there is a way of extending σ to G so that quite a lot of particularly pretty properties of the extension are inherited from σ. The construction is by no means trivial, and we will just sketch the general contour of it in order to work with it, and then examine it in some special cases. Proofs and further details can be found in [10, Section E.1] and [25, Section 6.1].

Letting G, H and σ be as above, let $\pi\colon G \to G/H$ denote the canonical surjective map onto the left coset space of H. We let X denote the vector space of functions $f\colon G \to \mathcal{H}$ such that

(i) f is continuous;
(ii) $\pi(\operatorname{supp} f)$ is compact;
(iii) $f(sh) = \sigma(h^{-1})f(s)$ for all $s \in G$ and $h \in H$.

The idea is to turn X into an inner product space, then complete it and after that introduce the representation itself. The following result is absolutely essential in order to obtain the wanted inner product.

Theorem 1.6.1. *There always exists a Radon measure ν on the quotient space G/H that is* quasi-invariant*, i.e., for all Borel sets $E \subseteq G/H$ it holds that $\nu(E) = 0$ if and only if $\nu(sE) = 0$ for all $s \in G$, and the support of ν is the entire space G/H.*

Any quasi-invariant Radon measure ν on G/H corresponds to a unique *rho-function*, i.e., a continuous function $\rho\colon G \to \mathbb{R}_{>0}$ satisfying

$$\rho(sh) = \frac{\Delta_H(h)}{\Delta_G(h)}\rho(s), \quad s \in G,\ h \in H,$$

where Δ_G and Δ_H denote the modular functions of G and H respectively. The correspondence is obtained by means of the equality

$$\int_{G/H}\int_H f(sh)\,\mathrm{d}\mu_H(h)\,\mathrm{d}\nu(sH) = \int f(s)\rho(s)\,\mathrm{d}\mu(s), \quad f \in C_c(G),$$

where μ_H is a fixed left Haar measure of H. For all $f, g \in X$ we now note that

$$\langle f(sh), g(sh)\rangle = \langle \rho(h^{-1})f(s), \rho(h^{-1})g(s)\rangle = \langle f(s), g(s)\rangle$$

for all $s \in G$ and $h \in H$, so that the function $s \mapsto \langle f(s), g(s)\rangle$ is constant on left cosets of H. Hence it induces a function on G/H, and this function is continuous with compact support by (i) and (ii) above. Thus we can define

$$\langle f, g\rangle = \int \langle f(s), g(s)\rangle\,\mathrm{d}\nu(sH), \quad f, g \in X.$$

It turns out that this defines an inner product on X, and we let $\mathcal{H}_\nu$ denote the completion of X with respect to this inner product. For the representation σ to be extended, we need the result that there exists a continuous function $R_\nu\colon G \times (G/H) \to \mathbb{R}_{>0}$, called the *Radon-Nikodym derivative*, such that

$$\int f(sx)R_\nu(s, x)\,\mathrm{d}\nu(x) = \int f(x)\,\mathrm{d}\nu(x), \quad f \in C_c(G/H),\ s \in G.$$

We usually write

$$R_\nu(s, x) = \frac{\mathrm{d}s\nu(x)}{\mathrm{d}\nu(x)}, \quad s \in G,\ x \in G/H,$$

and it can shown that if ρ is the rho-function for ν, then

$$\frac{\mathrm{d}s\nu(tH)}{\mathrm{d}\nu(tH)} = \frac{\rho(st)}{\rho(t)}, \quad s, t \in G.$$

For all $s \in G$ we then define

$$\sigma_\nu(s)f(t) = \left(\frac{\mathrm{d}s^{-1}\nu(tH)}{\mathrm{d}\nu(tH)}\right)^{1/2} f(s^{-1}t), \quad f \in X.$$

This in fact defines an isometric linear surjection $\sigma_\nu(s)\colon X \to X$ which then naturally extends to $\mathcal{H}_\nu$. The map $\sigma_\nu\colon G \to \mathcal{U}(\mathcal{H}_\nu)$ is a unitary representation and if ν' is another quasi-invariant Radon measure on G/H, then the unitary representations σ_ν and $\sigma_{\nu'}$ are equivalent. We call σ_ν the representation of G *induced* by σ, and we denote it by $\mathrm{Ind}_H^G\sigma$.

Example 1.6.2. (i) If $H = \{1\}$ and $\sigma = 1_H$, then X as defined above is simply the space $C_c(G)$. Since any left Haar measure on G can be interpreted as a measure on G/H, we see that the inner product on X can be taken to be the standard L^2-inner product of functions in $C_c(G)$, and it is clear that the completion of X with respect to this inner product is in fact $L^2(G)$. The rho-function corresponding to the measure on G/H is necessarily constant by uniqueness of Haar measure, so the Radon-Nikodym derivative is always equal to 1, and thus we see that the representation on G induced by 1_H is in fact the left regular representation, i.e., $\mathrm{Ind}_H^G 1_H = \lambda_G$.

(ii) Let μ_H be a fixed left Haar measure on H. We say that a Borel measure ν on G/H is *G-invariant* if $\nu(sE) = \nu(E)$ for all $s \in G$ and Borel sets $E \subseteq G/H$. If ν is a G-invariant Radon measure on G/H, then

$$f \mapsto \int_{G/H}\int_H f(sh)\,\mathrm{d}\mu_H(h)\,\mathrm{d}\nu(sH), \quad f \in C_c(G)$$

is a non-zero left invariant positive linear functional on $C_c(G)$, so uniqueness of Haar measure on G yields a positive scalar c such that

$$\int_{G/H}\int_H f(sh)\,\mathrm{d}\mu_H(h)\,\mathrm{d}\nu(sH) = c\int f\,\mathrm{d}\mu$$

for all $f \in C_c(G)$. The rho-function corresponding to ν is therefore equal to c locally almost everywhere and hence everywhere by continuity of ρ. If we now let $\sigma = 1_H$, then the map $C_c(G/H) \to X$ given by $f \mapsto f \circ \pi$ is an isomorphism. The same considerations as in (i) now yield that the completion of X with respect to the inner product is in fact $L^2(G/H, \nu)$. As the rho-function is constant, the Radon-Nikodym derivative is equal to 1, and hence

$$(\mathrm{Ind}_H^G 1_H)(s)\xi(tH) = \xi(s^{-1}tH), \quad \xi \in L^2(G/H,\nu),\ s,t \in G.$$

This representation of G on $L^2(G/H, \nu)$ is usually called the *quasi-regular representation.* If H is normal, then ν can just be taken to be a left Haar measure on G/H.

The mentioning of a few flabbergasting properties of the induced representation is in order:

Theorem 1.6.3. *Let G be a locally compact group and let H be a closed subgroup of G.*

(i) *If σ_1, σ_2 are equivalent unitary representations of H, then $\mathrm{Ind}_H^G\sigma_1$ and $\mathrm{Ind}_H^G\sigma_2$ are equivalent.*
(ii) (Induction by stages.) *If K is a closed subgroup of H and σ is a unitary representation of K, then $\mathrm{Ind}_H^G(\mathrm{Ind}_K^H\sigma)$ and $\mathrm{Ind}_K^G\sigma$ are equivalent.*
(iii) (Continuity of induction.) *Suppose that σ_1, σ_2 are two unitary representations of H. If $\sigma_1 \prec \sigma_2$, then*
$$\mathrm{Ind}_H^G\sigma_1 \prec \mathrm{Ind}_H^G\sigma_2.$$

As an immediate consequence of Example 1.6.2 (i) and Theorem 1.6.3 (ii), we see that $\mathrm{Ind}_H^G\lambda_H$ is in fact equivalent to λ_G.

1.7 Amenability and the amenable radical

Amenability of locally compact groups can be defined in a wide variety of different ways, of which the most common is by using the notion of left invariant means on subspaces of $L^\infty(G)$, once again denoting the space of locally measurable functions $f : G \to \mathbb{C}$ that are bounded except on a locally null set, in which functions are identified if they are equal locally almost everywhere.

Any locally compact group G has the trivial subgroup as an amenable, normal, closed subgroup. As we shall prove now, G in fact has an amenable, normal, closed subgroup that is maximal in the sense that it contains all other such subgroups of G. To see this, we only need to know about some specific permanence properties of amenability, for which proofs can be easily attained by mentioning a few equivalent properties of G itself. A *left invariant mean* on a translation-invariant subspace X of $L^\infty(G)$ is a positive linear functional $\mathfrak{m}: X \to \mathbb{C}$ such that $\mathfrak{m}(1) = 1$ and $\mathfrak{m}(s.f) = \mathfrak{m}(f)$ for all $s \in G$ and $f \in X$, where translation invariance of X means that $f \in X$ implies $s.f \in X$ for all $s \in G$. The most important non-trivial translation-invariant subspaces of $L^\infty(G)$ are $C_b(G)$, the space of continuous bounded functions on G, and $LUC_b(G)$, the space of left uniformly continuous bounded functions of G.

Amenability can also be described by the so-called *fixed point property*, namely that any continuous affine action of G on a compact, convex subset of a locally convex space has a fixed point. To summarize all of this in one fell swoop, we have the following result due to Day, Rickert and Namioka:

Theorem 1.7.1. *The following conditions are equivalent:*

(i) *There exists a left invariant mean on $L^\infty(G)$.*
(ii) *There exists a left invariant mean on $C_b(G)$.*
(iii) *There exists a left invariant mean on $LUC_b(G)$.*
(iv) *The group G has the fixed point property.*

Proof. See [29, Sections 2.2 and 3.3]. □

We say that the locally compact group G is *amenable* if it satisfies any of the above conditions, and with the above theorem at hand it is in fact quite easy to prove the following permanence properties of amenability.

Theorem 1.7.2. (i) *If G is amenable and π is a continuous homomorphism with dense range in another locally compact group G', then G' is amenable.*

(ii) *If H is an amenable, closed, normal subgroup of G and G/H is amenable, then G is amenable.*

(iii) *If $(H_\alpha)_{\alpha\in A}$ is a directed system of amenable, closed subgroups of G in the sense that for all $\alpha,\beta\in A$ there exists $\gamma\in A$ such that $H_\alpha\cup H_\beta\subseteq H_\gamma$, then $H=\overline{\bigcup_{\alpha\in A} H_\alpha}$ is an amenable, closed subgroup of G.*

Proof. The fixed point property applies easily to prove (i). For details on (ii) and (iii), see [29, Theorems 2.3.3 and 2.3.4]. ☐

For any topological group G, we will let $\mathrm{Aut}(G)$ denote the group of continuous automorphisms of G. Let G and H are locally compact groups, and let $\varphi\colon H\to\mathrm{Aut}(G)$ be a group homomorphism. Equipping $G\times H$ with the product topology, $G\times H$ is a locally compact space. Writing $\varphi_t=\varphi(t)$ for all $t\in H$, we then define a binary operation on $G\times H$ by the rule

$$(s,t)(s',t') := (s\varphi_t(s'),tt'),\quad s,s'\in G,\ t,t'\in H.$$

Then $G\times H$ is a group with respect to this composition. This is the so-called *semidirect product* of G and H with respect to φ, and we denote it by $G\rtimes_\varphi H$ (when φ is clear from the context, we shall just write $G\rtimes H$). It is immediate that the binary operation and inverse map are continuous maps, so $G\rtimes_\varphi H$ is a locally compact group.

Corollary 1.7.3. *Let G be a locally compact group. Then G has a largest amenable, closed, normal subgroup.*

Proof. Suppose first that H_1 and H_2 are two amenable, closed, normal subgroups of G. Since H_2 acts on H_1 by conjugation, we can construct the semidirect product $H_1\rtimes H_2$ with respect to this action. Then $H_1\rtimes H_2$ is amenable by Theorem 1.7.2 (ii), as it contains H_1 as a closed, normal subgroup (by the inclusion $g\mapsto(g,1)$) with the quotient group being isomorphic to H_2. As a result of the semidirect product being constructed with respect to conjugation, we have a continuous group homomorphism $H_1\rtimes H_2\to\overline{H_1H_2}$ given by $(h_1,h_2)\mapsto h_1h_2$. By Theorem 1.7.2 (i), $\overline{H_1H_2}$ is amenable, and since H_1 and H_2 are normal, $\overline{H_1H_2}$ is also normal.

If we now let $(H_\alpha)_{\alpha\in A}$ be the family of all amenable, closed, normal subgroups of G, then we have just shown that this is in fact a directed system in the sense of Theorem 1.7.2 (iii). Therefore $H=\overline{\bigcup_{\alpha\in A}H_\alpha}$ is a closed amenable subgroup of G. It is easy to see that H is normal, and it thus follows that H is the largest amenable, closed, normal subgroup of G. ☐

Definition 1.7.4. For any locally compact group G, the largest amenable, closed, normal subgroup of G obtained by means of Corollary 1.7.3 is called *the amenable radical* of G and is denoted by AR_G.

It now turns out that we have the following consequence of C^*-simplicity:

Theorem 1.7.5 (de la Harpe, 2007)**.** *If G is a locally compact group and $\mathrm{AR}_G\neq\{1\}$, then G is not C^*-simple.*

To realize this in the easiest possible way, we record yet another characterization of amenability, both beautiful and *extremely* surprising.

Theorem 1.7.6 (Godement, 1948)**.** *The locally compact group G is amenable if and only if $1_G\prec\lambda_G$.*

We refer to [10, Theorem G.3.2] for a proof of the above result. To make preparations for a proof of Theorem 1.7.5, it is enough to consider this very simple lemma:

Lemma 1.7.7. *For any $s_0\neq 1$ in a locally compact group G, there exist continuous functions $f_1,f_2\colon G\to[0,1]$ with disjoint compact supports such that $f_1(1)=f_2(s_0)=1$, $s_0\cdot\operatorname{supp}f_1\cap\operatorname{supp}f_1=\emptyset$ and $s_0\cdot\operatorname{supp}f_2\cap\operatorname{supp}f_2=\emptyset$.*

Proof. Let U_1 and U_2 be disjoint neighbourhoods of 1 and s respectively, and define $V_1 = U_1 \cap s_0^{-1}U_2$ and $V_2 = U_2 \cap s_0 U_1$. By the locally compact version of Urysohn's lemma, we can take continuous functions $f_1, f_2 \colon G \to [0,1]$ such that $f_1(1) = f_2(s_0) = 1$ and f_i vanishes outside a compact subset of V_i for $i = 1, 2$. In particular,

$$s_0 \cdot \operatorname{supp} f_1 \cap \operatorname{supp} f_1 \subseteq V_1 \cap s_0 V_1 \subseteq U_1 \cap U_2 = \emptyset$$

and similarly $s_0 \cdot \operatorname{supp} f_2 \cap \operatorname{supp} f_2 = \emptyset$. □

Proof of Theorem 1.7.5. Let $N = \mathrm{AR}_G$ and let $\rho = \mathrm{Ind}_N^G 1_N$ be the quasi-regular representation; recall from Example 1.6.2 that σ is then given by

$$\rho(s)\xi(tN) = \xi(s^{-1}tN), \quad \xi \in L^2(G/N),\ s, t \in G.$$

Since N is amenable, we have $1_N \prec \lambda_N$. By continuity of induction, this implies

$$\rho = \mathrm{Ind}_N^G 1_N \prec \mathrm{Ind}_N^G \lambda_N = \lambda_G.$$

We will now show that λ_G is not weakly contained in ρ, so that we can conclude by Corollary 1.5.10 that G is not C^*-simple. Let $s_0 \in N$ with $s_0 \neq 1$. By the above lemma there exist continuous non-zero functions $f_1, f_2 \colon G \to [0,1]$ with disjoint compact supports such that $s_0 \cdot \operatorname{supp} f_1 \cap \operatorname{supp} f_1 = \emptyset$ and $s_0 \cdot \operatorname{supp} f_2 \cap \operatorname{supp} f_2 = \emptyset$. We can assume that $\|f_1\|_2 = \|f_2\|_2 = 1$. Defining $\xi = f_1 + i f_2 \in L^2(G)$, we then have

$$\operatorname{Re} \langle \lambda_G(s_0)\xi, \xi\rangle = \int (f_1(s_0^{-1}t) f_1(t) - f_2(s_0^{-1}t) f_2(t))\, \mathrm{d}\mu(t) = 0$$

by construction, and

$$\langle \lambda_G(1)\xi, \xi\rangle = \int |f_1(t) + i f_2(t)|^2\, \mathrm{d}\mu(t) = \int (f_1(t)^2 + f_2(t)^2)\, \mathrm{d}\mu(t) = 2.$$

For any $s \in N$, we have $\rho(s)\eta(tN) = \eta(s^{-1}tN) = \eta(tN)$ and hence $\rho(s)\eta = \eta$ for all $\eta \in L^2(G/N)$ by normality of N. If λ_G were weakly contained in ρ, then for all $\varepsilon > 0$ there would exist functions $\eta_1, \ldots, \eta_n \in L^2(G/N)$ such that

$$\left| \langle \lambda_G(s)\xi, \xi\rangle - \sum_{i=1}^n \|\eta_i\|^2 \right| = \left| \langle \lambda_G(s)\xi, \xi\rangle - \sum_{i=1}^n \langle \rho(s)\eta_i, \eta_i\rangle \right| < \varepsilon$$

for all $s \in \{1, s_0\}$. In particular, we would have $|\langle \lambda_G(s_0)\xi, \xi\rangle - \langle \lambda_G(1)\xi, \xi\rangle| < 2\varepsilon$ for all $\varepsilon > 0$, which is clearly a contradiction. □

A wealth of different groups belong to the class of amenable groups, some of these being the class of compact groups (the Haar measure provides a left-invariant mean straight away), along with solvable groups and abelian groups. We refer to [53, Proposition 0.15] for a proof of that abelian groups are indeed amenable, from which amenability of solvable groups follows by a simple induction argument used with Theorem 1.7.2 (ii). In particular, we have the following useful consequence of C^*-simplicity.

Corollary 1.7.8. *Any C^*-simple locally compact group G is centerless and icc.*

Proof. The center $Z(G)$ of G is a closed normal abelian subgroup, so we must have $Z(G) = \{1\}$ by Theorem 1.7.5. To verify the second statement, let F be the normal subgroup of G consisting of all elements with finite conjugacy class. We claim that F is amenable; once that has been established it will follow from Theorem 1.7.2 (i) that $\overline{F} \subseteq \mathrm{AR}_G$, so that $F = \{1\}$. By the same theorem it is enough to show that any finitely generated subgroup Λ of F is amenable. If $s_1, \ldots, s_n \in F$ is a generating set for a subgroup Λ of F, note that the centralizer Z_i in G of each s_i is a subgroup of finite index in G. Indeed, if $i \in \{1, \ldots, n\}$ and $a_1, \ldots, a_k \in G$ are elements such that

$$\{s s_i s^{-1} \mid s \in G\} = \{a_i s_i a_i^{-1} \mid i = 1, \ldots, k\},$$

then for all $s \in G$ we have $s s_i s^{-1} = a_j s_i a_j^{-1}$ for some $j = 1, \ldots, k$. Hence $a_j^{-1} s \in Z_i$ and $s \in a_j Z_i$. In particular, if we let

$$Z = \bigcap_{i=1}^n Z_i$$

then Z is of finite index in G, so that $Z \cap \Lambda$ is of finite index in Λ. But $Z \cap \Lambda$ is the center of Λ, so since the quotient group $\Lambda/(Z \cap \Lambda)$ is finite and $Z \cap \Lambda$ is normal, abelian and closed in Λ, it follows from Theorem 1.7.2 (ii) that Λ is amenable. Hence G is icc. □

Remark 1.7.9. Theorem 1.7.6 suggests that amenability has something to say when discussing functoriality of the reduced group C^*-algebra in certain cases. Indeed, let G be a locally compact group, let N be a normal, closed subgroup of G and define $\sigma = \mathrm{Ind}_N^G 1_N$. If N is amenable, then $1_N \prec \lambda_N$, so that continuity of induction implies

$$\sigma \prec \mathrm{Ind}_N^G \lambda_N = \lambda_G.$$

Hence we obtain a *-homomorphism $C_r^*(G) \to C_\sigma^*(G)$. Note now that if $\pi\colon G \to G/N$ is the quotient map, then $\sigma = \lambda_{G/N} \circ \pi$. Letting ν denote a left Haar measure on G/N such that the constant c of Example 1.6.2 (ii) equals 1, then for any $f \in C_c(G)$ and $\xi, \eta \in L^2(G/N)$ we have

$$\begin{aligned}\int \langle f(s)\sigma(s)\xi, \eta\rangle \,\mathrm{d}\mu(s) &= \int \langle f(s)\lambda_{G/N}(sN)\xi, \eta\rangle \,\mathrm{d}\mu(s)\\ &= \int_{G/N}\int_N \langle f(sn)\lambda_{G/N}(sN)\xi, \eta\rangle \,\mathrm{d}\mu_N(n)\,\mathrm{d}\nu(sN)\\ &= \int_{G/N}\left\langle \left(\int_N f(sn)\,\mathrm{d}\mu_N(n)\right)\lambda_{G/N}(sN)\xi, \eta\right\rangle \mathrm{d}\nu(sN)\end{aligned}$$

by Fubini's theorem. The linear map $P\colon C_c(G) \to C_c(G/N)$ given by

$$Pf(sN) = \int_N f(sn)\,\mathrm{d}\mu_N(n), \quad s \in G,$$

is surjective by [25, Proposition 2.48], so by the above computations we see that $C_\sigma^*(G) = C_r^*(G/N)$. Hence we have a *-homomorphism $\varphi\colon C_r^*(G) \to C_r^*(G/N)$ satisfying

$$\varphi(\lambda_G(f)) = \lambda_{G/N}(Pf) \tag{1.7.1}$$

for all $f \in C_c(G)$. It is fairly easy to see that the kernel of φ is a non-trivial ideal of $C_r^*(G)$ if $N \neq \{1\}$, so we have in fact given another proof of Theorem 1.7.5.

In fact, a converse holds: if N is a normal, *open* (and hence closed) subgroup of G and there exists a homomorphism $\varphi\colon C_r^*(G) \to C_r^*(G/N)$ satisfying (1.7.1), then N is amenable. Indeed, let $J\colon C_r^*(N) \to C_r^*(G)$ be the isometric embedding of Proposition 1.3.11. As G/N is discrete, $C_r^*(G/N)$ has an identity (see also Lemma 1.8.2), and it is easy to check that

$$\varphi(J(\lambda_N(f))) = \left(\int_N f(s)\,\mathrm{d}\mu(s)\right)1$$

for all $f \in C_c(N)$. As $C_c(N)$ is dense in $C^*(N)$, it follows that $C_r^*(N)$ has a character α such that $\alpha(\lambda_N(x)) = 1_N(x)$ for all $x \in C^*(N)$. Therefore $1_N \prec \lambda_N$ by Theorem 1.5.9, so that N is amenable. ✺

In general, it is not possible to say a whole lot about traces on reduced group C^*-algebras of locally compact groups. However, if a locally compact group contains a non-trivial amenable open subgroup N, then $C_r^*(G)$ has a trace τ satisfying

$$\tau(\lambda_G(f)) = \int_N f(s)\,\mathrm{d}\mu(s)$$

for all $f \in L^1(G)$ (cf. [45, Corollary 4.1]).

1.8 Discrete groups and their reduced group C^*-algebras

Up until now, we may have found a lot of useful properties of C^*-simple locally compact groups, but we haven't actually given examples of any. The hinderance is mainly topological in nature; as we shall see in the next chapters, a wealth of *discrete* groups are C^*-simple and have the unique trace property. As mentioned in the prologue, it is an open question whether there actually exist non-discrete locally compact groups that are either C^*-simple or have unique trace, posed by de la Harpe in [32].

One reason that it is a wise idea to stick with discrete groups to produce examples of as well as positive results about C^*-simplicity, comes from the following result originally proven by Bekka, Cowling and de la Harpe in [9]:

Theorem 1.8.1. *Any C^*-simple locally compact group is disconnected.*

The proof requires a deep understanding of the structure theory of Lie groups, and we do not have time (nor resources) to give a proof here.

Another reason is that whenever one wishes to determine simplicity of a C^*-algebra, it is always convenient to be able to work with a multiplicative identity in the algebra itself. The next result is well-known:

Proposition 1.8.2. *A locally compact group G is discrete if and only if the reduced group C^*-algebra $C_r^*(G)$ is unital.*

To realize this, a lemma is needed.

Lemma 1.8.3. *Let G be a non-discrete locally compact group with a left Haar measure μ. Then there exists a decreasing sequence $(V_n)_{n\geq 1}$ of compact, symmetric neighbourhoods of 1 such that $\mu(V_n) \to 0$.*

Proof. Let V_1 be a compact symmetric neighbourhood of 1. Then $V_1 \neq \{1\}$ since G is non-discrete, so there exists $x_2 \in V_1 \setminus \{1\}$. As G is Hausdorff, there exist disjoint neighbourhoods $U_2 \subseteq V_1$ and $U_2' \subseteq V_1$ of 1 and x_2, respectively. Then

$$W_2 = \overline{x_2^{-1}U_2' \cap U_2} \cap x_2^{-1}V_1 \quad \text{and} \quad W_2' = \overline{U_2' \cap x_2 U_2} \cap V_1$$

are disjoint compact neighbourhoods of 1 and x_2, respectively, and

$$2\mu(W_2) = \mu(W_2 \cup W_2') \leq \mu(V_1).$$

By defining $V_2 = W_2 \cap W_2^{-1}$, we obtain a compact symmetric neighbourhood V_2 of 1 with $V_2 \subseteq V_1$ and $\mu(V_2) \leq \frac{1}{2}\mu(V_1)$. The wanted sequence is then defined inductively. □

Proof of Proposition 1.8.2. If G is discrete, then the Dirac point mass of the identity element provides an identity element of $L^1(G) = \ell^1(G)$ and hence of $C_r^*(G)$. To prove the converse, assume that G is non-discrete and let μ denote a fixed left Haar measure on G. We claim that there is a sequence $(g_n)_{n\geq 1}$ of functions in $L^2(G)$ with $\|g_n\|_2 = 1$ for all $n \geq 1$ such that $\lambda_G(f)g_n \to 0$ in $L^2(G)$ for all $f \in L^1(G)$. Once that has been proved, assume for contradiction that $C_r^*(G)$ does have an identity. Then there exists $f \in L^1(G)$ such that $\|1 - \lambda_G(f)\| < \frac{1}{2}$. By the above claim, there exists a function $g \in L^2(G)$ such that $\|g\|_2 = 1$ and $\|\lambda_G(f)g\| < \frac{1}{2}$, but then

$$\|(1 - \lambda_G(f))g\|_2 = \|g - \lambda_G(f)g\|_2 \geq \|g\|_2 - \|\lambda_G(f)g\|_2 > \frac{1}{2}.$$

Hence $\|1 - \lambda_G(f)\| > \frac{1}{2}$, a contradiction.

By a density argument, it suffices to prove the claim for all $f \in C_c(G)$. Let $K = \operatorname{supp} f$, $C = \|f\|_\infty$ and let $(V_n)_{n\geq 1}$ be a decreasing sequence of compact, symmetric neighbourhoods of 1 such that $\mu(V_n) \to 0$. If we let

$$g_n = \frac{1_{V_n}}{\mu(V_n)^{1/2}},$$

then $(\lambda_G(f)g_n)(t) = (f * g_n)(t) = 0$ for all $t \notin KV_n$. Indeed, if $f(s)g_n(s^{-1}t) \neq 0$ for some $s \in G$, then $s \in K$ and $s^{-1}t \in V_n$, so that $t \in KV_n$. As

$$|(f * g)(t)| \leq C \int |g(s^{-1}t)| \,\mathrm{d}\mu(s) = C \int |g(s)| \,\mathrm{d}\mu(s) \leq C\mu(V_n)^{1/2}$$

for all $t \in G$, we therefore conclude that

$$\|\lambda_G(f)g_n\|_2^2 \leq \mu(KV_n) \sup\{|(f * g_n)(t)|^2 \,|\, t \in KV_n\} \leq C^2 \mu(KV_1)\mu(V_n) \to 0$$

as wanted, since KV_1 is compact. □

These two results provide us with a good enough argument for restricting our attention to the discrete case, and of course plenty of the results of the previous section are greatly simplified in doing so. If Γ is a discrete group and $(\rho, \mathcal{H})$ is a unitary representation of Γ, then $C^*_\rho(\Gamma)$ is the norm closure of the *-subalgebra

$$\mathcal{K}_\rho(\Gamma) = \left\{ \sum_{s \in F} \lambda_s \rho(s) \,\middle|\, F \subseteq \Gamma \text{ finite},\ \lambda_s \in \mathbb{C} \text{ for all } s \in F \right\} \subseteq B(\mathcal{H}),$$

as we noted in (1.3.1). In the case of the reduced group C^*-algebra of a discrete group Γ, it is useful to give this subalgebra a name:

Definition 1.8.4. The subset $\mathcal{K}_{\lambda_\Gamma}(\Gamma)$ of $B(\ell^2(\Gamma))$ is called the *complex group ring* of Γ and will *always* be denoted by $\mathbb{C}\Gamma$.

It is then evident that $L(\Gamma)$ is the von Neumann algebra generated by the subalgebra $\mathbb{C}\Gamma$. Moreover, the left regular representation λ_Γ also allows for a canonical faithful trace given by

$$x \mapsto \tau_\Gamma(x) = \langle x\delta_1, \delta_1 \rangle$$

for $x \in C^*_r(\Gamma)$ (cf. [14, Proposition 2.5.3]), where δ_1 denotes the Dirac measure of the singleton $\{1\} \subseteq \Gamma$. In fact, this also defines a faithful normal trace on the group von Neumann algebra $L(\Gamma)$ (which we will also refer to as the canonical trace), so that δ_1 is a cyclic, separating vector for $C^*_r(\Gamma)$ and $L(\Gamma)$. Therefore the question of unique trace for discrete groups becomes a question of showing that τ_Γ is the *only* trace on $C^*_r(\Gamma)$.

Related to Theorem 1.7.5, Paschke and Salinas proved the following proposition in 1979:

Proposition 1.8.5. *If Γ is a discrete group and $\mathrm{AR}_\Gamma \neq \{1\}$, then $C^*_r(\Gamma)$ is not simple and does not have a unique trace.*

Proof. If Λ is a subgroup of Γ, then there exists an isometric embedding $J\colon C^*_r(\Lambda) \to C^*_r(\Gamma)$ and a conditional expectation $E\colon C^*_r(\Gamma) \to J(C^*_r(\Lambda))$ (cf. [56, Proposition 8.5]) such that

$$E(\lambda_\Gamma(s)) = J(\lambda_\Lambda(s)) \text{ (resp. } 0) \text{ when } s \in \Lambda \text{ (resp. } s \in \Gamma \setminus \Lambda).$$

Let $\Lambda = \mathrm{AR}_\Gamma$. Since $C^*(\Lambda) = C^*_r(\Lambda)$ by [14, Theorem 2.6.8], the trivial representation of Λ produces a character $\tau\colon C^*_r(\Lambda) \to \mathbb{C}$. Defining $\tau_1 = \tau \circ J^{-1} \circ E$, then τ_1 is a state on $C^*_r(\Gamma)$ satisfying $\tau_1(\lambda_\Gamma(s)) = 1$ for all $s \in \Lambda$. Observe that because Λ is a normal subgroup of Γ, then $st \in \Lambda$ if and only if $ts \in \Lambda$ for all $s, t \in \Gamma$, in which case

$$\tau_1(\lambda_\Gamma(s)\lambda_\Gamma(t)) = \tau_1(\lambda_\Gamma(t)\lambda_\Gamma(s)) = 1 \text{ (resp. } 0) \text{ when } st \in \Lambda \text{ (resp. } st \notin \Lambda).$$

Since the linear span of $\{\lambda_\Gamma(s) \,|\, s \in \Gamma\}$ is dense in $C^*_r(\Gamma)$, it follows that τ_1 is a trace on $C^*_r(\Gamma)$, and it does not coincide with the canonical trace. Indeed, for any $s \neq 1$ in Λ, we have $\langle \lambda_\Gamma(s)\delta_1, \delta_1 \rangle = 0$ whilst $\tau_1(\lambda_\Gamma(s)) = 1$.

Concerning non-C^*-simplicity of Γ, we can either use Theorem 1.7.5 or observe that

$$\mathfrak{I} = \{x \in C^*_r(\Gamma) \,|\, \tau_1(x^*x) = 0\}$$

is an ideal of $C^*_r(\Gamma)$. Since $\tau_1(\lambda_\Gamma(1)) = 1$, $\mathfrak{I}$ is proper, and if we define $x_s = 1 - \lambda_\Gamma(s)$ for all $s \in \Lambda$ we see that

$$\tau_1(x_s^* x_s) = 2 - \tau_1(\lambda_\Gamma(s^{-1})) - \tau_1(\lambda_\Gamma(s)) = 0.$$

Hence $x_s \in \mathfrak{I}$ for all $s \in \Lambda$, so since $\Lambda \neq \{1\}$, it follows that $\mathfrak{I}$ is a non-trivial ideal. $\square$

As in the previous section, we record the following consequence of the above proposition.

Proposition 1.8.6. *A discrete group with the unique trace property is centerless and icc.*

CHAPTER 2

THE DIXMIER PROPERTY

Before we head on into the quest of finding examples of groups that are either C^*-simple or have unique trace, it might be a good idea to have a tactic for how to establish these two properties. The first example of a non-trivial C^*-simple group with unique trace was given by Powers in [58], namely the free non-abelian group $\Gamma = \mathbb{F}_2$ on two generators. The essential idea of the proof was to realize that for any element $a \in \mathbb{C}\Gamma$ and $\varepsilon > 0$ there would exist a positive integer $n \geq 1$, elements $s_1, \ldots, s_n \in \Gamma$ and positive numbers $\lambda_1, \ldots, \lambda_n$ with $\sum_{i=1}^n \lambda_i = 1$ such that

$$\left\| \tau(x)1 - \sum_{i=1}^n \lambda_i \lambda_\Gamma(s_i) a \lambda_\Gamma(s_i)^* \right\| < \varepsilon,$$

where τ is the faithful trace on $C_r^*(\Gamma)$. Once that had been shown, an uninvolved argument yielded that $C_r^*(\Gamma)$ was in fact simple with unique trace. As we shall see in the next chapter, other ideas of Powers could be remolded into generating *many* examples of discrete groups of this sort.

The study of elements of the form $\sum_{i=1}^n \lambda_i u_i a u_i^*$ in a unital C^*-algebra $\mathcal{A}$, where $a \in \mathcal{A}$, $\lambda_1, \ldots, \lambda_n \geq 0$ with $\sum_{i=1}^n \lambda_i = 1$ and $u_1, \ldots, u_n \in \mathcal{U}(\mathcal{A})$, harks back to the paper [21] by Dixmier, wherein he proved an approximation theorem for von Neumann algebras, stated as follows: for all von Neumann algebras $\mathcal{A}$, the norm closure of the convex set $O_{\mathcal{A}}(a)$ generated by all elements of the form uau^* where $u \in \mathcal{U}(\mathcal{A})$, i.e.,

$$O_{\mathcal{A}}(a) = \left\{ \sum_{i=1}^n \lambda_i u_i a u_i^* \;\middle|\; n \geq 1,\ u_1, \ldots, u_n \in \mathcal{U}(\mathcal{A}),\ \lambda_1, \ldots, \lambda_n \geq 0,\ \sum_{i=1}^n \lambda_i = 1 \right\}, \tag{2.1}$$

always contains a central element. With this settled, Dixmier was able to prove a lot of results about norm-closed ideals in von Neumann algebras (see [23, Chapter III.5]). In Propositions 2.9 and 2.11, we give a proof of Dixmier's approximation theorem for all finite *factors*, i.e., for all elements x in a finite factor $\mathscr{M}$, the set $\overline{O_{\mathscr{M}}(x)} \cap \mathbb{C}1_{\mathscr{M}}$ is non-empty. Inspired by this, we define:

Definition 2.1. We say that a unital C^*-algebra $\mathcal{A}$ satisfies the *Dixmier property* if it holds for all $a \in \mathcal{A}$ that

$$\overline{O_{\mathcal{A}}(a)} \cap \mathbb{C}1_{\mathcal{A}} \neq \emptyset,$$

where $O_{\mathcal{A}}(a)$ is defined as in (2.1) and $\overline{O_{\mathcal{A}}(a)}$ is the norm closure of $O_{\mathcal{A}}(a)$ in $\mathcal{A}$.

It is evident that $b \in O_{\mathcal{A}}(a)$ implies $O_{\mathcal{A}}(b) \subseteq O_{\mathcal{A}}(a)$. Moreover, if $\tau \colon \mathcal{A} \to \mathbb{C}$ is a trace on $\mathcal{A}$, then for any $a \in \mathcal{A}$, we have

$$\tau\left(\sum_{i=1}^n \lambda_i u_i a u_i^* \right) = \sum_{i=1}^n \lambda_i \tau(u_i a u_i^*) = \sum_{i=1}^n \lambda_i \tau(u_i^* u_i a) = \sum_{i=1}^n \lambda_i \tau(a) = \tau(a)$$

for all $u_1, \ldots, u_n \in \mathcal{U}(\mathcal{A})$ and $\lambda_1, \ldots, \lambda_n \geq 0$ satisfying $\sum_{i=1}^n \lambda_i = 1$. Hence $\tau(O_{\mathcal{A}}(a)) = \{\tau(a)\}$, so by continuity, we obtain

$$\tau\left(\overline{O_{\mathcal{A}}(a)} \right) = \{\tau(a)\}. \tag{2.2}$$

We see now that Powers proved that $C_r^*(\mathbb{F}_2)$ in fact satisfied the Dixmier property, and as previously mentioned, this was enough to conclude simplicity and uniqueness of trace for $C_r^*(\mathbb{F}_2)$. Here's why.

Proposition 2.2. *Let $\mathcal{A}$ be a unital C^*-algebra with a faithful trace τ, and assume that $\mathcal{A}$ satisfies the Dixmier property. Then $\mathcal{A}$ is simple and has a unique trace.*

Proof. First, let $\mathfrak{I}$ be a non-zero, closed, two-sided ideal of $\mathcal{A}$; we will show that $\mathfrak{I} = \mathcal{A}$. As $\mathfrak{I} \neq \{0\}$, we can take some $x \neq 0$ in $\mathfrak{I}$. Letting $a = x^*x$, note that $a > 0$ and that faithfulness of τ implies $\tau(a) > 0$. We deduce that $0 \notin \overline{O_{\mathcal{A}}(a)}$, as otherwise we would have $0 = \tau(0) \in \tau(\overline{O_{\mathcal{A}}(a)}) = \{\tau(a)\}$ which would imply $\tau(a) = 0$, a contradiction. Since $\mathfrak{I}$ is a two-sided ideal, we have $O_{\mathcal{A}}(a) \subseteq \mathfrak{I}$ and hence $\overline{O_{\mathcal{A}}(a)} \subseteq \mathfrak{I}$ because $\mathfrak{I}$ is closed. Since $\mathcal{A}$ satisfies the Dixmier property and $0 \notin \overline{O_{\mathcal{A}}(a)}$, there exists some non-zero $\lambda \in \mathbb{C}$ such that $\lambda 1_{\mathcal{A}} \in \overline{O_{\mathcal{A}}(a)} \subseteq \mathfrak{I}$. Hence $\mathfrak{I} = \mathcal{A}$, so $\mathcal{A}$ is simple.

Assume now that φ is a trace on $\mathcal{A}$. For any $a \in \mathcal{A}$, then because $\mathcal{A}$ satisfies the Dixmier property, there is some $\lambda \in \mathbb{C}$ such that $\lambda 1_{\mathcal{A}} \in \overline{O_{\mathcal{A}}(a)}$. It now follows from (2.2) that

$$\varphi(a) = \varphi(\lambda 1_{\mathcal{A}}) = \lambda = \tau(\lambda 1_{\mathcal{A}}) = \tau(a).$$

Therefore τ is the only trace on $\mathcal{A}$. □

In fact, a converse of the above statement also holds: if $\mathcal{A}$ is a simple unital C^*-algebra and τ is a trace on $\mathcal{A}$ then τ is necessarily faithful, as the subset $\mathfrak{I} = \{x \in \mathcal{A} \,|\, \tau(x^*x) = 0\}$ is a closed, two-sided ideal of $\mathcal{A}$. Less obvious is the fact that $\mathcal{A}$ also has the Dixmier property, and the rest of this chapter is devoted to a proof of this.

Theorem 2.3 (Haagerup-Zsidó, 1984). *Let $\mathcal{A}$ be a simple unital C^*-algebra with at most one trace. Then $\mathcal{A}$ satisfies the Dixmier property. Furthermore, if $\mathcal{A}$ does have a unique trace τ then*

$$\overline{O_{\mathcal{A}}(a)} \cap \mathbb{C}1_{\mathcal{A}} = \{\tau(a)1_{\mathcal{A}}\}.$$

Our proof will differ a little from the one originally given in [30], and Mikael Rørdam must be thanked for providing us with it. The theorem, along with Proposition 2.2, yields the following consequence for reduced group C^*-algebras of discrete groups:

Corollary 2.4. *Let Γ be a discrete group. Then $C^*_r(\Gamma)$ is simple with unique trace if and only if $C^*_r(\Gamma)$ satisfies the Dixmier property.*

In order to prove the theorem, let us define first a useful term related to $O_{\mathcal{A}}(a)$.

Definition 2.5. Let $\mathcal{A}$ be a unital C^*-algebra, $\lambda_1, \dots, \lambda_n$ be positive numbers with $\sum_{i=1}^n \lambda_i = 1$ and $u_1, \dots, u_n \in \mathcal{U}(\mathcal{A})$ be unitaries. Then the linear contraction $f\colon \mathcal{A} \to \mathcal{A}$ given by

$$f(x) = \sum_{i=1}^{n} \lambda_i u_i x u_i^*, \quad x \in \mathcal{A},$$

is called an *averaging process*. The set of averaging processes on $\mathcal{A}$ is denoted by $\mathfrak{F}(\mathcal{A})$.

It is clear that $O_{\mathcal{A}}(a) = \{f(a) \,|\, f \in \mathfrak{F}(\mathcal{A})\}$ for all $a \in \mathcal{A}$. Moreover, a simple calculation shows that $f, g \in \mathfrak{F}(\mathcal{A})$ implies $g \circ f \in \mathfrak{F}(\mathcal{A})$.

Remark 2.6. Before proving Theorem 2.3, we note that it suffices to verify that $\overline{O_{\mathcal{A}}(a)}$ meets the scalars in $\mathcal{A}$ for all *self-adjoint* elements in $\mathcal{A}$. Here's the reason why: assuming that the Dixmier property holds for all self-adjoint elements in $\mathcal{A}$, let $a \in \mathcal{A}$ and write $a = a_1 + ia_2$ where $a_1, a_2 \in \mathcal{A}$ are the self-adjoint real and imaginary parts of a. Then for any $\varepsilon > 0$ there exists $f \in \mathfrak{F}(\mathcal{A})$ and $\lambda_1 \in \mathbb{C}$ such that

$$\|f(a_1) - \lambda_1 1_{\mathcal{A}}\| < \frac{\varepsilon}{2}.$$

Since $f(a_2)$ is also self-adjoint, there exists $g \in \mathfrak{F}(\mathcal{A})$ and $\lambda_2 \in \mathbb{C}$ such that

$$\|g(f(a_2)) - \lambda_2 1_{\mathcal{A}}\| < \frac{\varepsilon}{2}.$$

By defining $\lambda = \lambda_1 + i\lambda_2$, we see that

$$\|g(f(a)) - \lambda 1_{\mathcal{A}}\| \leq \|g(f(a_1) - \lambda_1 1_{\mathcal{A}})\| + \|g(f(a_2)) - \lambda_2 1_{\mathcal{A}}\| < \varepsilon,$$

as g is a unital contraction. Since $g \circ f \in \mathfrak{F}(\mathcal{A})$, it follows that $\lambda 1_{\mathcal{A}} \in \overline{O_{\mathcal{A}}(a)}$. ✺

The main idea of the proof is to pass to the enveloping von Neumann algebra and decompose it into finite and properly infinite summands. Once we prove results related to the Dixmier property for finite and properly infinite von Neumann algebras, the proof itself will then translate these results back to a result for the C^*-algebra in question.

First we prove that we only need consider a specific class of finite von Neumann algebras when dealing with the property of having a unique *normal* trace.

Lemma 2.7. *Let $\mathscr{M}$ be a finite von Neumann algebra. If $\mathscr{M}$ has a unique normal trace, then $\mathscr{M}$ is a factor.*

Proof. Let $\mathscr{C}$ denote the center of $\mathscr{M}$ and let $T\colon \mathscr{M} \to \mathscr{C}$ denote the canonical center-valued trace on $\mathscr{M}$ (cf. [40, Theorem 8.2.8]). If $\mathscr{M}$ acts on the Hilbert space $\mathcal{H}$, then for all $\xi \in (\mathcal{H})_1$ we can define a linear functional $\varphi_\xi\colon \mathscr{M} \to \mathbb{C}$ by

$$\varphi_\xi(x) = \langle T(x)\xi, \xi\rangle, \quad x \in \mathscr{M}.$$

By the properties of T, each φ_ξ is a normal trace on $\mathscr{M}$. By assumption we must then have $\varphi_\xi = \varphi_\eta$ for all $\xi, \eta \in (\mathcal{H})_1$. Supposing that $\mathscr{C} \neq \mathbb{C}1_{\mathscr{M}}$, then there must exist some non-zero projection $p \in \mathscr{C}$ with $p \neq 1_{\mathscr{M}}$. Hence we can take $\xi, \eta \in (\mathcal{H})_1$ such that $p\xi = \xi$ and $p\eta = 0$. Since $T(p) = p$ we have

$$1 = \|\xi\|^2 = \langle p\xi, \xi\rangle = \varphi_\xi(p) = \varphi_\eta(p) = \langle p\eta, \eta\rangle = 0,$$

a contradiction. Hence $\mathscr{C} = \mathbb{C}1_{\mathscr{M}}$, so $\mathscr{M}$ is a factor. □

If $a \in \mathcal{A}$ is self-adjoint, then $\sigma(a)$ is a compact subset of $\mathbb{R}$. By defining

$$\alpha(a) = \inf \sigma(a), \quad \beta(a) = \sup \sigma(a), \quad d(a) = \beta(a) - \alpha(a),$$

note that $\sigma(a) \subseteq [\alpha(a), \beta(a)]$. We employ this notation for the following results.

Lemma 2.8. *For any state φ on a unital C^*-algebra $\mathcal{A}$, it holds for all self-adjoint elements $a \in \mathcal{A}$ that $\alpha(a) \leq \varphi(a) \leq \beta(a)$ and that*

$$\|\varphi(a)1_{\mathcal{A}} - a\| \leq d(a).$$

Proof. If $a \in \mathcal{A}$ is self-adjoint, then $\alpha(a)1_{\mathcal{A}} \leq a \leq \beta(a)1_{\mathcal{A}}$. Hence if $\varphi\colon \mathcal{A} \to \mathbb{C}$ is a state, we have $\alpha(a)1_{\mathcal{A}} \leq \varphi(a)1_{\mathcal{A}} \leq \beta(a)1_{\mathcal{A}}$. Together, these two pairs of inequalities yield

$$(\alpha(a) - \beta(a))1_{\mathcal{A}} \leq \varphi(a)1_{\mathcal{A}} - a \leq (\beta(a) - \alpha(a))1_{\mathcal{A}},$$

or $\|\varphi(a)1_{\mathcal{A}} - a\| \leq \beta(a) - \alpha(a) = d(a)$. □

We now examine the two types of finite factors, using fervently that such von Neumann algebras have a unique trace (in other words, you cannot say “finite factor” without saying “unique trace”).

Proposition 2.9. *Let $\mathscr{M}$ be a I_n-factor with unique trace τ. Then $\tau(a)1_{\mathscr{M}} \in O_{\mathscr{M}}(a)$ for all $a \in \mathscr{M}$.*

Proof. Let $\{E_{jk} \mid j, k = 1, \dots, n\}$ be a system of matrix units for $\mathscr{M}$. Letting S denote the symmetric group consisting of all permutations of the set $\Omega = \{1, \dots, n\}$ and F denote the set of maps $\Omega \to \{-1, 1\}$, we define

$$V(\xi, \pi) = \sum_{i=1}^{n} \xi(i) E_{\pi(i)i} \in \mathscr{M}, \quad \xi \in F, \ \pi \in S.$$

Then $\Gamma = \{V(\xi, \pi) \mid \xi \in F, \ \pi \in S\}$ is a finite subgroup of the unitary group $\mathcal{U}(\mathscr{M})$. Moreover, if $j, k \in \Omega$, then by letting π denote some permutation of Ω mapping k to j and defining $\xi_1, \xi_2 \in F$ by $\xi_1(m) = 1$ for all $m \in \Omega$ and

$$\xi_2(m) = \begin{cases} 1 & \text{if } m = k \\ -1 & \text{else,} \end{cases}$$

we then have

$$V(\xi_1, \pi) + V(\xi_2, \pi) = (\xi_1(k) + \xi_2(k))E_{\pi(k)k} = 2E_{jk},$$

so that E_{jk} belongs to the linear span of Γ for all $j,k\in\Omega$, implying that the linear span of Γ is all of $\mathscr{M}$. Let $a\in\mathscr{M}$ and define

$$x=\frac{1}{|\Gamma|}\sum_{u\in\Gamma}uau^*.$$

Then $x\in O_{\mathscr{M}}(a)$ and for all $v\in\Gamma$, we have

$$vxv^*=\frac{1}{|\Gamma|}\sum_{u\in\Gamma}(vu)a(vu)^*=\frac{1}{|\Gamma|}\sum_{w\in\Gamma}waw^*=x,$$

since left translation by w in Γ is an isomorphism. Therefore $vx=xv$, so x commutes with everything in Γ and hence everything in $\mathscr{M}$ by linearity. Since $\mathscr{M}$ is a factor, there exists $\lambda\in\mathbb{C}$ such that $x=\lambda 1_{\mathscr{M}}$, but then $\tau(a)=\tau(x)=\lambda$, yielding what we wanted. □

Lemma 2.10. *Let $\mathscr{M}$ be a II_1-factor and let $a\in\mathscr{M}$ be self-adjoint. Then there exists $u\in\mathcal{U}(\mathscr{M})$ such that*

$$d\left(\frac{1}{2}a+\frac{1}{2}uau^*\right)\le\frac{1}{2}d(a).$$

Proof. Let $\alpha=\alpha(a)$, $\beta=\beta(a)$ and let $\tau\colon\mathscr{M}\to\mathbb{C}$ be the canonical normal trace on $\mathscr{M}$. By defining

$$B_\tau=\left\{s\in\mathbb{R}\;\middle|\;\tau(1_{[\alpha,s)}(a))\le\frac{1}{2}\right\},\quad t=\sup B_\tau$$

and letting $p_0=1_{[\alpha,t)}(a)$ and $p_1=1_{[\alpha,t]}(a)$, note that $p_0\le p_1$. Moreover, we have $\tau(p_0)\le\frac{1}{2}\le\tau(p_1)$. Indeed, observe first that $(-\infty,t)\subseteq B_\tau\subseteq(-\infty,t]$. Moreover,

$$1_{[\alpha,t-\frac{1}{n})}(a)\to 1_{[\alpha,t)}(a)\quad\text{and}\quad 1_{[\alpha,t+\frac{1}{n}]}(a)\to 1_{[\alpha,t]}(a)$$

weakly by the Borel functional calculus. As these sequences are bounded, it follows that

$$\tau(p_0)=\lim_{n\to\infty}\tau(1_{[\alpha,t-\frac{1}{n})}(a))\le\frac{1}{2},\quad \tau(p_1)=\lim_{n\to\infty}\tau(1_{[\alpha,t+\frac{1}{n})}(a))\ge\frac{1}{2}$$

by normality of τ.

As $\mathscr{M}$ is a II_1-factor, we can take a projection $q\in\mathscr{M}$ with $\tau(q)=\frac{1}{2}-\tau(p_0)$ [40, Theorem 8.5.7]. Since $\tau(q)\le\tau(p_1-p_0)$, we have $q\precsim p_1-p_0$ by the comparability theorem, so there exists a projection $q_0\in\mathscr{M}$ with $q_0\sim q$ and $q_0\le p_1-p_0$. Defining $p=p_0+q_0$, we have $\tau(p)=\frac{1}{2}$ and $p_0\le p\le p_1$. Since $x1_{\{t\}}(x)=t1_{\{t\}}(x)$ for all $x\in\mathbb{R}$, the Borel functional calculus tells us that $a(p_1-p_0)=t(p_1-p_0)$. On the grounds that $q_0\le p_1-p_0$, it now follows that $aq_0=q_0a=tq_0$, so that a commutes with q_0. Hence a also commutes with p.

We now define $a_1=ap$ and $a_2=ap^\perp$ in the following, so that $a=a_1+a_2$. It follows from the Borel functional calculus that $\alpha p_0\le ap_0\le tp_0$ and $tp_0^\perp\le ap_0^\perp\le\beta p_0^\perp$, and because $\alpha 1_{\mathscr{M}}\le a\le\beta 1_{\mathscr{M}}$, we also have $\alpha q_0\le aq_0\le\beta q_0$. Recalling that $aq_0=tq_0$, we then obtain the inequalities

$$\alpha p\le a_1\le tp,\quad tp^\perp\le a_2\le\beta p^\perp. \tag{2.3}$$

Since $\tau(p)=\tau(p^\perp)=\frac{1}{2}$, it follows that $p\sim p^\perp$. Taking a partial isometry $v\in\mathscr{M}$ such that $p=vv^*$ and $p^\perp=v^*v$, we define $u=v+v^*$. Note that $v^*=p^\perp v^*$ and hence $v=vp^\perp$, so that $vp=0$. Hence $v^2=(pv)^2=0$, and therefore

$$(v+v^*)^2=v^2+vv^*+v^*v+(v^*)^2=1_{\mathscr{M}}+v^2+(v^2)^*=1_{\mathscr{M}}.$$

Hence u is a self-adjoint unitary. We claim that u has the desired property. Indeed, since $vp=pv^*=0$, we have

$$upu=(vp+v^*p)u=v^*pu=v^*(pv+pv^*)=v^*pv=p^\perp$$

and $up^\perp u=1_{\mathscr{M}}-upu=p$. By (2.3) we then find

$$\alpha p^\perp\le ua_1u\le tp^\perp,\quad tp\le ua_2u\le\beta p. \tag{2.4}$$

Adding (2.3) and (2.4) together, we obtain

$$\alpha 1_{\mathscr{M}} \leq a_1 + ua_1u \leq t1_{\mathscr{M}}, \quad t1_{\mathscr{M}} \leq a_2 + ua_2u \leq \beta 1_{\mathscr{M}},$$

and by adding the above inequalities together as well, we find

$$(\alpha + t)1_{\mathscr{M}} \leq a + uau \leq (\beta + t)1_{\mathscr{M}}.$$

Multiplying by $\frac{1}{2}$, we finally get

$$d\left(\frac{1}{2}a + \frac{1}{2}uau\right) \leq \frac{1}{2}(\beta - \alpha) = \frac{1}{2}d(a),$$

as wanted. $\square$

Proposition 2.11. *If $\mathscr{M}$ is a II_1-factor with unique trace τ and $a \in \mathscr{M}$ is self-adjoint, then*

$$\tau(a)1_{\mathscr{M}} \in \overline{O_{\mathscr{M}}(a)},$$

the closure being in the norm.

Proof. Define $a_0 = a$. By Lemma 2.10 there exists $a_1 \in O_{\mathscr{M}}(a_0)$ such that $d(a_1) \leq \frac{1}{2}d(a_0)$. Iterating this process, we obtain a sequence $(a_n)_{n\geq 0}$ of elements in $\mathscr{M}$ satisfying $a_{n+1} \in O_{\mathscr{M}}(a_n)$ and

$$d(a_{n+1}) \leq \frac{1}{2}d(a_n)$$

for all $n \geq 0$. It follows by induction that $a_n \in O_{\mathscr{M}}(a)$ for all $n \geq 0$, and from (2.2) we then have $\tau(a_n) = \tau(a)$ for all $n \geq 0$. Hence by Lemma 2.8, we have

$$\|\tau(a)1_{\mathscr{M}} - a_n\| = \|\tau(a_n)1_{\mathscr{M}} - a_n\| \leq d(a_n) \leq \frac{1}{2^n}d(a) \to 0,$$

completing the proof. $\square$

In fact the two previous propositions combined with Proposition 2.2 now give us the following result:

Corollary 2.12. *Let $\mathscr{M}$ be a finite factor. Then $\mathscr{M}$ is simple.*

In the case of properly infinite von Neumann algebras, we turn to a new notion.

Definition 2.13. Let $\mathcal{A}$ be a unital C^*-algebra. An element $a \in \mathcal{A}$ is *full* if it is not contained in any proper, closed, two-sided ideal in $\mathcal{A}$, i.e., if $\mathfrak{I}$ is a closed two-sided ideal of $\mathcal{A}$ such that $a \in \mathfrak{I}$, then $\mathfrak{I} = \mathcal{A}$.

Various elementary results about full projections are proved in Section A.2. It is easy to check that if p is a full projection in a unital C^*-algebra $\mathcal{A}$, then all projections in $\mathcal{A}$ majorizing p or equivalent to p are full.

Lemma 2.14. *Let $\mathscr{M}$ be a properly infinite von Neumann algebra. Then for any full projection $p \in \mathscr{M}$ and any $n \geq 1$, there exist unitaries $u_1, \ldots, u_n \in \mathscr{M}$ such that*

$$\frac{1}{n}\sum_{i=1}^{n} u_i p u_i^* \geq \frac{n-1}{n}1_{\mathscr{M}}.$$

Proof. Let $p \in \mathscr{M}$ be a full projection. Since p is properly infinite in $\mathscr{M}$ by Proposition A.2.8, then by the Schröder-Bernstein theorem there exist mutually orthogonal projections $q_1, \ldots, q_{n-1} \in \mathscr{M}$ such that $q_i \sim p$ and $q_i \leq p$ for all i and $\sum_{i=1}^{n} q_n = p$. Define projections $p_i = q_i$ for $i = 1, \ldots, n-1$ and $p_n = q_n + 1_{\mathscr{M}} - p$. Since $p_n \geq q_n$ and q_n is full, it follows that p_n is full. Hence by Corollary A.2.7, we have $p_n \sim p$. Also, since $p_n \geq 1_{\mathscr{M}} - p$ we have $p_i \perp p_n$ for all $i = 1, \ldots, n-1$, so that $(p_i)_{i=1}^{n}$ is a family of mutually orthogonal equivalent projections, satisfying

$$\sum_{i=1}^{n} p_i = 1_{\mathscr{M}} \quad \text{and} \quad \sum_{i=1}^{n-1} p_i \leq p.$$

Define a function $r\colon \mathbb{N} \to \{1,\dots,n\}$ by letting $r(x)$ be the unique integer in $\{1,\dots,n\}$ such that $r(x) \equiv x \pmod n$ for $x \in \mathbb{N}$. Fixing $i \in \{1,\dots,n\}$, then for all $j = 1,\dots,n$ we take a partial isometry $v_{ij} \in \mathscr{M}$ such that

$$v_{ij}v_{ij}^* = p_{r(i+j)}, \quad v_{ij}^*v_{ij} = p_j.$$

Note that for $j,k \in \{1,\dots,n\}$ with $j \neq k$, then $r(i+j) \neq r(i+k)$ and so

$$v_{ij}v_{ik}^* = v_{ij}p_jp_kv_{ik}^* = 0, \quad v_{ij}^*v_{ik} = v_{ij}^*p_{r(i+j)}p_{r(i+k)}v_{ik} = 0.$$

Define $u_i = \sum_{j=1}^n v_{ij}$. Then each u_i is a unitary, as we have

$$u_iu_i^* = \sum_{j=1}^n\sum_{k=1}^n v_{ij}v_{ik}^* = \sum_{j=1}^n v_{ij}v_{ij}^* = \sum_{j=1}^n p_{r(i+j)} = \sum_{j=1}^n p_j = 1_{\mathscr{M}}$$

and $u_iu_i^* = 1_{\mathscr{M}}$ similarly. Observe that if $i,j \in \{1,\dots,n\}$ then

$$u_ip_ju_i^* = \sum_{k=1}^n\sum_{l=1}^n v_{ik}p_jv_{il}^* = \sum_{k=1}^n\sum_{l=1}^n v_{ik}p_jp_lv_{il}^* = \sum_{k=1}^n v_{ik}v_{ij}^* = v_{ij}v_{ij}^* = p_{r(i+j)}.$$

Therefore

$$\sum_{i=1}^n u_ipu_i^* \geq \sum_{i=1}^n u_i\left(\sum_{j=1}^{n-1} p_j\right)u_i^* = \sum_{j=1}^{n-1}\sum_{i=1}^n u_ip_ju_i^* = \sum_{j=1}^{n-1}\sum_{i=1}^n p_{r(i+j)} = \sum_{j=1}^{n-1} 1_{\mathscr{M}} = (n-1)1_{\mathscr{M}},$$

completing the proof. □

Proposition 2.15. *Let $\mathscr{M}$ be a properly infinite von Neumann algebra, and let $\mathcal{A} \subseteq \mathscr{M}$ be a simple C^*-subalgebra with $1_{\mathscr{M}} \in \mathcal{A}$. Then for all self-adjoint $a \in \mathcal{A}$, we have*

$$\overline{O_{\mathscr{M}}(a)} \cap \mathbb{C}1_{\mathscr{M}} = [\alpha(a), \beta(a)]1_{\mathscr{M}},$$

the closure of $O_{\mathscr{M}}(a)$ being in the norm.

Proof. Since $\overline{O_{\mathscr{M}}(a)} \cap \mathbb{C}1_{\mathscr{M}}$ is convex for all $a \in \mathcal{A}$, it suffices to show that $\alpha(a)1_{\mathscr{M}}, \beta(a)1_{\mathscr{M}} \in \overline{O_{\mathscr{M}}(a)}$ for all self-adjoint $a \in \mathcal{A}$. Assume first that we have shown that $\beta(a)1_{\mathscr{M}} \in \overline{O_{\mathscr{M}}(a)}$ for all *positive* $a \in \mathcal{A}$. Then for any self-adjoint $x \in \mathcal{A}$ the fact that $\beta(x) = \beta(x + \alpha(x)1_{\mathscr{M}}) - \alpha(x)$ yields $\beta(x)1_{\mathscr{M}} \in \overline{O_{\mathscr{M}}(x)}$, and since this must also hold for $-x$, we have

$$\alpha(x)1_{\mathscr{M}} = -\beta(-x)1_{\mathscr{M}} \in -\overline{O_{\mathscr{M}}(-x)} = \overline{O_{\mathscr{M}}(x)}$$

as well, so that the statement follows. Henceforth we therefore assume that $a \in \mathcal{A}$ is positive, and we need to prove that $\beta(a)1_{\mathscr{M}} \in \overline{O_{\mathscr{M}}(a)}$.

Let $\beta = \beta(a)$. If $\beta = 0$, then $a = 0$ and $\beta 1_{\mathscr{M}} \in \overline{O_{\mathscr{M}}(a)} = \{0\}$. Therefore assume that $\beta > 0$. Let $\varepsilon > 0$ with $\varepsilon < \beta$, and take $n \geq 2$ such that $n\varepsilon > \beta$. Defining

$$\varepsilon_0 = \frac{n\varepsilon - \beta}{n-1},$$

it is easy to see that $0 < \varepsilon_0 < \beta$. Now define a projection $p = 1_{[\beta-\varepsilon_0,\beta]}(a) \in \mathscr{M}$. By the Borel functional calculus, we then have

$$ap^\perp \leq (\beta - \varepsilon_0)p^\perp. \tag{2.5}$$

We now claim that p is full in $\mathscr{M}$. Indeed, suppose that $\mathfrak{J}$ is a proper, closed, two-sided ideal in $\mathscr{M}$ containing p. Then the natural quotient map $\mathscr{M} \to \mathscr{M}/\mathfrak{J}$ induces a *-homomorphism $\pi\colon \mathcal{A} \to \mathscr{M}/\mathfrak{J}$. Since $\mathcal{A}$ is simple and $\pi(1_{\mathscr{M}}) \neq 0$, π must be injective, so $\sigma(a) = \sigma(\pi(a))$. Applying π to (2.5), we see that $\pi(a) \leq (\beta - \varepsilon_0)1_{\mathscr{M}/\mathfrak{J}}$ because $\pi(p) = 0$. Hence $\sigma(a) = \sigma(\pi(a)) \subseteq (-\infty, \beta - \varepsilon_0]$, contradicting the fact that $\beta \in \sigma(a)$.

Observe that $pa = ap$ and $ap \geq (\beta - \varepsilon_0)p$ by the Borel functional calculus. Since $a = ap + ap^\perp$ and $ap^\perp \geq 0$ (a being positive), it follows that $(\beta - \varepsilon_0)p \leq ap \leq a$. Since p is full, Lemma 2.14 provides $u_1,\dots,u_n \in \mathcal{U}(\mathscr{M})$ such that

$$\frac{1}{n}\sum_{i=1}^n u_ipu_i^* \geq \frac{n-1}{n}1_{\mathscr{M}}.$$

As $a \leq \beta 1_{\mathscr{M}}$, we now have

$$\beta 1_{\mathscr{M}} \geq \frac{1}{n}\sum_{i=1}^{n} u_i a u_i^* \geq (\beta - \varepsilon_0)\frac{1}{n}\sum_{i=1}^{n} u_i p u_i^* \geq (\beta - \varepsilon_0)\frac{n-1}{n} 1_{\mathscr{M}}.$$

Defining $x = \frac{1}{n}\sum_{i=1}^{n} u_i a u_i^*$, then $x \in O_{\mathscr{M}}(a)$. The above inequalities now imply

$$0 \leq \beta 1_{\mathscr{M}} - x \leq \left(\frac{\beta}{n} + \frac{n-1}{n}\varepsilon_0\right) 1_{\mathscr{M}} = \left(\frac{\beta}{n} + \frac{n\varepsilon - \beta}{n}\right) 1_{\mathscr{M}} = \varepsilon 1_{\mathscr{M}},$$

in turn implying $\|\beta 1_{\mathscr{M}} - x\| \leq \varepsilon$. Hence $\beta 1_{\mathscr{M}} \in \overline{O_{\mathscr{M}}(a)}$, and the proof is complete. □

We are now almost ready to prove the main theorem; however we need to know what happens to operator closures in $\mathcal{A}^{**}$ of convex subsets in a C^*-algebra $\mathcal{A}$.

Lemma 2.16. *Let $\mathcal{A}$ be a C^*-algebra and consider $\mathcal{A}$ as a C^*-subalgebra of its enveloping von Neumann algebra $\mathcal{A}^{**}$. If $\mathscr{S} \subseteq \mathcal{A}$ is a convex subset, then*

$$\overline{\mathscr{S}}^{\text{ultraweak}} \cap \mathcal{A} = \overline{\mathscr{S}}^{\text{norm}}$$

in $\mathcal{A}^{**}$. If $\mathscr{S}$ is bounded, then the ultraweak closure of $\mathscr{S}$ can be replaced by the weak and strong operator closure.

Proof. The inclusion "$\supseteq$" is clear. Assume therefore that $x \in \mathcal{A} \subseteq \mathcal{A}^{**}$ belongs to the ultraweak closure of $\mathscr{S}$. Then there exists a net $(x_\alpha)_{\alpha\in A}$ in $\mathscr{S}$ such that $\omega(x_\alpha) \to \omega(x)$ for all $\omega \in (\mathcal{A}^{**})_* = \mathcal{A}^*$, so that x belongs to the $\sigma(\mathcal{A}, \mathcal{A}^*)$-closure of $\mathscr{S}$ in $\mathcal{A}$. Since $\mathscr{S}$ is convex, x belongs to the norm closure of $\mathscr{S}$ in $\mathcal{A}$ [63, Theorem 3.12]. The final statement follows immediately, as the weak operator topology, strong operator topology and the ultraweak topology coincide on bounded sets [15, Proposition 2.1]. □

The above lemma even holds for Banach spaces, if we consider the weak* topology on $\mathcal{A}^{**}$.

Proof of Theorem 2.3. Assume first that $\mathcal{A}$ has a unique trace τ. As noted in Remark 2.6, it suffices to show that

$$\tau(a)1_{\mathcal{A}} \in \overline{O_{\mathcal{A}}(a)}$$

for all self-adjoint elements $a \in \mathcal{A}$. Once that is proved, (2.2) tells us that $\tau(a)$ is the *only* $\lambda \in \mathbb{C}$ such that $\lambda 1_{\mathcal{A}} \in \overline{O_{\mathcal{A}}(a)}$. Therefore, let $a \in \mathcal{A}$ be self-adjoint. By viewing $\mathcal{A}$ as a C^*-subalgebra of its enveloping von Neumann algebra $\mathscr{M} = \mathcal{A}^{**}$ (so that $1_{\mathscr{M}} = 1_{\mathcal{A}}$), we have

$$\mathscr{M} \cong \mathscr{N} := \mathscr{M}_1 \oplus \mathscr{M}_2,$$

where $\mathscr{M}_1$ is either a finite von Neumann algebra (or $\{0\}$) and $\mathscr{M}_2$ is a properly infinite von Neumann algebra (or $\{0\}$). Let $\pi_i \colon \mathcal{A} \to \mathscr{M}_i$ denote the *-homomorphisms arising from this isomorphism for $i = 1, 2$.

Since τ is the only trace on $\mathcal{A}$, its extension $\tilde{\tau}$ to $\mathscr{M}$ is the only normal trace on $\mathscr{M}$, by $\mathcal{A}$ being ultraweakly dense in $\mathscr{M}$. Properly infinite von Neumann algebras have no traces, so $\mathscr{M}_1$ cannot be zero and must therefore be a finite von Neumann algebra. Because *-isomorphisms of von Neumann algebras are ultraweak-to-ultraweak homeomorphisms, there is only one normal trace $\tilde{\tau}'$ on $\mathscr{N}$ as well. Hence any normal trace on $\mathscr{M}_1$ is necessarily given by $x \mapsto \tilde{\tau}'((x, 0))$ for $x \in \mathscr{M}_1$, so by Lemma 2.7, $\mathscr{M}_1$ is a finite factor. Letting φ denote the unique trace of $\mathscr{M}_1$, then $\varphi \circ \pi_1$ is a trace on $\mathcal{A}$ and hence $\varphi \circ \pi_1 = \tau$ by uniqueness of τ. Regardless of the type of $\mathscr{M}_1$, it now follows from Propositions 2.9 and 2.11 that $\tau(a)1_{\mathscr{M}_1} \in \overline{O_{\mathscr{M}_1}(\pi_1(a))}$.

Supposing now that $\mathscr{M}_2$ is non-zero, then $\mathcal{A}$ being simple implies that π_2 is injective, so that $\pi_2(\mathcal{A})$ is a simple C^*-subalgebra of $\mathscr{M}_2$ and $\pi_2(1_{\mathcal{A}}) = 1_{\mathscr{M}_2}$. Moreover, π_2 preserves spectra, so Proposition 2.15 then tells us that

$$\overline{O_{\mathscr{M}_2}(\pi_2(a))} \cap \mathbb{C}1_{\mathscr{M}_2} = [\alpha(\pi_2(a)), \beta(\pi_2(a))]1_{\mathscr{M}_2} = [\alpha(a), \beta(a)]1_{\mathscr{M}_2},$$

and therefore $\tau(a)1_{\mathscr{M}_2} \in \overline{O_{\mathscr{M}_2}(\pi_2(a))}$. This proves that $\tau(a)1_{\mathscr{N}} \in \overline{O_{\mathscr{N}}((\pi_1(a), \pi_2(a))}$, and hence, by going back to $\mathscr{M}$, that

$$\tau(a)1_{\mathcal{A}} \in \overline{O_{\mathscr{M}}(a)}.$$

If $\mathscr{M}_2 = \{0\}$, then the above inclusion obviously still holds.

We are almost home, but we still need to prove that $\tau(a)1_{\mathcal{A}}$ also belongs to $\overline{O_{\mathcal{A}}(a)}$. Let $z \in O_{\mathscr{M}}(a)$. Then there exist positive numbers $\lambda_1, \dots, \lambda_n \geq 0$ summing to 1 and unitaries $u_1, \dots, u_n \in \mathcal{U}(\mathscr{M})$ such that $z = \sum_{i=1}^n \lambda_i u_i^* a u_i$. Since the set of unitaries in $\mathcal{A}$ is strongly dense in $\mathscr{M}$ by Kaplansky's density theorem [39, Corollary 5.3.7], then for all $i = 1, \dots, n$ there exists a net $(u_\nu^i)_{\nu \in I_i}$ of unitaries in $\mathcal{A}$ such that $u_\nu^i \to u_i$ strongly. For any $i = 1, \dots, n$ and $\xi, \eta \in \mathcal{H}$ where $\mathcal{H}$ is the Hilbert space on which $\mathscr{M}$ is represented, we have

$$\begin{aligned}|\langle (u_\nu^{i*} a u_\nu^i - u_i^* a u_i)\xi, \eta\rangle| &\leq |\langle (u_\nu^{i*} - u_i^*) a u_\nu^i \xi, \eta\rangle| + |\langle u_i^* a (u_\nu^i - u_i)\xi, \eta\rangle| \\ &\leq \|a u_\nu^i \xi\| \|(u_\nu^i - u_i)\eta\| + |\langle (u_\nu^i - u_i)\xi, a u_i \eta\rangle|,\end{aligned}$$

so that $u_\nu^{i*} a u_\nu^i \to u_i^* a u_i$ weakly. Hence

$$\sum_{i=1}^n \lambda_i u_\nu^{i*} a u_\nu^i \to \sum_{i=1}^n \lambda_i u_i^* a u_i = z$$

weakly (as a net indexed by the directed set $\prod_{i=1}^n I_i$). As this net consists of elements in $O_{\mathcal{A}}(a)$, we find that z belongs to the weak operator (WOT) closure of $O_{\mathcal{A}}(a)$. Therefore we have

$$\tau(a)1_{\mathcal{A}} \in \overline{O_{\mathscr{M}}(a)} \cap \mathcal{A} \subseteq \overline{O_{\mathcal{A}}(a)}^{\mathrm{WOT}} \cap \mathcal{A} = \overline{O_{\mathcal{A}}(a)}$$

by Lemma 2.16, completing the first part of the proof.

Perhaps not surprisingly, the second part is almost exactly the same. Assume that $\mathcal{A}$ does not have a trace. Considering the embedding $\mathcal{A} \subseteq \mathscr{M} = \mathcal{A}^{**}$ and letting $\mathscr{M}_1$ and $\mathscr{M}_2$ be the finite and properly infinite summands as before, note that $\mathscr{M}$ does not have a trace, as a trace on $\mathscr{M}$ would restrict to a trace on $\mathcal{A}$. If $\mathscr{M}_1$ were a finite von Neumann algebra, the center-valued trace on $\mathscr{M}_1$ would induce a trace on $\mathscr{M}_1 \oplus \mathscr{M}_2$. Hence $\mathscr{M}_1 = \{0\}$, so $\mathscr{M}$ is properly infinite. Therefore, if $a \in \mathcal{A}$ is self-adjoint, Proposition 2.15 again yields $\overline{O_{\mathscr{M}}(a)} \cap \mathbb{C}1_{\mathcal{A}} = [\alpha(a), \beta(a)]1_{\mathcal{A}}$. By virtue of what we proved in the first part, we have

$$\overline{O_{\mathcal{A}}(a)} \cap \mathbb{C}1_{\mathcal{A}} \subseteq \overline{O_{\mathscr{M}}(a)} \cap \mathbb{C}1_{\mathcal{A}} \subseteq \overline{O_{\mathcal{A}}(a)}^{\mathrm{WOT}} \cap \mathbb{C}1_{\mathcal{A}} = \overline{O_{\mathcal{A}}(a)} \cap \mathbb{C}1_{\mathcal{A}},$$

proving that $\overline{O_{\mathcal{A}}(a)} \cap \mathbb{C}1_{\mathcal{A}} = [\alpha(a), \beta(a)]1_{\mathcal{A}}$ for all self-adjoint elements $a \in \mathcal{A}$. Hence $\mathcal{A}$ has the Dixmier property. $\square$

Chapter 3

Powers groups

With the Dixmier property in hand, it is now time to give some examples of C^*-simple groups with unique trace. We will once again use the already-mentioned paper [58] by Powers as a guiding light, but just that. As Pierre de la Harpe noted in his 1985 paper [31], the proof of Powers was in fact so robust that it could be used to uncover a wide variety of positive results on C^*-simplicity and uniqueness of trace, simply by extracting the property of $\mathbb{F}_2$ that made the proof work, thereby neologizing the notion of a *Powers group*. In doing so, de la Harpe laid the foundation for a new generation of similar, but weaker definitions, still yielding C^*-simplicity and uniqueness of trace (but we will get to that).

3.1 C^*-simplicity and uniqueness of trace of Powers groups

We first give the definition of a Powers group as originally given by de la Harpe.

Definition 3.1.1. A non-trivial discrete group Γ is a *Powers group* if for any non-empty finite subset $F \subseteq \Gamma \setminus \{1\}$ and any integer $N \geq 1$ there exist a partition $\Gamma = C \sqcup D$ and elements $s_1, \ldots, s_N \in \Gamma$ such that

(i) $fC \cap C = \emptyset$ for all $f \in F$ and
(ii) $s_i D \cap s_j D = \emptyset$ for all $i, j = 1, \ldots, N$ with $i \neq j$.

Example 3.1.2. Any non-abelian free group $\mathbb{F}$ is a Powers group. Indeed, let F and N be as in the definition, and let x and y be two distinct elements of a free generating set for $\mathbb{F}$. Then there exists an integer $k \geq 1$ such that $x^k f x^{-k}$ begins and ends with a non-zero power of x; for instance, if $m(f)$ resp. $n(f)$ denote the power of x occurring at the beginning resp. the end of x after reduction for $f \in F$, take k such that

$$k > \max\{-m(f), n(f)\}$$

for all $f \in F$. If we let C be the set of words $s \in \mathbb{F}$ such that $x^k s$ does not begin with a non-zero power of x after reduction, then for all $f \in F$ and $s \in C$ the word $x^k(fs) = (x^k f x^{-k}) x^k s$ begins with a power of x, so that $fC \cap C = \emptyset$. If we now define $D = \mathbb{F} \setminus C$ and $s_i = y^i x^k$ for $i = 1, \ldots, N$, then for all i and $s \in D$, the word $s_i s$ begins with $y^i x^r$ for some non-zero integer r, so that all $s_i D$ are disjoint. Hence $\mathbb{F}$ is a Powers group.

One can obtain some useful properties of Powers group, just by using the definition.

Proposition 3.1.3. *Any Powers group Γ is icc. If Λ is a subgroup of Γ of finite index, then Λ is also a Powers group.*

Proof. To show that Γ is icc, assume for contradiction that there exists an element $f \in \Gamma \setminus \{1\}$ with finite conjugacy class F. As $1 \notin F$, then we can take a partition $\Gamma = C \sqcup D$ and elements $s_1, s_2, s_3 \in \Gamma$ such that the conditions of the definition are fulfilled. Since $s_1 D \cap s_j D = \emptyset$ for $j \in \{2, 3\}$, then for all $f \in F$ we have

$$f s_1 D \subseteq f s_j C = s_j (s_j^{-1} f s_j) C \subseteq s_j D,$$

as $s_j^{-1} f s_j \in F$ and $f' C \cap C = \emptyset$ for all $f' \in F$. Therefore $f s_1 D \subseteq s_2 D \cap s_3 D$, contradicting the assumption.

Assume now that Λ is a subgroup of Γ of finite index and let T be a left transversal for Λ in Γ, i.e., any $s \in \Gamma$ can be written $s = ts'$ for unique $t \in T$ and $s' \in \Lambda$. We define $c = |T|$. Let $F' \subseteq \Lambda \setminus \{1\}$ be a finite subset and let $N' \geq 1$ be an integer. Letting $N = cN'$, then because Γ is a Powers group there exist a partition $\Gamma = C \sqcup D$ and elements $s_1, \ldots, s_N \in \Gamma$ satisfying the conditions of the definition for

F' and N. For all $i = 1, \dots, N$, write $s_i = t_i s_i'$ for $t_i \in T$ and $s_i' \in \Lambda$. There must be at least N' of the t_i's which are equal, so by reordering the s_i's we can assume that $t_1 = \cdots = t_{N'}$. Defining $C' = \Lambda \cap C$ and $D' = \Lambda \cap D$, then for all $f' \in F'$ we have $f'C' \cap C' \subseteq f'C \cap C = \emptyset$, and for any $i, j \in \{1, \dots, N'\}$ with $i \neq j$ we have

$$s_i'D' \cap s_j'D' = t_1^{-1}(s_i D' \cap s_j D') \subseteq t_1^{-1}(s_i D \cap s_j D) = \emptyset.$$

Hence with C', D' and $s_1', \dots, s_n'$, we see that Λ is a Powers group. □

In a similar vein, we also have the following result:

Proposition 3.1.4. *Let I be a directed set and let $(\Gamma_i)_{i \in I}$ be an increasing family of subgroups of a group Γ such that $\Gamma = \bigcup_{i \in I} \Gamma_i$. If all Γ_i are Powers groups, then Γ is a Powers group.*

Proof. Let $F \subseteq \Gamma \setminus \{1\}$ be a finite subset and let $N \geq 1$. Then there exists an $i \in I$ such that $F \subseteq \Gamma_i \setminus \{1\}$, so there exists a partition $\Gamma_i = C_i \sqcup D_i$ and elements $s_1, \dots, s_N \in \Gamma_i$ such that $fC_i \cap C_i = \emptyset$ for all $f \in F$ and $s_j D_i \cap s_k D_i = \emptyset$ for all distinct $j, k = 1, \dots, N$. Letting T be a right transversal for Γ_i in Γ, then by defining $C = C_i T$ and $D = D_i T$, then $\Gamma = C \sqcup D$ is the wanted partition and $s_1, \dots, s_N$ satisfy condition (ii) of the Powers property. Hence Γ is a Powers group. □

As we shall prove now, Powers groups are also C^*-simple with unique trace. By Proposition 1.8.5, this implies that Powers groups have trivial amenable radical and are thus highly non-amenable (but this does not mean that Powers groups are easy to recognize).

The next result is due to Powers, emanating from his 1975 paper, albeit with some modifications.

Lemma 3.1.5. *Let Γ be a Powers group. If a is a self-adjoint element in the complex group ring $\mathbb{C}\Gamma$ with $\tau(a) = 0$, where τ is the canonical faithful trace on $C_r^*(\Gamma)$, then for all $N \geq 1$ there exist $s_1, \dots, s_N \in \Gamma$ such that*

$$\left\| \frac{1}{N} \sum_{i=1}^{N} \lambda_\Gamma(s_i) a \lambda_\Gamma(s_i)^* \right\| \leq \frac{2}{\sqrt{N}} \|a\|.$$

Proof. First and foremost we know that there exist $z_1, \dots, z_n \in \mathbb{C}$ and $f_1, \dots, f_n \in \Gamma$ such that $a = \sum_{i=1}^n z_i \lambda_\Gamma(f_i)$. Since $\tau(a) = 0$, we can assume that $f_i \in \Gamma \setminus \{1\}$ for all $i = 1, \dots, n$. Let $F = \{f_1, \dots, f_n\}$. By Γ being a Powers group, we can take a partition $\Gamma = C \sqcup D$ and elements $s_1, \dots, s_N \in \Gamma$ satisfying the conditions of the definition for F and N. Note that we also have $f^{-1}C \cap C = \emptyset$ for all $f \in F$. If $\eta \in \ell^2(s_j C)$ and $s \in s_j C$, then

$$s_j f_i^{-1} s_j^{-1} s \in s_j f_i^{-1} C \subseteq s_j D,$$

so that

$$\lambda_\Gamma(s_j f_i s_j^{-1}) \eta(s) = \eta(s_j f_i^{-1} s_j^{-1} s) = 0.$$

Hence $\lambda_\Gamma(s_j f_i s_j^{-1})$ maps $\ell^2(s_j C)$ into $\ell^2(s_j D)$ for all j, and it therefore follows that

$$b_j = \lambda_\Gamma(s_j) a \lambda_\Gamma(s_j)^* = \sum_{i=1}^{n} z_i \lambda_\Gamma(s_j f_i s_j^{-1})$$

maps $\ell^2(s_j C)$ into $\ell^2(s_j D)$. Letting p_j be the projection onto $\ell^2(s_j D)$ for $j = 1, \dots, N$, note that all the projections p_j are orthogonal and that $(1 - p_j) b_j (1 - p_j) = 0$, since b_j maps $\ell^2(s_j C)$ into $\ell^2(s_j D)$.

Now let $\xi \in \ell^2(\Gamma)$ with $\|\xi\| = 1$. Then we find that

$$\begin{aligned} |\langle b_j \xi, \xi \rangle| &\leq |\langle b_j \xi, p_j \xi \rangle| + |\langle b_j p_j \xi, (1 - p_j)\xi \rangle + \langle b_j (1 - p_j)\xi, (1 - p_j)\xi \rangle| \\ &\leq \|b_j\| \|p_j \xi\| + |\langle b_j p_j \xi, (1 - p_j)\xi \rangle| \\ &\leq 2\|a\| \|p_j \xi\| \end{aligned}$$

for all $j = 1, \dots, N$, as $\|(1-p_j)\xi\| \le 1$. Now define $b = \frac{1}{N}\sum_{j=1}^N b_j$. Since the projections $p_1, \dots, p_N$ are orthogonal, we then have

$$\begin{aligned} |\langle b\xi, \xi\rangle| &\le \frac{1}{N}\sum_{j=1}^N |\langle b_j\xi, \xi\rangle| \\ &\le \frac{2}{N}\|a\|\sum_{j=1}^N \|p_j\xi\| \\ &\le \frac{2}{N}\|a\|\left(\sum_{j=1}^N \|p_j\xi\|^2\right)^{1/2}\left(\sum_{j=1}^N 1\right)^{1/2} \\ &\le \frac{2}{\sqrt{N}}\|a\|. \end{aligned}$$

Since b is self-adjoint, we have $\|b\| = \sup\{|\langle b\xi, \xi\rangle| \mid \xi \in \ell^2(\Gamma), \|\xi\| = 1\}$ (cf. [73, Theorem 7.18]), completing the proof. □

Proposition 3.1.6. *The reduced group C^*-algebra $C_r^*(\Gamma)$ of a Powers group Γ satisfies the Dixmier property, so $C_r^*(\Gamma)$ is simple and has a unique trace.*

Proof. Recall from Remark 2.6 that we only need to show that every self-adjoint element satisfies the Dixmier property. Let a be a self-adjoint element of $C_r^*(\Gamma)$ and $\varepsilon > 0$. Then there exists $a_0 \in \mathbb{C}\Gamma$ such that $\|a - a_0\| < \frac{\varepsilon}{3}$. By replacing a_0 with $\frac{1}{2}(a_0 + a_0^*)$ if necessary, we can assume that a_0 is self-adjoint. Applying Lemma 3.1.5 to $\tau(a_0)1 - a_0$, there exists an averaging process $f \in \mathfrak{F}(C_r^*(\Gamma))$ such that $\|\tau(a_0)1 - f(a_0)\| = \|f(\tau(a_0)1 - a_0)\| < \frac{\varepsilon}{3}$. Therefore

$$\begin{aligned} \|\tau(a)1 - f(a)\| &\le \|\tau(a-a_0)1\| + \|\tau(a_0)1 - f(a_0)\| + \|f(a_0 - a)\| \\ &\le 2\|a - a_0\| + \|\tau(a_0)1 - f(a_0)\| < \varepsilon, \end{aligned}$$

since both τ and f are linear contractions. This shows that $\tau(a)1 \in \overline{O_{C_r^*(\Gamma)}(a)}$, so $C_r^*(\Gamma)$ satisfies the Dixmier property. □

We proceed to find our next example of Powers groups: (most) free products. Recall that if Γ_1 and Γ_2 are groups, then the *free product* $\Gamma_1 * \Gamma_2$ of Γ_1 and Γ_2 is the group of all *words* of elements in Γ_1 and Γ_2, i.e., combinations $s_1 s_2 \cdots s_n$ where each s_i is either an element of Γ_1 and Γ_2. The identity elements of Γ_1 and Γ_2 are identified and become the identity element of the free product. Any word can be *reduced* by replacing pairs $s_i s_{i+1}$, where s_i and s_{i+1} belong to the same group, by its product in that group, and then removing identity elements. Consequently, any reduced word $s \in \Gamma_1 * \Gamma_2$ is either the identity element or of the form $s = s_1 \cdots s_n$, where $s_1, \dots, s_n$ are non-identity elements of either Γ_1 and Γ_2 and satisfy the condition $s_i \in \Gamma_j \Rightarrow s_{i+1} \in \Gamma_{3-j}$ for $1 \le i \le n-1$ and $1 \le j \le 2$. In this case the *length* $\ell(s)$ is equal to n. Elements of $\Gamma_1 * \Gamma_2$ are multiplied in the obvious way.

That free products are in fact Powers groups follows from a proof originally given by Paschke and Salinas (cf. [52]):

Theorem 3.1.7. *Let Γ_1 and Γ_2 be non-trivial groups satisfying $(|\Gamma_1| - 1)(|\Gamma_2| - 1) \ge 2$. Then the free product $\Gamma_1 * \Gamma_2$ is a Powers group, and consequently $C_r^*(\Gamma_1 * \Gamma_2)$ is simple with a unique trace.*

Proof. Assume without loss of generality that $|\Gamma_2| \ge 3$. Let

$$F = \{f_1, \dots, f_m\} \subseteq (\Gamma_1 * \Gamma_2) \setminus \{1\}$$

be a finite subset and let $N \ge 1$. Take $x \in \Gamma_1$ and distinct $y_1, y_2 \in \Gamma_2$ such that none of the elements are the identity, and define $s = xy_1$ and $t = xy_2$. Then there exists an integer $k \ge 1$ of s such that the word $s^k f s^{-k}$ begins and ends (after reduction) with a non-zero power of s for any $f \in F$. For instance, we can let $k = 1 + \max\{\ell(f) \mid f \in F\}$. If $f \in F$ is a power of s, then the result is trivial; otherwise, f is (after reduction) a product of $\ell(f)$ elements of Γ_1 and Γ_2, and these factors cannot cancel out s entirely.

Let D be the set of all words $r \in \Gamma_1 * \Gamma_2$ such that $x^{-1}ts^kr$ either is the identity element or begins with a non-identity element of Γ_2 *different* from y_1 after reduction. Defining $s_i = s^its^k$ for all $i = 1, \ldots, N$, then for all such i and $r \in D$, the word $s_ir = (s^ix)(x^{-1}ts^kr)$ begins with s^ix and does *not* begin with s^j for any $j > i$. Hence for all $i, j = 1, \ldots, N$ with $i \neq j$, the sets s_iD and s_jD are disjoint.

Finally, let $C = (\Gamma_1 * \Gamma_2) \setminus D$ and let $r \in C$. Then $x^{-1}ts^kr$ either begins with a non-identity element of Γ_1 after reduction or begins with y_1. By writing

$$s^kr = y_2^{-1}(x^{-1}ts^kr),$$

we conclude that s^kr either begins with y_2^{-1} or $y_2^{-1}y_1$ and hence always begins with a non-identity element of Γ_2. Therefore if $f \in F$, then $s^kfr = (s^kfs^{-k})(s^kr)$ begins with x, so $fr \notin C$ and hence $fC \cap C = \emptyset$. Hence $\Gamma_1 * \Gamma_2$ is a Powers group. □

Remark 3.1.8. The requirement that $(|\Gamma_1| - 1)(|\Gamma_2| - 1) \geq 2$ is not only sufficient, but necessary for $\Gamma_1 * \Gamma_2$ to be a Powers group. Indeed, $\mathbb{Z}_2 * \mathbb{Z}_2$ is not icc: if a and b denote the generators of each copy of $\mathbb{Z}_2$, then the conjugacy class of $ab \in \mathbb{Z}_2 * \mathbb{Z}_2$ consists only of ab and $ba = (ab)^{-1}$. ✺

In his original exposition on Powers groups, the main discovery of de la Harpe was that in considering quite specific group actions on Hausdorff spaces, one could find lots of examples of Powers groups. The essential notion was that of a group element acting *hyperbolically.*

Definition 3.1.9. Let γ be a homeomorphism of a Hausdorff space Ω. We say that γ is *hyperbolic* if there are two distinct fixed points $s_\gamma, r_\gamma \in \Omega$ so that for any neighbourhoods S of s_γ and R of r_γ there exists an integer $N \geq 1$ such that $\gamma^n(\Omega \setminus S) \subseteq R$ for all $n \geq N$. The points s_γ and r_γ are called the *source* and *range* of γ, respectively. Two hyperbolic homeomorphisms of Ω are said to be *transverse* if they have no fixed points in common.

Note that if γ is as above, then γ has no other fixed points. Indeed, if $x \in \Omega$ were a fixed point of γ and $x \neq s_\gamma$, then there would exist a neighbourhood S of s_γ not containing x. For any neighbourhood R of r_γ there would then exist an integer $n \geq 1$ such that $\gamma^n(X \setminus S) \subseteq R$, implying $x \in R$. Hence $x = r_\gamma$.

Remark 3.1.10. If γ_1 is a hyperbolic homeomorphism of Ω and γ_2 is any homeomorphism of Ω, consider the conjugates $g_k = \gamma_2^k\gamma_1\gamma_2^{-k}$ where $k \geq 1$ is an integer. First of all, each g_k is hyperbolic with source $s_{g_k} = \gamma_2^k(s_{\gamma_1})$ and range $r_{g_k} = \gamma_2^k(r_{\gamma_1})$. Hence g_k and g_j are transverse for $k > j$ if and only if

$$\{\gamma_2^{k-j}(s_{\gamma_1}), \gamma_2^{k-j}(r_{\gamma_1})\} \cap \{s_{\gamma_1}, r_{\gamma_1}\} = \emptyset.$$

If γ_1 and γ_2 are transverse hyperbolic homeomorphisms, there exists $N \geq 1$ such that

$$\gamma_2^n(\{s_{\gamma_1}, r_{\gamma_1}\}) \subseteq \Omega \setminus \{s_{\gamma_1}, r_{\gamma_1}\}$$

for all $n \geq N$. Hence for any increasing sequence $(n_k)_{k\geq 1}$ of $\mathbb{N}$ such that $n_{k+1} \geq n_k + N$ for all $k \geq 1$, the sequence $(g_{n_k})_{k\geq 1}$ is one of pairwise transverse hyperbolic homeomorphisms. Moreover, if γ is some hyperbolic homeomorphism of Ω, then γ is at most non-transverse to two g_{n_k}. By throwing these out of the sequence, we obtain a sequence of pairwise transverse hyperbolic homeomorphisms transverse to γ. ✺

Lemma 3.1.11. *For any two transverse hyperbolic homeomorphisms γ_1, γ_2 of a Hausdorff space Ω and any neighbourhood U of r_{γ_2}, there exists $N \geq 1$ such that $\gamma_2^n\gamma_1\gamma_2^{-n}$ has source and range contained in U for all $n \geq N$.*

Proof. Since $s_{\gamma_2} \in \Omega \setminus \{s_{\gamma_1}, r_{\gamma_1}\}$, we can take $N \geq 1$ such that

$$\gamma_2^n(\{s_{\gamma_1}, r_{\gamma_1}\}) \subseteq U$$

for all $n \geq N$. Since $\gamma_2^n(s_{\gamma_1})$ and $\gamma_2^n(r_{\gamma_1})$ are the source and range of $\gamma_2^n\gamma_1\gamma_2^{-n}$, the result follows. □

Whenever we say that a group Γ acts on a Hausdorff space Ω by homeomorphisms, we mean that the map $x \mapsto \gamma x$ is a homeomorphism of Ω for any $\gamma \in \Gamma$. We say that $\gamma \in \Gamma$ is a homeomorphism for short.

Definition 3.1.12. If Γ acts by homeomorphisms on a Hausdorff space Ω and Γ contains two transverse hyperbolic homeomorphisms of Ω, the action of Γ on Ω is called *strongly hyperbolic.*

Now comes the central result in de la Harpe's paper [31]; we reformulate it as in [33].

Proposition 3.1.13. *Let Γ be a group acting by homeomorphisms on a Hausdorff space Ω satisfying the following conditions:*

(i) *The action of Γ on Ω is strongly hyperbolic.*
(ii) *For any finite subset $F \subseteq \Gamma \setminus \{1\}$, there exists a fixed point $x \in \Omega$ of some hyperbolic homeomorphism in Γ such that $fx \neq x$ for all $f \in F$.*

Then Γ is a Powers group.

Proof. Let $F \subseteq \Gamma \setminus \{1\}$ be finite and $N \geq 1$ be an integer. By hypothesis, there is a hyperbolic homeomorphism γ of Γ with $fr_\gamma \neq r_\gamma$ for all $f \in F$. In particular, by Ω being Hausdorff and Γ acting by homeomorphisms, there is a neighbourhood C_Ω of r_γ such that $fC_\Omega \cap C_\Omega = \emptyset$ for all $f \in F$. Since Γ is strongly hyperbolic, it follows from Remark 3.1.10 that there are N pairwise transverse hyperbolic homeomorphisms $\gamma_1, \ldots, \gamma_N$ transverse to γ. By Lemma 3.1.11, we can assume that the source and range of each γ_i are contained in C_Ω simply by conjugating γ_i by a large enough power of γ, in which case the $\gamma_1, \ldots, \gamma_N$ remain pairwise transverse. Now take neighbourhoods R_i of r_{γ_i} such that $R_1, \ldots, R_N$ are pairwise disjoint. By the definition of each γ_i being hyperbolic, then by replacing γ_i by a larger power of itself we can further assume that $\gamma_i(\Omega \setminus C_\Omega) \subseteq R_i$ for all $i = 1, \ldots, N$.

Fix an $x_0 \in \Omega$ and define

$$C = \{\gamma \in \Gamma \,|\, \gamma x_0 \in C_\Omega\}, \quad D = \{\gamma \in \Gamma \,|\, \gamma x_0 \in \Omega \setminus C_\Omega\}.$$

Then $C \sqcup D = \Gamma$, $fC \cap C = \{\gamma \in \Gamma \,|\, \gamma x_0 \in fC_\Omega \cap C_\Omega\} = \emptyset$ for all $f \in F$ and for any $i, j \in \{1, \ldots, N\}$ with $i \neq j$, we have

$$\gamma_i D \cap \gamma_j D = \{\gamma \in \Gamma \,|\, \gamma x_0 \in \gamma_i(\Omega \setminus C_\Omega) \cap \gamma_j(\Omega \setminus C_\Omega)\} \subseteq \{\gamma \in \Gamma \,|\, \gamma x_0 \in R_i \cap R_j\} = \emptyset.$$

Hence Γ is a Powers group. □

There are plenty of ways one can modify the conditions of the above proposition to suit the needs of a particular investigation, and we note two of the most common ways to do this.

Corollary 3.1.14. *Let Γ be a group acting by homeomorphisms on a Hausdorff space Ω such that the action is strongly hyperbolic and Γx is dense in Ω for all $x \in \Omega$. If for any finite subset $F \subseteq \Gamma \setminus \{1\}$ there exists $x \in \Omega$ such that $fx \neq x$ for all $f \in F$, then Γ is a Powers group.*

Proof. Let $F \subseteq \Gamma \setminus \{1\}$ be a finite subset and take $x \in \Omega$ such that $fx \neq x$ for all $f \in F$. By Ω being Hausdorff and Γ acting by homeomorphisms, we can take a neighbourhood U of x such that $fU \cap U = \emptyset$ for all $f \in F$. Now, since Γ is strongly hyperbolic there exists an $x_0 \in \Omega$ that is fixed by a hyperbolic homeomorphism $\gamma_0 \in \Gamma$. Moreover, since Γx_0 is dense in Ω, there exists $\gamma \in \Gamma$ such that $\gamma x_0 \in U$. Now $f(\gamma x_0) \neq \gamma x_0$ for all $f \in F$ and γx_0 is fixed by the hyperbolic homeomorphism $\gamma\gamma_0\gamma^{-1}$, so Proposition 3.1.13 applies. □

For instance, if the group action is transitive the above result applies.

Corollary 3.1.15. *Let Γ be a group acting by homeomorphisms on a Hausdorff space Ω such that all non-identity elements of Γ have only finitely many fixed points in Ω. If the action of Γ on Ω is strongly hyperbolic, then Γ is a Powers group.*

Proof. By Remark 3.1.10 there exists a sequence $(\gamma_n)_{n \geq 1}$ of transverse hyperbolic homeomorphisms in Γ. If $F \subseteq \Gamma \setminus \{1\}$ is a finite subset, then there must exist a fixed point of some γ_n that is not fixed by any element of F. Hence Γ is a Powers group by Proposition 3.1.13. □

Example 3.1.16. In a paper from 2011, de la Harpe and Préaux gave the following examples of Powers groups by means of considering group actions on trees [34, Theorem 2]:

(i) Let $\Gamma_1, \Gamma_2, \Lambda$ be groups for which there exists an injective homomorphism $\iota_1 \colon \Lambda \to \Gamma_1$ and $\iota_2 \colon \Lambda \to \Gamma_2$ such that $[\Gamma_1 : \iota_1(\Lambda)] \geq 3$ and $[\Gamma_2 : \iota_2(\Lambda)] \geq 2$. Letting N denote the normal subgroup generated by elements of the form $\iota_1(s)\iota_2(s)^{-1}$ for $s \in \Lambda$, we obtain the *free product with amalgamation*

$$\Gamma_1 *_\Lambda \Gamma_2 := \Gamma_1 * \Gamma_2 / N = \langle \Gamma_1, \Gamma_2 \,|\, \iota_1(s) = \iota_2(s) \text{ for all } s \in \Lambda \rangle.$$

If we identify Λ with $\iota_1(\Lambda)$ and $\iota_2(\Lambda)$, we can define a decreasing sequence of subsets of Λ inductively by setting $\Lambda_0 = \Lambda$ and

$$\Lambda_{i+1} = \left[\bigcap_{s \in \Gamma_1} s\Lambda s^{-1} \right] \cap \left[\bigcap_{t \in \Gamma_2} t\Lambda t^{-1} \right], \quad i \geq 0.$$

If $\Gamma_1 *_\Lambda \Gamma_2$ is countable and $\Lambda_i = \{1\}$ for some $i \geq 0$, then $\Gamma_1 *_\Lambda \Gamma_2$ is a Powers group.

(ii) Let Γ be a group, let Λ be a proper subgroup of Γ and let θ be an isomorphism of Λ onto some subgroup of Γ. If τ denotes a new symbol (i.e., an element not in Γ, we can define the *HNN extension*

$$H = \mathrm{HNN}(\Gamma, \Lambda, \theta) = \langle \Gamma, \tau \,|\, \tau^{-1} s \tau = \theta(s) \text{ for all } s \in \Lambda \rangle,$$

a construction originally devised by Higman, Neumann and Neumann in [37], into which Γ embeds naturally. Identifying Γ with a subgroup of H by means of this embedding, we then define a decreasing sequence of subsets of Λ by setting $\Lambda_0 = \Lambda$ and

$$\Lambda_i' = \Lambda_i \cap \theta(\Lambda_i), \quad \Lambda_{i+1} = \left[\bigcap_{s \in \Gamma} s\Lambda_i' s^{-1} \right] \cap \tau \left[\bigcap_{s \in \Gamma} s\Lambda_i' s^{-1} \right] \tau^{-1}, \quad i \geq 0.$$

If H is countable and $\Lambda_i = \{1\}$ for some $i \geq 0$, then H is a Powers group.

(iii) For $m, n \in \mathbb{Z}$, the *Baumslag-Solitar group* $\mathrm{BS}(m,n)$ is the group with presentation

$$\mathrm{BS}(m,n) = \langle s, t \,|\, ts^m t^{-1} = s^n \rangle = \mathrm{HNN}(\langle s \rangle, \langle s^m \rangle, s^{mk} \mapsto s^{nk}).$$

Then $\mathrm{BS}(m,n)$ is a Powers group if and only if $|m| \geq 2$, $|n| \geq 2$ and $|m| \neq |n|$.

Plenty of other examples, though somewhat complex, are given in [32].

3.2 Non-elementary subgroups of $\mathrm{PSL}(2,\mathbb{R})$

We will now find examples of Powers groups where the Powers property can be obtained with the results about group actions from the previous section. As it turns out, this requires some knowledge of hyperbolic geometry – Möbius transformations in particular. For the sake of completeness, we will provide the reader with a recap of the most important facts about these and then go on to bring their connection to Powers groups to light.

For any field $\mathbb{K}$ of characteristic zero and $n \geq 1$, the *general linear group* $\mathrm{GL}(n,\mathbb{K})$ is the group of all invertible $n \times n$ matrices. We let $\mathrm{GL}(n,\mathbb{K})_{>0}$ denote the invertible matrices with positive determinant. Restricting the determinant function to $\mathrm{GL}(n,\mathbb{K})$ yields a multiplicative group homomorphism $\det \colon \mathrm{GL}(n,\mathbb{K}) \to \mathbb{K} \setminus \{0\}$, the kernel of which is called the *special linear group* $\mathrm{SL}(n,\mathbb{K})$. Hence an invertible matrix A over $\mathbb{K}$ belongs to $\mathrm{SL}(n,\mathbb{K})$ if and only if A has determinant 1.

In the following, let $\mathbb{H}$ denote the upper half-plane $\{z \in \mathbb{C} \,|\, \mathrm{Im}\, z > 0\}$ and let $\partial\mathbb{H}$ denote the one-point compactification $\mathbb{R} \cup \{\infty\}$ of $\mathbb{R}$. It is well-known that $\partial\mathbb{H}$ is then homeomorphic to the unit circle $\partial\mathbb{D} = \{z \in \mathbb{C} \,|\, |z| = 1\}$ by means of the stereographic projection

$$z \mapsto \begin{cases} \frac{\mathrm{Re}\, z}{1 - \mathrm{Im}\, z} & \text{if } z \neq i, \\ \infty & \text{if } z = i, \end{cases} \qquad z \in \partial\mathbb{D}.$$

If $\mathbb{C} \cup \{\infty\}$ is the one-point compactification of $\mathbb{C}$ (or the *extended complex plane*), then the subspace topology of $\mathbb{R} \cup \{\infty\}$ as a subspace of $\mathbb{C} \cup \{\infty\}$ coincides with the topology of the one-point compactification of $\mathbb{R}$. Hence we can view $\partial\mathbb{H}$ as a subspace of $\mathbb{C} \cup \{\infty\}$. Lastly, we define the *extended upper half plane* $\tilde{\mathbb{H}} = \mathbb{H} \cup \partial\mathbb{H} \subseteq \mathbb{C} \cup \{\infty\}$.

It will be useful for us to define some elementary operations on $\mathbb{C} \cup \{\infty\}$:

$$\frac{a}{0} = \infty, \ \frac{a}{\infty} = 0, \ b + \infty = \infty, \quad a \in \mathbb{C} \setminus \{0\}, \ b \in \mathbb{C}.$$

For any $A \in \mathrm{GL}(2, \mathbb{C})$, we consider its corresponding *Möbius transformation*

$$\gamma(z) = \frac{az+b}{cz+d}, \quad \text{for } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{3.2.1}$$

The domain of γ is usually the extended complex plane $\mathbb{C} \cup \{\infty\}$ (with the topology of the one-point compactification of $\mathbb{C}$), and we define

$$\gamma\left(\frac{-d}{c}\right) = \infty, \quad \gamma(\infty) = \frac{a}{c}$$

in accordance with the extended division on $\mathbb{C} \cup \{\infty\}$ (note that the numbers $\frac{-d}{c}$ and $\frac{a}{c}$ rely only on this division if $c = 0$, in which case $a \neq 0$ and $d \neq 0$). It is a standard result from complex analysis that γ is a homeomorphism of $\mathbb{C} \cup \{\infty\}$.

If we restrict our attention to $\mathrm{GL}(2, \mathbb{R})$ and assume that $A \in \mathrm{GL}(2, \mathbb{R})$ has strictly positive determinant, then γ as defined above maps $\mathbb{H}$ bijectively onto $\mathbb{H}$ and $\partial\mathbb{H}$ bijectively onto $\partial\mathbb{H}$; in particular, γ is a homeomorphism of $\mathbb{H}$ and of $\partial\mathbb{H}$. The set of Möbius transformations on $\tilde{\mathbb{H}}$ arising from such matrices is denoted by $\mathrm{M\ddot{o}b}(\mathbb{H})$, i.e.,

$$\mathrm{M\ddot{o}b}(\mathbb{H}) = \left\{ \gamma \colon \tilde{\mathbb{H}} \to \tilde{\mathbb{H}} \;\middle|\; \gamma(z) = \frac{az+b}{cz+d}, \ a, b, c, d \in \mathbb{R}, \ ad - bc > 0 \right\},$$

and the usual composition turns $\mathrm{M\ddot{o}b}(\mathbb{H})$ into a group. The map $A \mapsto \gamma$ given as above then yields a surjective group homomorphism $\mathrm{GL}(2, \mathbb{R})_{>0} \to \mathrm{M\ddot{o}b}(\mathbb{H})$ for which the kernel consists of all real scalar matrices. In particular, since $\mathrm{SL}(2, \mathbb{R})$ maps onto $\mathrm{M\ddot{o}b}(\mathbb{H})$ under the above isomorphism, we obtain a group isomorphism $\mathrm{SL}(2, \mathbb{R})/\{\pm 1\} \to \mathrm{M\ddot{o}b}(\mathbb{H})$. We then define the *projective special linear group* $\mathrm{PSL}(2, \mathbb{R})$ by

$$\mathrm{PSL}(2, \mathbb{R}) = \mathrm{SL}(2, \mathbb{R})/\{\pm 1\}.$$

Note that $\mathrm{SL}(2, \mathbb{R})$ can be topologized simply by letting it inherit the topology from $\mathbb{R}^4$, in which $\mathrm{SL}(2, \mathbb{R})$ is the closed subspace obtained by taking the pre-image of $\{1\}$ under the determinant homomorphism. We then endow $\mathrm{PSL}(2, \mathbb{R})$ with the quotient topology, under which it becomes a locally compact group, and $\mathrm{PSL}(2, \mathbb{R})$ acts by homeomorphisms on $\tilde{\mathbb{H}}$ by virtue of the one-to-one correspondence with Möbius transformations in $\mathrm{M\ddot{o}b}(\mathbb{H})$.

We now consider fixed points of $\gamma \in \mathrm{M\ddot{o}b}(\mathbb{H})$ where γ is not the identity. There are two cases:

(i) If $\infty \in \partial\mathbb{H}$ is a fixed point, then because $\gamma(\infty) = \frac{a}{c}$ we must have $c = 0$. This implies that

$$\gamma(z) = \frac{a}{d}z + \frac{b}{d},$$

and thus that $-\frac{b}{d-a}$ is a fixed point, which may or may not also be ∞ since it is possible that $a = d$. If this is the case, note that $b \neq 0$ since we assumed that γ was not the identity, in which case $\frac{-b}{d-a}$ is well-defined.

(ii) If ∞ is not a fixed point, then $c \neq 0$. We then have that $z_0 \in \mathbb{H} \cup \partial\mathbb{H}$ is a fixed point if and only if

$$az_0 + b = z_0(cz_0 + d) \Leftrightarrow cz_0^2 + (d-a)z_0 - b = 0.$$

(Note that $cz_0 + d \neq 0$, since $z_0 \neq -\frac{d}{c}$ by assumption.) Note that because γ arises from a real matrix, this quadratic equation has either (a) two real solutions, (b) one real solution or (c) two non-real complex conjugate solutions (of which only one belongs to $\mathbb{H}$), in all cases given by

$$z_0 = \frac{(a-d) \pm \sqrt{(a-d)^2 + 4bc}}{2c}.$$

This completely determines the possible fixed point properties of γ, and it is appropriate to split Möbius transformations into three *types*:

Definition 3.2.1. Let $\gamma \in \text{Möb}(\mathbb{H})$.

(a) If γ has two fixed points in $\partial\mathbb{H}$ and none in $\mathbb{H}$, we say that γ is *hyperbolic* (we will see in a moment that this coincides with our original definition of hyperbolicity, but for now, this is a different notion).
(b) If γ has one fixed point in $\partial\mathbb{H}$ and none in $\mathbb{H}$, we say that γ is *parabolic*.
(c) If γ has one fixed point in $\mathbb{H}$ and none in $\partial\mathbb{H}$, we say that γ is *elliptic*.

For all $A \in \text{SL}(2,\mathbb{R})$, let us now consider the *trace*

$$\text{tr}(A) = a + d, \quad \text{for } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Suppose that $\gamma \in \text{Möb}(\mathbb{H})$ corresponds to A. Then:

(i) If ∞ is a fixed point of γ, then $c = 0$ and $ad = ad - bc = 1$. We have seen that $a = d$ if and only if ∞ is the only fixed point, and in this case we have $|\text{tr}(A)| = 2$. If $a \neq d$, then $|\text{tr}(A)| = |a + \frac{1}{a}| > 2$.
(ii) If ∞ is not a fixed point of γ, then $c \neq 0$. We then know that

$$\text{tr}(A)^2 - 4 = (a+d)^2 - 4 = (a-d)^2 + 4ad - 4 = (a-d)^2 + 4bc.$$

In short, we have the following:

Proposition 3.2.2. *Let $A \in \text{SL}(2,\mathbb{R})$ and let $\gamma \in \text{Möb}(\mathbb{H})$ be the Möbius transformation corresponding to A. Assume that γ is not the identity map. Then the following holds:*

(i) *γ is hyperbolic (in the sense of Definition 3.2.1) if and only if $|\text{tr}(A)| > 2$.*
(ii) *γ is parabolic if and only if $|\text{tr}(A)| = 2$.*
(iii) *γ is elliptic if and only if $|\text{tr}(A)| < 2$.*

We say that $s \in \text{PSL}(2,\mathbb{R})$ is *hyperbolic* (resp. *parabolic*, *elliptic*) if the unique homeomorphism $\gamma \in \text{Möb}(\mathbb{H})$ corresponding to s is hyperbolic (resp. parabolic, elliptic). The function $A \mapsto |\text{tr}(A)|$ on $\text{SL}(2,\mathbb{R})$ clearly factors through $\text{PSL}(2,\mathbb{R})$ and thus becomes a continuous map on $\text{PSL}(2,\mathbb{R})$, so the above result tells us that the hyperbolic elements and elliptic elements in $\text{PSL}(2,\mathbb{R})$ constitute open sets, and that the set of parabolic elements in $\text{PSL}(2,\mathbb{R})$ is closed.

We now ask what hyperbolicity on $\tilde{\mathbb{H}}$ means from a topological point of view. A particular type of Möbius transformation on $\tilde{\mathbb{H}}$ is a *dilation*, namely a map of the form $z \mapsto kz$ where $k > 0$ and $k \neq 1$. Note that dilations have fixed points 0 and ∞. In fact, one can characterize all Möbius transformations with two fixed points by means of dilations:

Lemma 3.2.3. *Let $\gamma \in \text{Möb}(\mathbb{H})$. Then γ is hyperbolic in the sense of Definition 3.2.1 if and only if there exists $\gamma_1, \gamma_2 \in \text{Möb}(\mathbb{H})$ such that $\gamma = \gamma_1^{-1}\gamma_2\gamma$ and γ_2 is a dilation.*

Proof. It is easily seen that conjugation of Möbius transformations preserves the number of fixed points, from which the "if" implication clearly follows. We now remark that if γ' is a Möbius transformation fixing 0 and ∞, then γ' is a dilation. Indeed, if $\gamma'(z) = \frac{az+b}{cz+d}$, then it is clear that $c = 0$ and $b = 0$, and assuming that γ' is not the identity, then $\frac{a}{d} \neq 1$.

Supposing now that $\gamma \in \text{Möb}(\mathbb{H})$ fixes two points $z_1, z_2 \in \partial\mathbb{H}$, then there are two cases:

(i) $z_1 \in \mathbb{R}$ and $z_2 = \infty$. Define $\gamma_1(z) = z - z_1$.
(ii) $z_1, z_2 \in \mathbb{R}$. Assuming that $z_1 < z_2$, define

$$\gamma_1(z) = \frac{z - z_2}{z - z_1}.$$

In both cases, $\gamma_1 \in \text{Möb}(\mathbb{H})$. As $\gamma_1\gamma\gamma_1^{-1}$ has fixed points 0 and ∞, it is a dilation. $\square$

Let $0 < k < 1$, define $\gamma(z) = kz$ for $z \in \tilde{\mathbb{H}}$ and let R and S be open neighbourhoods of 0 and ∞ respectively in $\tilde{\mathbb{H}}$. We then know that $S = \tilde{\mathbb{H}} \setminus K$ for some compact set $K \subseteq \mathbb{C}$, and therefore

$$\gamma^n(\tilde{\mathbb{H}} \setminus S) = \gamma^n(K \cap (\mathbb{H} \cup \mathbb{R})) = k^n(K \cap (\mathbb{H} \cup \mathbb{R})).$$

It is then clear that there exists $N \geq 1$ such that $n \geq N$ implies $\gamma^n(\tilde{\mathbb{H}} \setminus S) \subseteq R$. Therefore γ is a hyperbolic homeomorphism of $\tilde{\mathbb{H}}$ in the sense of Definition 3.1.9, with source ∞ and range 0. Since the inverse of a hyperbolic homeomorphism is also hyperbolic, we then know that any dilation is a hyperbolic homeomorphism of $\tilde{\mathbb{H}}$. Since conjugation by any homeomorphism preserves hyperbolicity, Lemma 3.2.3 yields the following result:

Proposition 3.2.4. *Let $\gamma \in \text{Möb}(\mathbb{H})$. Then γ is hyperbolic in the sense of Definition 3.2.1 if and only if γ is a hyperbolic homeomorphism on $\tilde{\mathbb{H}}$ in the sense of Definition 3.1.9.*

We now consider another way of visualizing $\mathbb{H}$. The *Poincaré disc* $\mathbb{D}$ is the unit ball $\{z \in \mathbb{C} \mid |z| < 1\}$, and $\partial\mathbb{D}$ denotes the boundary of $\mathbb{D}$ or simply the unit circle. The map $h\colon \tilde{\mathbb{H}} \to \overline{\mathbb{D}}$ given by

$$h(z) = \begin{cases} \frac{z-i}{iz-1} & \text{if } z \in \mathbb{H} \cup \mathbb{R} \\ -i & \text{if } z = \infty \end{cases}$$

is then a homeomorphism. The point $i \in \overline{\mathbb{D}}$ corresponds to $0 \in \tilde{\mathbb{H}}$, and the real axis is the boundary of $\mathbb{D} \setminus \{1\}$, surrounding the Poincaré disc from the top down. The closer one goes to $-i$ inside $\overline{\mathbb{D}}$, the further one tends to ∞ inside $\tilde{\mathbb{H}}$. If $\gamma \in \text{Möb}(\mathbb{H})$ is given by $\gamma(z) = \frac{az+b}{cz+d}$, then $\gamma' = h\gamma h^{-1}$ is given by

$$\gamma'(z) = \frac{\alpha z + \beta}{\bar{\beta} z + \bar{\alpha}}, \quad z \in \overline{\mathbb{D}} \tag{3.2.2}$$

where $\alpha = \frac{a+d}{2} + \frac{b+c}{2} i$ and $\beta = -\frac{b+c}{2} - \frac{a-d}{2} i$, and these numbers satisfy the relation

$$|\alpha|^2 - |\beta|^2 = ad - bc > 0.$$

Then γ' is a homeomorphism and it maps $\mathbb{D}$ bijectively to $\mathbb{D}$ and $\partial\mathbb{D}$ to $\partial\mathbb{D}$. We say that γ' is a Möbius transformation of $\overline{\mathbb{D}}$, and the set of these is denoted by Möb($\mathbb{D}$). Some tedious calculations show that

$$\text{Möb}(\mathbb{D}) = \left\{ \gamma'\colon \overline{\mathbb{D}} \to \overline{\mathbb{D}} \;\middle|\; \gamma'(z) = \frac{\alpha z + \beta}{\bar{\beta} z + \bar{\alpha}},\ \alpha, \beta \in \mathbb{C},\ |\alpha|^2 - |\beta|^2 > 0 \right\},$$

as we have for any γ' given by (3.2.2), where $\alpha, \beta \in \mathbb{C}$ with $|\alpha|^2 - |\beta|^2 > 0$, that $h^{-1}\gamma' h \in \text{Möb}(\mathbb{H})$ and that $h^{-1}\gamma' h$ corresponds to the matrix $B \in \text{SL}(2, \mathbb{R})$ given by

$$\begin{pmatrix} \operatorname{Re}\alpha - \operatorname{Im}\beta & \operatorname{Im}\alpha - \operatorname{Re}\beta \\ -\operatorname{Im}\alpha - \operatorname{Re}\beta & \operatorname{Re}\alpha + \operatorname{Im}\beta \end{pmatrix}.$$

One of the bigger advantages of working with the Poincaré disc instead of the upper half-plane is that Möbius transformations on the Poincaré disc correspond to “nicer” matrices. Indeed, in a similar manner to what we have already seen, the expression (3.2.2) yields a surjective group homomorphism of the subgroup

$$\text{SU}(1,1) = \left\{ \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \;\middle|\; \alpha, \beta \in \mathbb{C},\ |\alpha|^2 - |\beta|^2 = 1 \right\}$$

of GL(2, ℂ) onto Möb($\mathbb{D}$). The kernel of this homomorphism is $\{\pm 1\}$, and $\text{SU}(1,1)/\{\pm 1\}$ is therefore isomorphic to PSL(2, ℝ).

Finally, fixed point properties of Möbius transformations in Möb($\mathbb{H}$) also transfer to Möb($\mathbb{D}$) in a nice way. Let us say that $\gamma' \in \text{Möb}(\mathbb{D})$ is *hyperbolic* (resp. *parabolic*, *elliptic*) if $h^{-1}\gamma' h \in \text{Möb}(\mathbb{H})$ is hyperbolic (resp. parabolic, elliptic). Then we have:

(i) γ' is hyperbolic if and only if γ' has two fixed points in $\partial\mathbb{D}$ and none in $\mathbb{D}$.
(ii) γ' is parabolic if and only if γ' has one fixed point in $\partial\mathbb{D}$ and none in $\mathbb{D}$.
(iii) γ' is elliptic if and only if γ' has no fixed points in $\partial\mathbb{D}$ and one in $\mathbb{D}$.

Moreover, if $A \in \text{SU}(1,1)$ and $\gamma' \in \text{Möb}(\mathbb{D})$ is defined as in (3.2.2), then $\operatorname{tr}(A) = 2\operatorname{Re}\alpha$. Since the trace of $h^{-1}\gamma h \in \text{Möb}(\mathbb{H})$ is also $2\operatorname{Re}\alpha$ by what we saw earlier, we conclude that

(i) γ' is hyperbolic if and only if $|\mathrm{tr}(A)| > 2$.
(ii) γ' is parabolic if and only if $|\mathrm{tr}(A)| = 2$.
(iii) γ' is elliptic if and only if $|\mathrm{tr}(A)| < 2$.

The bottom line is this: if we are to consider how elements of $\mathrm{PSL}(2, \mathbb{R})$ act on $\tilde{\mathbb{H}}$ by Möbius transformations, we can just as well consider the action of the corresponding Möbius transformations on $\overline{\mathbb{D}}$. Indeed, we have constructed the Poincaré disc model by means of the map h, which allows us to jump back and forth between maps on $\tilde{\mathbb{H}}$ and maps on $\overline{\mathbb{D}}$ without any hesitation.

We will henceforth identify matrices in $\mathrm{PSL}(2, \mathbb{R})$ with their corresponding Möbius transformations. For instance, saying that a matrix fixes some point simply means that the corresponding Möbius transformation fixes this point. Translating Lemma 3.2.3 to $\mathrm{PSL}(2, \mathbb{R})$, a matrix $s \in \mathrm{PSL}(2, \mathbb{R})$ is hyperbolic if and only if it is conjugate in $\mathrm{PSL}(2, \mathbb{R})$ to a matrix of the form

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \tag{3.2.3}$$

for some $\lambda > 0$ with $\lambda \neq 1$.

We note two small results before going any further, characterizing the parabolic and elliptic elements of $\mathrm{Möb}(\mathbb{H})$.

Lemma 3.2.5. *Let $\gamma \in \mathrm{Möb}(\mathbb{H})$. Then γ is parabolic if and only if there exist $\gamma_1, \gamma_2 \in \mathrm{Möb}(\mathbb{H})$ such that $\gamma_2(z) = z + k$ for some non-zero $k \in \mathbb{R}$ and $\gamma = \gamma_1^{-1}\gamma_2\gamma_1$.*

Proof. If γ_2 is given as above, then the only fixed point of γ_2 is ∞, so because any $\gamma_1 \in \mathrm{Möb}(\mathbb{H})$ maps $\partial\mathbb{H}$ bijectively onto $\partial\mathbb{H}$, it follows that $\gamma_1^{-1}\gamma_2\gamma_1$ is parabolic. Conversely assume that γ is parabolic. Then γ has one fixed point, say $s_0 \in \partial\mathbb{H}$. If $s_0 \neq \infty$, we can define a Möbius transformation

$$\gamma_1(z) = \frac{z + 1 - s_0}{z - s_0}, \quad z \in \tilde{\mathbb{H}}$$

taking s_0 to ∞. In any case, there exists $\gamma_1 \in \mathrm{Möb}(\mathbb{H})$ such that $\gamma_1\gamma\gamma_1^{-1} \in \mathrm{Möb}(\mathbb{H})$ fixes only ∞. Now write

$$\gamma_1\gamma\gamma_1^{-1}(z) = \frac{az + b}{cz + d}$$

for real numbers a, b, c, d with $ad - bc = 1$. Since ∞ is fixed, we must have $c = 0$. We must also have $b \neq 0$, since 0 is not a fixed point. Since $\gamma_1\gamma\gamma_1^{-1}$ then has a fixed point at $\frac{b}{d-a}$, we must have $a = d$, so

$$\gamma_1\gamma\gamma_1^{-1}(z) = z + k$$

for some non-zero $k \in \mathbb{R}$. □

Consequently, a matrix $s \in \mathrm{PSL}(2, \mathbb{R})$ is parabolic if and only if it is conjugate to

$$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$$

for some non-zero $k \in \mathbb{R}$.

Remark 3.2.6. Parabolic transformations have a very useful limit property. Let $\gamma \in \mathrm{Möb}(\mathbb{H})$ be a parabolic transformation and let $\gamma_1, \gamma_2 \in \mathrm{Möb}(\mathbb{H})$ such that $\gamma_2(z) = z + k$ for some $k \in \mathbb{R} \setminus \{0\}$ and $\gamma = \gamma_1\gamma_2\gamma_1^{-1}$. Then $\gamma_2^n(z) = z + nk \to \infty$ for all $z \in \tilde{\mathbb{H}}$, so since γ has one fixed point, namely $z_0 = \gamma_1(\infty)$, we conclude that $\gamma^n(z)$ converges to its fixed point z_0 for all $z \in \tilde{\mathbb{H}}$. ✺

Lemma 3.2.7. *Let $\gamma' \in \mathrm{Möb}(\mathbb{D})$. Then γ' is elliptic if and only if there exist $\gamma_1, \gamma_2 \in \mathrm{Möb}(\mathbb{D})$ such that $\gamma_2(z) = kz$ for some $k \in \partial\mathbb{D} \setminus \{1\}$ and $\gamma' = \gamma_1^{-1}\gamma_2\gamma_1$.*

Proof. Any γ_2 as described has 0 as its only fixed point. Since γ_1 maps $\mathbb{D}$ onto $\mathbb{D}$ bijectively, $\gamma_1\gamma_2\gamma_1^{-1}$ has a fixed point in $\mathbb{D}$ and is therefore elliptic. Conversely, assume that $\gamma' \in \mathrm{Möb}(\mathbb{D})$ is elliptic and let

$z_0 \in \mathbb{D}$ denote its fixed point. To obtain a Möbius transformation of $\overline{\mathbb{D}}$ mapping z_0 to 0, we pass to the upper half-plane $\mathbb{H}$. Letting $y_0 = h^{-1}(z_0) \in \mathbb{H}$, we can write $y_0 = a + ib$ for real numbers $a > 0$. Then

$$\delta(z) = \frac{z-a}{b}$$

is a Möbius transformation on $\tilde{\mathbb{H}}$ mapping y_0 to i, so that $\gamma_1 = h\gamma h^{-1} \in \text{Möb}(\mathbb{D})$ maps z_0 to 0. Therefore $\gamma_1\gamma'\gamma_1^{-1}$ is a Möbius transformation on $\overline{\mathbb{D}}$ with 0 as its only fixed point. Write

$$\gamma_1\gamma'\gamma_1^{-1}(z) = \frac{\alpha z + \beta}{\bar{\beta} z + \bar{\alpha}}$$

for $\alpha, \beta \in \mathbb{C}$ with $|\alpha|^2 - |\beta|^2 = 1$. Since 0 is fixed, we must have $\beta = 0$. Defining $k = \alpha/\bar{\alpha}$, then $|k| = 1$, and $k \neq 1$ since $\gamma_1\gamma'\gamma_1^{-1}$ is not the identity. The proof is complete. □

If $\theta \in (0, \pi)$ and $k = e^{2i\theta}$, then the Möbius transformation $z \mapsto kz$ on $\overline{\mathbb{D}}$ can be represented by the matrix in SU(1, 1) given by

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}.$$

Passing back to $\text{PSL}(2, \mathbb{R})$, we see that a matrix $s \in \text{PSL}(2, \mathbb{R})$ is elliptic if and only if it is conjugate to

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

for some $\theta \in (0, \pi)$.

Definition 3.2.8. A subgroup Γ of $\text{PSL}(2, \mathbb{R})$ is called *elementary* if there exists $x \in \tilde{\mathbb{H}}$ such that the orbit Γx is finite.

Note that because any element of $\text{PSL}(2, \mathbb{R})$ maps $\mathbb{H}$ onto $\mathbb{H}$ and $\partial\mathbb{H}$ onto $\partial\mathbb{H}$, any orbit of a point in $\tilde{\mathbb{H}}$ must either belong to $\mathbb{H}$ or $\partial\mathbb{H}$. It is easy to check that if $s \in \text{PSL}(2, \mathbb{R})$ and Γ is a subgroup of $\text{PSL}(2, \mathbb{R})$, then Γ is elementary if and only if $s\Gamma s^{-1}$ is elementary.

If Γ is a group, the *commutator* of two elements $s, t \in \Gamma$ is the element

$$[s, t] = sts^{-1}t^{-1}.$$

Supposing that Γ is a subgroup of $\text{PSL}(2, \mathbb{R})$, then it is easy to see that $\text{tr}[s, t]$ does not depend on the matrices in $\text{SL}(2, \mathbb{R})$ chosen to represent $s, t \in \Gamma$.

Proposition 3.2.9. *Let Γ be a non-trivial subgroup of* $\text{PSL}(2, \mathbb{R})$ *that contains only elliptic elements beside the identity element. Then all elements of Γ have the same fixed point, and Γ is an abelian, elementary subgroup.*

Proof. It will be convenient here to use the Poincaré disc model. As Γ contains at least one elliptic element s, fixing some element of $\mathbb{D}$, we can assume that s fixes 0, simply by conjugating Γ by an appropriate element (cf. Lemma 3.2.7). Take $t \in \Gamma$ with $t \neq s$ and write

$$s = \begin{pmatrix} \alpha & 0 \\ 0 & \overline{\alpha} \end{pmatrix}, \quad t = \begin{pmatrix} \omega_1 & \overline{\omega_2} \\ \omega_2 & \overline{\omega_1} \end{pmatrix},$$

where $\alpha, \omega_1, \omega_2 \in \mathbb{C}$ satisfy the relations $|\alpha| = 1$ and $|\omega_1|^2 - |\omega_2|^2 = 1$. Then one can show that $\text{tr}[s, t] = 2 + 4|\omega_2|^2(\text{Im}\,\alpha)^2$. As Γ contains no hyperbolic elements, we note that $|\text{tr}[s, t]| \leq 2$, implying that either $\omega_2 = 0$ or that $\alpha \in \mathbb{R}$. The latter of these can be ruled out right away, since s is not the identity map. Hence

$$t = \begin{pmatrix} \omega_1 & 0 \\ 0 & \overline{\omega_1} \end{pmatrix},$$

so t also fixes 0. Hence all elements of Γ have the same fixed point.

If we still assume that all non-identity elements of Γ fix 0, then all the Möbius transformations on $\overline{\mathbb{D}}$ corresponding to elements s of Γ must be of the form $z \mapsto k(s)z$ for some $k(s) \in \partial\mathbb{D}$. The map $\Gamma \to \partial\mathbb{D}$ given by $s \mapsto k(s)$ is an injective group homomorphism, so Γ is abelian. Finally, the Γ-orbit of 0 is $\{0\}$, so Γ is elementary. □

Lemma 3.2.10. *Two non-identity elements in* $\mathrm{PSL}(2,\mathbb{R})$ *commute if and only if they have exactly the same set of fixed points.*

Proof. Let s, t be commuting non-identity elements of $\mathrm{PSL}(2,\mathbb{R})$ (so that they have at most two fixed points). If z is a fixed point of t, then $sz = stz = tsz$, so that sz is also a fixed point of t and therefore s maps the set of fixed points of t to itself. In particular, if t has only one fixed point z_0, then z_0 is a fixed point of s. Since t also maps the fixed point set of s to itself, then if s had one other fixed point z_1, then t would fix z_1 as well, a contradiction. Therefore, if $s, t \in \mathrm{PSL}(2,\mathbb{R})$ are commuting non-identity elements and t has only one fixed point, then s has the same fixed point set as t.

Recall that for all $z \in \tilde{\mathbb{H}}$, z is a fixed point of t if and only if rz is a fixed point of rtr^{-1}. If t has two fixed points, then t is hyperbolic and is therefore conjugate to a matrix u of the form (3.2.3) for some $\lambda > 0$ with $\lambda \neq 1$ by Lemma 3.2.3, so we can let $r \in \mathrm{PSL}(2,\mathbb{R})$ such that $rtr^{-1} = u$. Writing

$$rsr^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

then because rsr^{-1} and u commute, we have

$$\begin{pmatrix} \lambda a & \lambda^{-1} b \\ \lambda c & \lambda^{-1} d \end{pmatrix} = \begin{pmatrix} \lambda a & \lambda b \\ \lambda^{-1} c & \lambda^{-1} d \end{pmatrix}.$$

Since $\lambda > 0$ and $\lambda \neq 1$, we must have $b = c = 0$, so rsr^{-1} has fixed points 0 and ∞ which are exactly the fixed points of u. Hence s and t have equal fixed point sets.

For the converse, assume that two non-identity elements $s, t \in \mathrm{PSL}(2,\mathbb{R})$ have equal fixed point sets. Then s and t are of the same type, so from Lemmas 3.2.3, 3.2.5 and 3.2.7 we know that s and t are conjugate to matrices of the three types

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

where $\lambda > 0$ with $\lambda \neq 1$, $k \in \mathbb{R}$ with $k \neq 0$ and $\theta \in (0, \pi)$ (in the sense that s is conjugate to a matrix of one of the above types iff t is). It is easy to see that matrices of each of these types commute with matrices of the same type, and therefore s and t commute. □

Lemma 3.2.11. *If* s *and* t *are matrices in* $\mathrm{PSL}(2,\mathbb{R})$ *with exactly one fixed point in common and* s *is hyperbolic, then the commutator* $[s, t]$ *is parabolic.*

Proof. Let $z \in \partial\mathbb{H}$ denote the mutual fixed point of s and t. Then $[s, t]z = z$. We claim that $[s, t]$ is parabolic. First of all, since the fixed point sets of s and t are not equal, the above lemma tells us that $[s, t]$ is not the identity element. Conjugating s and t by an appropriate element, we can assume that s has fixed points 0 and ∞ and that t fixes ∞ (if t fixes 0, we conjugate further by the matrix mapping 0 to ∞ and vice versa). We can therefore write

$$s = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad t = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$$

for $\lambda > 0$ with $\lambda \neq 1$ and non-zero real numbers a, b (since t does not fix 0). Then

$$[s, t] = \begin{pmatrix} 1 & ab(\lambda^2 - 1) \\ 0 & 1 \end{pmatrix},$$

the trace of which is 2. Hence $[s, t]$ is parabolic. □

Proposition 3.2.12. *Let* Γ *be a non-elementary subgroup of* $\mathrm{PSL}(2,\mathbb{R})$. *Then the action of* Γ *on* $\partial\mathbb{H}$ *is strongly hyperbolic, so* Γ *is a Powers group.*

Proof. It is clear that Γ is non-trivial. Suppose for contradiction that Γ does not contain any hyperbolic element. Then by Proposition 3.2.9, Γ must contain at least one parabolic element s. By Lemma 3.2.5, we can assume that $s(z) = z + k$ for some non-zero $k \in \mathbb{R}$ after conjugating Γ by an appropriate element. Take any $t \in \Gamma$, and write

$$t = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Note now that

$$s^n t = \begin{pmatrix} 1 & nk \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a + cnk & b + dnk \\ c & d \end{pmatrix}$$

for $n \geq 1$. Then $\mathrm{tr}(s^n t) = a + cnk + d$. Since $s^n t$ is either parabolic or elliptic by assumption, we must have $|a+cnk+d| \leq 2$ for all $n \geq 1$, which implies $c = 0$. Then t must fix ∞ as well, so that any element of Γ fixes ∞. Hence the Γ-orbit of ∞ is $\{\infty\}$, so Γ is elementary. Therefore if Γ is non-elementary, it contains a hyperbolic element s.

Let z_1, z_2 be the fixed points of s. Since Γ is non-elementary, the Γ-orbit of z_1 is infinite, so in particular we can take $t \in \Gamma$ such that $tz_1 \notin \{z_1, z_2\}$. Then $S = tst^{-1} \in \Gamma$ is hyperbolic with fixed points tz_1 and tz_2. We now turn back to our first definition of hyperbolicity by means of Proposition 3.2.4 and aim to apply Corollary 3.1.15 to the action of $\mathrm{PSL}(2, \mathbb{R})$ on $\partial\mathbb{H}$, as non-identity elements of $\mathrm{PSL}(2, \mathbb{R})$ have at most two fixed points in $\partial\mathbb{H}$.

If s and S are transverse on $\partial\mathbb{H}$, we are done. If s and S have one fixed point in common, say z_0, and consider the commutator $[s, S]$ which then satisfies $[s, S]z_0 = z_0$. By Lemma 3.2.11, $[s, S]$ is parabolic and therefore fixes only z_0. Since Γ is non-elementary, the Γ-orbit of z_0 is infinite, so there exists an element $u \in \Gamma$ such that $uz_0 \notin \{z_1, z_2\}$. Then $U = u[s, S]u^{-1} \in \Gamma$ is parabolic with fixed point $uz_0 \notin \{z_1, z_2\}$. Since U is parabolic, Remark 3.2.6 tells us that $U^n(z) \to uz_0$ for all $z \in \partial\mathbb{H}$. Hence we can choose $n \geq 1$ such that $\{U^n(z_1), U^n(z_2)\}$ does not intersect $\{z_1, z_2\}$ and therefore s and $U^n s U^{-n}$ are transverse hyperbolic homeomorphisms of $\partial\mathbb{H}$. Therefore the action of Γ on $\partial\mathbb{H}$ is strongly hyperbolic, so Corollary 3.1.15 applies. □

It is proved in [50] that a subgroup of $\mathrm{PSL}(2, \mathbb{R})$ is elementary if and only if it is solvable. Non-solubility, however, does not provide examples of C^*-simple subgroups of $\mathrm{PSL}(2, \mathbb{R})$ as easily as the next example does:

Example 3.2.13. Let Γ be a subgroup of $\mathrm{PSL}(2, \mathbb{R})$ containing $\mathrm{PSL}(2, \mathbb{Z})$. Then Γ is a non-elementary subgroup of $\mathrm{PSL}(2, \mathbb{R})$ and hence a Powers group. Indeed, if we consider matrices of the form

$$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}$$

for $n \in \mathbb{Z}$ in Γ, then matrices of the first form map any $z \in \tilde{\mathbb{H}} \setminus \{\infty\}$ to $z + n$, and matrices of the second form map ∞ to $\frac{1}{n}$. Hence any $z \in \tilde{\mathbb{H}}$ has infinite orbit under Γ. For $\Gamma = \mathrm{PSL}(2, \mathbb{Z})$, the so-called *modular group*, this should not come as a total surprise: it is well-known that $\mathrm{PSL}(2, \mathbb{Z})$ is isomorphic to the free product $\mathbb{Z}_2 * \mathbb{Z}_3$.

In three chapters' time, we shall see that we need not restrict ourselves only to 2×2 matrices to find examples of C^*-simple subgroups of matrix groups.

3.3 Weak Powers groups

One might hope that Powers groups are stable when passing to other groups than subgroups of finite index, but no luck. In the already-mentioned exposition on Powers groups, de la Harpe asked whether Powers groups were stable by extensions; eight years later, a negative answer to the question was given by Promislow.

Lemma 3.3.1. *Let Γ be a group, let S and T be disjoint subsets of Γ and let $x, y \in \Gamma$ be commuting elements such that $x(\Gamma \setminus S) \subseteq S$ and $y(\Gamma \setminus T) \subseteq T$. Then $\Gamma = S \cup T$.*

Proof. Let $s \in \Gamma \setminus S$. Then $xs \in S$, so $xs \in \Gamma \setminus T$. Therefore $yxs \in T$ and $yxs \notin S$. Since $\Gamma \setminus S \subseteq x^{-1}S$ and $\Gamma \setminus T \subseteq y^{-1}T$, we have $x^{-1}(\Gamma \setminus S) \subseteq S$ and $y^{-1}(\Gamma \setminus T) \subseteq T$. Hence $ys = x^{-1}(yxs) \in S$, so $ys \notin T$ and finally $s = y^{-1}(ys) \in T$. This completes the proof. □

Proposition 3.3.2 (Promislow, 1993). *The direct product of two non-trivial groups is never a Powers group.*

Proof. Let Γ_1 and Γ_2 be non-trivial groups and let $s \in \Gamma_1$ and $t \in \Gamma_2$ be fixed non-identity elements. If $\Gamma_1 \times \Gamma_2$ were a Powers group, there would exist a partition $\Gamma_1 \times \Gamma_2 = C \sqcup D$ and elements $g_1, g_2, g_3 \in \Gamma_1 \times \Gamma_2$ such that

$$(s,1)C \cap C = (1,t)C \cap C = \emptyset$$

and $g_iD \cap g_jD = \emptyset$ for all distinct $i, j \in \{1,2,3\}$. Writing $g_i = (a_i, b_i)$ for $i = 1,2,3$, we would then have

$$(a_1 s a_1^{-1}, 1)g_1C = g_1(s,1)C \subseteq g_1D \quad \text{and} \quad (1, b_2 t b_2^{-1})g_2C = g_2(1,t)C \subseteq g_2D.$$

By Lemma 3.3.1, we then have $\Gamma_1 \times \Gamma_2 = g_1D \cup g_2D$, but this is a contradiction as g_3D is non-empty and intersects neither g_1D nor g_2D. □

Before Promislow's result, Boca and Niţică had tried weakening the definition of a Powers group in [12] in order to obtain more easily proven permanence properties and still retain C^*-simplicity and uniqueness of trace. Their gambit paid off, resulting in the following notion.

Definition 3.3.3. A non-trivial group Γ is a *weak Powers group* if it satisfies the following property: for any non-empty finite subset $F \subseteq \Gamma \setminus \{1\}$ contained in a conjugacy class and any integer $N \geq 1$, there exist a partition $\Gamma = C \sqcup D$ and elements $s_1, \dots, s_N \in \Gamma$ such that

(i) $fC \cap C = \emptyset$ for all $f \in F$ and
(ii) $s_iD \cap s_jD = \emptyset$ for all $i, j = 1, \dots, N$ with $i \neq j$.

It is evident that all Powers groups are weak Powers groups.

It will be convenient to introduce the following notation: for any group Γ, $s \in \Gamma$ and subset $M \subseteq \Gamma$, we define

$$\langle s \rangle_M = \{gsg^{-1} \mid g \in M\}.$$

To check groups for the weak Powers property, we therefore investigate subsets of the form $\langle s \rangle_M$ for $s \in \Gamma \setminus \{1\}$ and finite subsets $M \subseteq \Gamma$.

Proposition 3.3.4. *Let Γ be a group.*

(i) *If Γ is a weak Powers group, then Γ is icc and non-amenable, and any subgroup of Γ of finite index is also a weak Powers group.*
(ii) *If Γ is the union of an increasing family of weak Powers groups, then Γ is a weak Powers group.*

Proof. The proofs of Propositions 3.1.3 and 3.1.4 still apply; once we prove that Γ is C^*-simple, non-amenability is clear. □

One reason to consider weak Powers groups is that they are more stable than Powers groups under forming new groups.

Proposition 3.3.5. *If Γ_1 and Γ_2 are weak Powers groups, then the direct product $\Gamma = \Gamma_1 \times \Gamma_2$ is also a weak Powers group.*

Proof. Let $s = (s_1, s_2) \in \Gamma \setminus \{1\}$ and $M \subseteq \Gamma$ be a finite subset. Since $s \neq 1$, we can assume that $s_1 \neq 1$ (if not, we just shift our attention to Γ_2 instead in the sequel). Let $\pi\colon \Gamma \to \Gamma_1$ denote the projection homomorphism. Then $\pi(M)$ is a finite subset of Γ_1, so if $N \geq 1$ is a positive integer, then by Γ_1 being a weak Powers group there exists a partition $\Gamma_1 = C_1 \sqcup D_1$ and elements $t_1', \dots, t_N' \in \Gamma_1$ such that $f'C_1 \cap C_1 = \emptyset$ for all $f' \in \langle s_1 \rangle_{\pi(M)}$ and $t_i'D_1 \cap t_j'D_1 = \emptyset$ for all distinct $i, j \in \{1, \dots, N\}$. Defining $C = C_1 \times \Gamma_2$, $D = D_1 \times \Gamma_2$ and $t_j = (t_j', 1)$ for all $j = 1, \dots, N$, then

$$\pi(fC \cap C) \subseteq \pi(f)C_1 \cap C_1 = \emptyset$$

for all $f \in \langle s \rangle_M$, since $\pi(\langle s \rangle_M) = \langle s_1 \rangle_{\pi(M)}$. Clearly, we also have $t_iD \cap t_jD = \emptyset$ for all $i, j \in \{1, \dots, N\}$ with $i \neq j$, so Γ is a weak Powers group. □

Corollary 3.3.6. *The inclusion of the class of Powers groups into the class of weak Powers groups is strict.*

Proof. Powers groups do exist, and any direct product of Powers groups is a weak Powers group, but not a Powers group by Proposition 3.3.2. □

Proposition 3.3.7. *Let*

$$1 \longrightarrow \Gamma' \longrightarrow \Gamma \overset{\pi}{\longrightarrow} \Gamma'' \longrightarrow 1$$

be a short exact sequence of groups, where Γ' *is a Powers group and* Γ'' *is a weak Powers group. Then* Γ *is a weak Powers group.*

Proof. By exactness, we can assume that Γ' is a normal subgroup of Γ and that Γ'' is the quotient group Γ/Γ'. Let T be a right transversal for Γ' in Γ, and let $s \in \Gamma \setminus \{1\}$, $M \subseteq \Gamma$ be a finite subset and $N \geq 1$ be a positive integer. There are now two possible cases. If $s \in \Gamma'$, then $\langle s \rangle_M \subseteq \Gamma' \setminus \{1\}$ by normality. Because Γ' is a Powers group, there now exist a partition $\Gamma' = C' \sqcup D'$ and elements $s_1, \ldots, s_N \in \Gamma'$ such that $fC' \cap C' = \emptyset$ for all $f \in \langle s \rangle_M$ and $s_i D' \cap s_j D' = \emptyset$ for all distinct $i, j \in \{1, \ldots, N\}$. By defining $C = \bigcup_{t \in T} C't$ and $D = \bigcup_{t \in T} D't$, we obtain a partition $\Gamma = C \sqcup D$, and furthermore,

$$fC \cap C = \bigcup_{r,t \in T} (fC'r \cap C't) = \emptyset \quad \text{and} \quad s_i D \cap s_j D = \bigcup_{r,t \in T} (s_i D'r \cap s_j D't) = \emptyset$$

for all $f \in \langle s \rangle_M$ and $i, j \in \{1, \ldots, N\}$ with $i \neq j$.

If $s \notin \Gamma'$, then $\pi(s) \neq 1$ and $\langle \pi(s) \rangle_{\pi(M)} \subseteq \Gamma'' \setminus \{1\}$ by normality of Γ'. Since Γ'' is a weak Powers group, there exist a partition $\Gamma'' = C'' \sqcup D''$ and elements $t_1, \ldots, t_N \in T$ such that $fC'' \cap C'' = \emptyset$ for all $f \in \langle \pi(s) \rangle_{\pi(M)}$ and $\pi(t_i)D'' \cap \pi(t_j)D'' = \emptyset$ for all $i, j \in \{1, \ldots, N\}$ with $i \neq j$. If we now define $C = \pi^{-1}(C'')$ and $D = \pi^{-1}(D'')$, then $\Gamma = C \sqcup D$. Furthermore, for all $f \in \langle s \rangle_M$ we have

$$\pi(fC \cap C) \subseteq \pi(f)C'' \cap C'' = \emptyset$$

since $\pi(\langle s \rangle_M) = \langle \pi(s) \rangle_{\pi(M)}$, so that $fC \cap C = \emptyset$. Similarly one sees that $t_i D \cap t_j D = \emptyset$ for all distinct i, j, so in any case, Γ is a weak Powers group. □

Remark 3.3.8. Proposition 3.3.7 does not hold if Γ' is a weak Powers group, but not a Powers group. Indeed, consider the short exact sequence

$$1 \longrightarrow \mathbb{F}_2 \times \mathbb{F}_2 \longrightarrow \Gamma \longrightarrow \mathbb{Z}_2 * \mathbb{Z}_3 \longrightarrow 1$$

constructed as follows. Letting $a \in \mathbb{Z}_2$ and $b \in \mathbb{Z}_3$ be generators of their separate groups, we define an action of $\mathbb{Z}_2 * \mathbb{Z}_3$ on $\mathbb{F}_2 \times \mathbb{F}_2$ by letting a take the element (x, y) to (y, x) in $\mathbb{F}_2 \times \mathbb{F}_2$ whilst b is the identity map. We already know that $\mathbb{F}_2 \times \mathbb{F}_2$ is a weak Powers group and that $\mathbb{Z}_2 * \mathbb{Z}_3$ is in fact a Powers group.

Let $x_0 \in \mathbb{F}_2$ be some non-identity element. If ι is the monomorphism $\mathbb{F}_2 \times \mathbb{F}_2 \to \Gamma$, then by defining $F = \{\iota(x_0, 1), \iota(1, x_0)\} \subseteq \Gamma$ we see that the elements in F are conjugate, as

$$(1_2, a)\iota(x_0, 1)(1_2, a) = (1_2, a)((x_0, 1), 1)(1_2, a) = ((1, x_0), a)(1_2, a) = \iota(1, x_0),$$

where 1_2 denotes the identity element in $\mathbb{F}_2 \times \mathbb{F}_2$. If Γ were a weak Powers group, there would exist a partition $\Gamma = C \sqcup D$ and elements $g_i = ((r_i, s_i), t_i) \in \Gamma$, $i = 1, 2, 3$, satisfying $fC \cap C = \emptyset$ for all $f \in F$ and $g_i D \cap g_j D = \emptyset$ for distinct $i, j \in \{1, 2, 3\}$. There must now be either two t_i's that contain an even number of a's after reduction, or two t_i's with an odd number of a's. If the number of occurences of a in, say, t_1 and t_2 is even, then

$$\iota(r_1 x_0 r_1^{-1}, 1)g_1 C = g_1 \iota(x_0, 1)C \subseteq g_1 D \quad \text{and} \quad \iota(1, s_2 x_0 s_2^{-1})g_2 C = g_1 \iota(1, x_0)C \subseteq g_2 D,$$

and the argument of Proposition 3.3.2 applies to yield a contradiction. Similarly, if the number of a's in t_1 and t_2 are odd, then

$$\iota(r_1 x_0 r_1^{-1}, 1)g_1 C = g_1 \iota(1, x_0)C \subseteq g_1 D$$

and similarly, $\iota(1, s_2 x_0 s_2^{-1})g_2 C = g_2 \iota(x_0, 1)C \subseteq g_2 D$, yielding a contradiction in any case: Γ is not a weak Powers group. However, we shall see soon that Γ *is* nonetheless C^*-simple with unique trace. ✺

Before showing that the reduced group C^*-algebra of a weak Powers group is simple with unique trace, it is a question worth asking whether there is a condition on discrete groups, perhaps weaker than the weak Powers property, that both implies C^*-simplicity and unique trace *and* is stable under extensions. As it turns out, Promislow gave an answer in the affirmative in [60], by defining the so-called PH groups (named after Powers and de la Harpe).

We define the PH property slightly differently from Promislow (see also [2, Section 5.1]).

Definition 3.3.9. A non-trivial group Γ is said to be *PH* if for all non-empty finite subsets $F \subseteq \Gamma \setminus \{1\}$ there is an ordering $F = \{s_1, \ldots, s_n\}$ and an increasing sequence of subgroups $\Gamma_1 \subseteq \cdots \subseteq \Gamma_n \subseteq \Gamma$ such that for all $1 \leq i \leq n$, finite subsets $M \subseteq \Gamma_i$ and integers $m \geq 1$ there exist $t_1, \ldots, t_m \in \Gamma_i$ and pairwise disjoint subsets $T_1, \ldots, T_m$ of Γ such that

$$(t_j t) s_i (t_j t)^{-1} (\Gamma \setminus T_j) \cap (\Gamma \setminus T_j) = \emptyset$$

for all $j \in \{1, \ldots, m\}$ and $t \in M$.

Not surprisingly, the class of PH groups, though slightly complicated formulation-wise, contains another (by now) well-known class of groups:

Proposition 3.3.10. *If Γ is a weak Powers group, then Γ is PH.*

Proof. Let $f \in \Gamma \setminus \{1\}$ and let $N \subseteq \Gamma$ be a finite subset. If we order $\langle f \rangle_N = \{f_1, \ldots, f_n\}$, then define

$$\Gamma_1 = \cdots = \Gamma_n = \Gamma.$$

If $1 \leq i \leq n$, $M \subseteq \Gamma$ is finite and $m \geq 1$ is an integer, then since Γ is weak Powers and $f_i \in \Gamma \setminus \{1\}$, there exists a partition $\Gamma = C \sqcup D$ and elements $t_1, \ldots, t_m \in \Gamma$ such that $s f_i s^{-1} C \cap C = \emptyset$ for all $s \in M$ and $t_j D \cap t_k D = \emptyset$ for all $j, k = 1, \ldots, m$ with $j \neq k$. By defining $T_j = t_j D$ for all $j = 1, \ldots, m$, we see that

$$t_j s f_i s^{-1} t_j^{-1} (\Gamma \setminus T_j) \subseteq t_j s f_i s^{-1} (\Gamma \setminus D) = t_j s f_i s^{-1} C \subseteq t_j D = T_j$$

for all $s \in M$ and $j \in \{1, \ldots, m\}$. Hence Γ is PH. □

Hence our desired property of weak Powers groups will follow once we prove the following:

Proposition 3.3.11. *If Γ is a PH group, then $C^*_r(\Gamma)$ is simple with a unique trace. In particular, weak Powers groups are C^*-simple and have unique trace.*

To give a proof, we require two lemmas. The first is a "generalization" of [35, Lemma 1] and, to some extent, [2, Lemma 5.1].

Lemma 3.3.12. *Let $\mathcal{H}$ be a Hilbert space, let $x \in B(\mathcal{H})$ be self-adjoint and let $n \geq 1$. If there exist unitaries $u_1, \ldots, u_n$ and projections $p_1, \ldots, p_n$ in $B(\mathcal{H})$ such that $p_i x p_i = 0$ for all $j = 1, \ldots, k$ and $u_i(1 - p_i)u_i^*$ are pairwise orthogonal projections for all $i = 1, \ldots, n$, then*

$$\left\| \frac{1}{n} \sum_{j=1}^{n} u_j x u_j^* \right\| \leq \frac{2}{\sqrt{n}} \|x\|.$$

Proof. Define $b = \frac{1}{n} \sum_{i=1}^{n} u_i x u_i^*$ and let $\xi \in \mathcal{H}$ be a unit vector. If we define $q_i = u_i(1 - p_i)u_i^*$ for all $i = 1, \ldots, n$, then we have

$$\begin{aligned}
\langle u_i x u_i^* \xi, \xi \rangle &= \langle q_i u_i x u_i^* \xi, \xi \rangle + \langle (1 - q_i) u_i x u_i^* \xi, \xi \rangle \\
&= \langle q_i u_i x u_i^* \xi, \xi \rangle + \langle u_i p_i x u_i^* \xi, \xi \rangle \\
&= \langle u_i x u_i^* \xi, q_i \xi \rangle + \langle q_i \xi, u_i x p_i u_i^* \xi \rangle + \langle (1 - q_i) \xi, u_i x p_i u_i^* \xi \rangle \\
&= \langle u_i x u_i^* \xi, q_i \xi \rangle + \langle q_i \xi, u_i x p_i u_i^* \xi \rangle + \langle \xi, u_i p_i x p_i u_i^* \xi \rangle \\
&= \langle u_i x u_i^* \xi, q_i \xi \rangle + \langle q_i \xi, u_i x p_i u_i^* \xi \rangle.
\end{aligned}$$

Therefore

$$\begin{aligned}|\langle b\xi,\xi\rangle| &\le \frac{1}{n}\sum_{i=1}^{n}\left[|\langle u_i x u_i^*\xi, q_i\xi\rangle| + |\langle q_i\xi, u_i x p_i u_i^*\xi\rangle|\right] \\ &\le \frac{1}{n}\sum_{i=1}^{n}\|q_i\xi\|\left(\|u_i x u_i^*\xi\| + \|u_i x p_i u_i^*\xi\|\right) \\ &\le \frac{2}{\sqrt{n}}\|x\|\left(\sum_{i=1}^{n}\|q_i\xi\|^2\right)^{1/2} \le \frac{2}{\sqrt{n}}\|x\|.\end{aligned}$$

As b is self-adjoint, this completes the proof. □

Lemma 3.3.13. *Let Γ be a PH group and let $a \in \mathbb{C}\Gamma$ be a self-adjoint element with $\tau(a) = 0$, where τ denotes the canonical faithful trace on $C_r^*(\Gamma)$. Then for all $\varepsilon > 0$, there exists an integer $N \ge 1$ and elements $s_1, \dots, s_N \in \Gamma$ such that*

$$\left\|\frac{1}{N}\sum_{i=1}^{N}\lambda_\Gamma(s_i)a\lambda_\Gamma(s_i)^*\right\| < \varepsilon. \tag{3.3.1}$$

Proof. Write $a = \sum_{i=1}^{n}(z_i\lambda_\Gamma(f_i) + \overline{z_i}\lambda_\Gamma(f_i)^*)$, where $z_1, \dots, z_n$ are complex numbers and $f_1, \dots, f_n \in \Gamma$, and define $a_i = z_i\lambda_\Gamma(f_i) + \overline{z_i}\lambda_\Gamma(f_i)^*$ for all i. By the assumption that $\tau(a) = 0$, we can further assume that $f_i \neq 1$ for all i. Choose integers $N_1, \dots, N_n \ge 1$ such that

$$\frac{2}{\sqrt{N_i}}\|a_i\| < \frac{\varepsilon}{n}, \quad i = 1, \dots, n.$$

Since Γ is PH, there is a sequence of subgroups $\Gamma_1 \subseteq \cdots \subseteq \Gamma_n \subseteq \Gamma$ such that the condition of Definition 3.3.9 holds with $F = \{f_1, \dots, f_n\}$. As $1 \in \Gamma_1$, so there exist $t_{1,1}, \dots, t_{1,N_1} \in \Gamma_1$ and pairwise disjoint subsets $T_{1,1}, \dots, T_{1,N_1}$ of Γ such that for all $1 \le j \le N_1$ we have

$$t_{1,j}f_1t_{1,j}^{-1}(\Gamma \setminus T_{1,j}) \cap (\Gamma \setminus T_{1,j}) = \emptyset.$$

Now fix $j \in \{1, \dots, N_1\}$. Defining $D_{1,j} = t_{1,j}^{-1}(\Gamma \setminus T_{1,j})$, let $p_{1,j}$ denote the projection of $\ell^2(\Gamma)$ onto $\ell^2(D_{1,j})$. Then for all $s, t \in \Gamma$ we have

$$p_{1,j}\lambda_\Gamma(s)p_{1,j}\delta_t = \begin{cases}\delta_{st} & \text{if } t \in s^{-1}D_{1,j} \cap D_{1,j} \\ 0 & \text{else.}\end{cases}$$

Since

$$\emptyset = t_{1,j}f_1t_{1,j}^{-1}(\Gamma \setminus T_{1,j}) \cap (\Gamma \setminus T_{1,j}) = t_{1,j}(f_1D_{1,j} \cap D_{1,j}),$$

it follows that $p_{1,j}\lambda_\Gamma(f_1)^*p_{1,j} = 0$ and hence $p_{1,j}a_1p_{1,j} = 0$. By defining $q_{1,j} = \lambda_\Gamma(t_{1,j})(1-p_{1,j})\lambda_\Gamma(t_{i,j})^*$, we have $q_{1,j}\delta_s = 1_{T_{1,j}}(s)\delta_s$ for all $s \in \Gamma$, so that the projections $q_{1,1}, \dots, q_{1,N_1}$ are mutually orthogonal. It now follows from Lemma 3.3.12 that

$$\left\|\frac{1}{N_1}\sum_{j=1}^{N_1}\lambda_\Gamma(t_{1,j})a_1\lambda_\Gamma(t_{1,j})^*\right\| \le \frac{2}{\sqrt{N_1}}\|a_1\|.$$

Now define

$$b_2 = \frac{1}{N_1}\sum_{j=1}^{N_1}\lambda_\Gamma(t_{1,j})a_2\lambda_\Gamma(t_{1,j})^*.$$

Since $S_1 = \{t_{1,1}, \dots, t_{1,N_1}\} \subseteq \Gamma_1 \subseteq \Gamma_2$ for all $j = 1, \dots, N_1$, there exist elements $t_{2,1}, \dots, t_{2,N_2} \in \Gamma_2$ and pairwise disjoint subsets $T_{2,1}, \dots, T_{2,N_2}$ of Γ such that for all $1 \le j \le N_2$ and $t \in S_1$ we have

$$t_{2,j_2}(tf_2t^{-1})t_{2,j}^{-1}(\Gamma \setminus T_{2,j}) \cap (\Gamma \setminus T_{2,j}) = \emptyset.$$

By defining $D_{2,j} = t_{2,j}^{-1}(\Gamma \setminus T_{2,j})$ and letting $p_{2,j}$ denote the projection of $\ell^2(\Gamma)$ onto $\ell^2(D_{2,j})$, then it is easy to show that $p_{2,j}\lambda_\Gamma(tf_2t^{-1})^*p_{2,j} = 0$ for all $t \in S_1$, so that $p_{2,j}b_2p_{2,j} = 0$. We can then define mutually orthogonal projections $q_{2,j} = \lambda_\Gamma(t_{2,j})(1-p_{2,j})\lambda_\Gamma(t_{2,j})^*$ for $j = 1,\dots,N_2$, and since b_2 is self-adjoint, Lemma 3.3.12 yields

$$\left\| \frac{1}{N_2}\sum_{j=1}^{N_2} \lambda_\Gamma(t_{2,j}) b_2 \lambda_\Gamma(t_{2,j})^* \right\| \leq \frac{2}{\sqrt{N_2}}\|b_2\|.$$

Inductively, we then define

$$b_{i+1} = \frac{1}{N_1\cdots N_i}\sum_{j_1=1}^{N_1}\cdots\sum_{j_i=1}^{N_i} \lambda_\Gamma(s_{i,j_i})\cdots\lambda_\Gamma(s_{1,j_1}) a_{i+1} \lambda_\Gamma(s_{1,j_1})^*\cdots\lambda_\Gamma(s_{i,j_i})^*$$

for $i = 2,\dots,n-1$. Letting $S_i = \{s_{i,j_i}\cdots s_{1,j_1} \mid j_k = 1,\dots,N_k,\ k = 1,\dots,i\} \subseteq \Gamma_i$ (recall that Γ_i is a subgroup) then yields elements $s_{i+1,1},\dots,s_{i+1,N_{i+1}} \in \Gamma$ satisfying

$$\left\| \frac{1}{N_{i+1}}\sum_{j=1}^{N_{i+1}} \lambda_\Gamma(s_{i+1,j}) b_{i+1} \lambda_\Gamma(s_{i+1,j})^* \right\| \leq \frac{2}{\sqrt{N_{i+1}}}\|b_{i+1}\|.$$

We then have

$$\begin{aligned}
&\left\| \frac{1}{N_1\cdots N_n}\sum_{j_1=1}^{N_1}\cdots\sum_{j_n=1}^{N_n} \lambda_\Gamma(s_{n,j_n})\cdots\lambda_\Gamma(s_{1,j_1}) a \lambda_\Gamma(s_{1,j_1})^*\cdots\lambda_\Gamma(s_{n,j_n})^* \right\| \\
\leq\ & \left\| \frac{1}{N_1}\sum_{j_1=1}^{N_1} \lambda_\Gamma(s_{1,j_1}) a_1 \lambda_\Gamma(s_{1,j_1})^* \right\| + \cdots + \left\| \frac{1}{N_n}\sum_{j_n=1}^{N} \lambda_\Gamma(s_{n,j_n}) b_n \lambda_\Gamma(s_{n,j_n})^* \right\| \\
\leq\ & \sum_{i=1}^{n} \frac{2}{\sqrt{N_i}}\|a_i\| < \varepsilon,
\end{aligned}$$

as $\|b_i\| \leq \|a_i\|$ for all $i = 2,\dots,n$. Hence we have found an integer $N = N_1\cdots N_n$ and elements $s_1,\dots,s_N \in \Gamma$ such that (3.3.1) holds. ☐

Proof of Proposition 3.3.11. The proof of Proposition 3.1.6 adapts verbatim. ☐

We now give another good reason that PH groups are worth our attention.

Proposition 3.3.14. *Let*

$$1 \longrightarrow \Gamma' \longrightarrow \Gamma \xrightarrow{\pi} \Gamma'' \longrightarrow 1$$

be a short exact sequence of groups, where Γ' and Γ'' are PH. Then Γ is PH.

Proof. We assume that Γ' is a normal subgroup of Γ and that $\Gamma'' = \Gamma/\Gamma'$; let T be a right transversal for Γ' in Γ. Let $F = \{s_1,\dots,s_n\} \subseteq \Gamma\setminus\{1\}$ be a finite subset. We can assume that there exists $1 \leq k \leq n$ such that after reordering, the first k elements $s_1,\dots,s_k$ belong to Γ' and $s_{k+1},\dots,s_n$ do not. The hypothesis that Γ' and Γ'' are PH yields a reordering of $s_1,\dots,s_k$ and $s_{k+1},\dots,s_n$ and increasing sequences of subgroups

$$\Lambda_1 \subseteq \cdots \subseteq \Lambda_k \subseteq \Gamma' \quad \text{resp.} \quad \Omega_{k+1} \subseteq \cdots \subseteq \Omega_n \subseteq \Gamma''$$

such that the condition of Definition 3.3.9 is satisfied for $s_1,\dots,s_k$ resp. $\pi(s_{k+1}),\dots,\pi(s_n)$. We claim that

$$\Lambda_1 \subseteq \cdots \subseteq \Lambda_k \subseteq \pi^{-1}(\Omega_{k+1}) \subseteq \cdots \subseteq \pi^{-1}(\Omega_n) \subseteq \Gamma$$

is the desired sequence of subgroups.

If $1 \leq i \leq k$, $M \subseteq \Lambda_i$ is a finite subset and $m \geq 1$ is an integer, there exist elements $t_1,\dots,t_m \in \Lambda_i$ and pairwise disjoint subsets $T_1,\dots,T_m$ of Γ' such that

$$(t_j t)s_i(t_j t)^{-1}(\Lambda \setminus T_j) \subseteq T_j$$

for all $1 \leq j \leq m$ and $t \in M$, and therefore also

$$(t_j t)s_i(t_j t)^{-1}(\Gamma \setminus T_j T) = (t_j t)s_i(t_j t)^{-1}(\Lambda \setminus T_j)T \subseteq T_j T.$$

Note that the sets $T_j T$ are also pairwise disjoint.

Now, for $k+1 \leq i \leq n$, any finite subset $M \subseteq \pi^{-1}(\Omega_i)$ and integer $m \geq 1$, there exist elements $\pi(u_1), \dots, \pi(u_m) \in \Omega_i$, where $u_1, \dots, u_m \in \Gamma$, and pairwise disjoint subsets $U_1, \dots, U_m$ of Γ'' such that

$$\pi(u_j t s_i (u_j t)^{-1})(\Gamma'' \setminus U_j) \subseteq U_j$$

for all $1 \leq j \leq m$ and $t \in M$. Then $u_j \in \pi(\Omega_i)$ for all j, the subsets $\pi^{-1}(U_j)$ are pairwise disjoint, and

$$u_j t s_i (u_j t)^{-1}(\Gamma \setminus \pi^{-1}(U_j)) \subseteq \pi^{-1}(U_j)$$

for all $1 \leq j \leq m$ and $t \in M$. Hence Γ is PH. □

Corollary 3.3.15. *The inclusion of the class of weak Powers groups into the class of PH groups is strict.*

Proof. The group Γ of Remark 3.3.8 is not weak Powers, but it is PH by Propositions 3.3.10 and 3.3.14. □

So far, so good: PH groups indeed have the properties we want, but finding them is an entirely different matter. One of the few known group-theoretical properties of PH groups (and hence Powers and weak Powers groups) is the content of the following result, due to Brin and Picioroaga.

Proposition 3.3.16. *A PH group Γ always contains a free non-abelian subgroup of rank two.*

The proof requires a famous lemma originating in the study of discrete subgroups of projective special linear groups:

Lemma 3.3.17 (Ping-pong lemma or Klein's criterion)**.** *Let Γ be a group acting on a set X and let Γ_1 and Γ_2 be subgroups of Γ of order at least 3 and 2 respectively. Assume that there exist distinct non-empty subsets X_1 and X_2 of X such that $sX_{3-i} \subseteq X_i$ for all $s \in \Gamma_i \setminus \{1\}$ for $i = 1, 2$. Then the subgroup Γ' of Γ generated by Γ_1 and Γ_2 is the free product $\Gamma_1 * \Gamma_2$.*

Proof. We can assume that both Γ_1 and Γ_2 have order greater than or equal to 2. Any word $s \in \Gamma_1 * \Gamma_2$ defines an element in Γ' (also written s), yielding a surjective homomorphism of $\Gamma_1 * \Gamma_2$ into Γ'. We have to show that any non-identity element $s \in \Gamma_1 * \Gamma_2$ defines a non-identity element in Γ'. Assume for contradiction that there exists an alternating product $s = s_1 \cdots s_n$ for $n \geq 2$ in $\Gamma_1 * \Gamma_2$ of non-identity elements in Γ_1 and Γ_2 such that $s = 1$ in Γ'. We can assume that $s_1, s_n \in \Gamma_1 \setminus \{1\}$. Indeed, if $s_1, s_n \in \Gamma_2$, conjugate s by an element $a \in \Gamma_1 \setminus \{1\}$ and consider the reduced word asa^{-1}, if $s_1 \in \Gamma_1$ and $s_n \in \Gamma_2$, conjugate s by an element $a \in \Gamma_1 \setminus \{1, s_1^{-1}\}$, and if $s_1 \in \Gamma_2$ and $s_n \in \Gamma_1$, conjugate s by an $a \in \Gamma_1 \setminus \{1, s_n\}$. With s of this form, we then have

$$X_2 = sX_2 \subseteq s_1 \cdots s_n X_2 \subseteq s_1 \cdots s_{n-1} X_1 \subseteq \ldots \subseteq s_1 X_2 \subseteq X_1.$$

Conjugating w by a non-identity element of Γ_2, we obtain a non-empty alternating product t beginning and ending with an element of $\Gamma_2 \setminus \{1\}$ such that $t = 1$ in Γ'. The same argument as above but with t implies $X_1 \subseteq X_2$, and hence our desired contradiction. □

Remark 3.3.18. Letting $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2$ and $s \in \mathbb{Z}_2$ be the generator, we can define $\Gamma_1 = \langle (s,1) \rangle$ and $\Gamma_2 = \langle (1,s) \rangle$. An action of Γ on $\{1,2\}$ can be defined by letting $(s,1)$ and $(1,s)$ interchange 1 and 2 and (s,s) be the identity map. Then $X_1 = \{1\}$ and $X_2 = \{2\}$ obviously satisfy the conditions of the ping-pong lemma, but $\Gamma_1 * \Gamma_2$ is infinite; hence the necessity of the requirement that at least one of the subgroups has order 3. ✺

Proof of Proposition 3.3.16. Let $s \in \Gamma \setminus \{1\}$. Then there exists a subgroup $\Gamma_1 \subseteq \Gamma$, $t_1, \dots, t_4 \in \Gamma_1$ and pairwise disjoint subsets $T_1, \dots, T_4 \subseteq \Gamma$ such that $t_i s t_i^{-1}(\Gamma \setminus T_i) \subseteq T_i$ for all $i = 1, \dots, 4$. Defining $x_i = t_i s t_i^{-1}$ for $i = 1, \dots, 4$ (so that $x_i(\Gamma \setminus T_i) \subseteq T_i$ and $x_i^{-1}(\Gamma \setminus T_i) \subseteq T_i$), it then follows for all $i, j \in \{1, \dots, 4\}$ with $i \neq j$ that

$$x_i x_j(\Gamma \setminus T_j) \subseteq x_i T_j \subseteq x_i(\Gamma \setminus T_i) \subseteq T_i \quad \text{and} \quad (x_i x_j)^{-1}(\Gamma \setminus T_i) \subseteq T_j.$$

Therefore if we define $u = x_1x_2$ and $v = x_3x_4$, we have

$$u(\Gamma \setminus T_2) \subseteq T_1, \quad u^{-1}(\Gamma \setminus T_1) \subseteq E_2, \quad v(\Gamma \setminus T_4) \subseteq T_3 \quad \text{and} \quad v^{-1}(\Gamma \setminus T_3) \subseteq T_4.$$

This implies that $u^n(T_3 \cup T_4) \subseteq T_1 \cup T_2$ and $v^n(T_1 \cup T_2) \subseteq T_3 \cup T_4$ for all $n \in \mathbb{Z} \setminus \{0\}$, so the subgroups generated by u and v have infinite order. By the ping-pong lemma, the subgroup of Γ generated by u and v is therefore isomorphic to $\mathbb{Z} * \mathbb{Z} = \mathbb{F}_2$. $\square$

A natural question is whether having a subgroup isomorphic to $\mathbb{F}_2$ actually classifies PH groups, but an even better question could be the following: *Does a group Γ contain $\mathbb{F}_2$ whenever Γ is either C^*-simple or has unique trace?* (The converse clearly doesn't hold; consider $\mathbb{F}_2 \times \mathbb{Z}_2$.)

Chapter 4

Reduced twisted crossed products

More than anything, this chapter is dedicated to the work of Erik Bédos on the subject of C^*-simple discrete groups. The results herein revolve around the concept of so-called ultraweak Powers groups and their relation to various discrete crossed products. The central objects of study are normal subgroups with trivial centralizer; in order to conclude properties of their overgroups, we investigate automorphisms of C^*-crossed products and von Neumann algebras, and we prove not just a lot of results on permanence of C^*-simplicity and uniqueness of trace for reduced group C^*-algebras, but also reduced crossed products as well. This generalizes a lot of the content of [35] and [1].

4.1 Ultraweak Powers groups

We first introduce an even weaker notion than that of a weak Powers group (originally from [5]).

Definition 4.1.1. A group Γ is called an *ultraweak Powers group* if it contains a normal weak Powers subgroup Λ with trivial centralizer, i.e., the only element of Γ commuting with all elements in Λ is the identity element.

An ultraweak Powers group generalizes the notion of a group of *Akemann-Lee type*, i.e., groups containing a normal, free, non-abelian subgroup with trivial centralizer which Akemann and Lee proved were C^*-simple in [1]. Note that all weak Powers groups are ultraweak Powers groups. Indeed, the centralizer of a weak Powers group Γ viewed as a subgroup of itself is the center of Γ, and we know from Section 1.8 that Γ is necessarily centerless.

Remark 4.1.2. An ultraweak Powers group need not be a weak Powers group. An example of this is the non-weak Powers group Γ constructed in Remark 3.3.8. Letting $a \in \mathbb{Z}_2$ and $b \in \mathbb{Z}_3$ be generators of their groups, note that all elements in $\mathbb{Z}_2 * \mathbb{Z}_3$ are of the form

$$b^{\delta_1} a b^{\delta_2} a \cdots b^{\delta_{n-1}} a b^{\delta_n}$$

for $n \geq 1$, where $\delta_i \in \{1,2\}$ for $i = 2, \ldots, n-1$ and $\delta_i \in \{0,1,2\}$ for $i \in \{1,n\}$. We define H to be the set of all reduced words in $\mathbb{Z}_2 * \mathbb{Z}_3$ of the above form where a occurs an even number of times, and H is easily checked to be a subgroup. The action used to define Γ can be now be used to define the semidirect product Λ of H by $\mathbb{F}_2 \times \mathbb{F}_2$. Since $\mathbb{Z}_2 * \mathbb{Z}_3$ is a Powers group and H is a subgroup of index 2, H is a Powers group. Moreover, H acts trivially on $\mathbb{F}_2 \times \mathbb{F}_2$, so Λ is the direct product of H and $\mathbb{F}_2 \times \mathbb{F}_2$ and is therefore a weak Powers group. As Λ is a subgroup of Γ of index 2, Λ is normal. Finally, Λ is easily seen to have trivial centralizer, so Γ is an ultraweak Powers group. We record this as an observation: ✺

Corollary 4.1.3. *The inclusion of weak Powers groups into the class of ultraweak Powers groups is strict.*

We now give some interesting examples of ultraweak Powers groups.

Example 4.1.4. Let $\mathbb{F}$ be a non-abelian free group of finite rank and let A be a group of outer automorphisms of $\mathbb{F}$, i.e., A is a subgroup of $\mathrm{Aut}(\mathbb{F})$ such that all non-identity automorphisms in A are outer. Forming the semi-direct product $\Gamma = \mathbb{F} \rtimes A$ by letting A act on $\mathbb{F}$ in the obvious way, then $\mathbb{F}$ is a normal subgroup of Γ and by the fact that all $\sigma \in A \setminus \{1\}$ are outer, it follows that $\mathbb{F}$ has trivial centralizer in Γ. As $\mathbb{F}$ is a Powers group, Γ is an ultraweak Powers group. In fact, it is also a PH group, as was shown by Promislow in [60, Theorem 8.1]. It is an open question whether the notions of a PH group and an ultraweak Powers group coincide.

Example 4.1.5. We consider the *braid group* B_n on n strands for $n \geq 3$, which has the presentation

$$B_n = \left\langle s_1, \dots, s_{n-1} \;\middle|\; \begin{array}{ll} s_i s_j s_i = s_j s_i s_j & \text{if } |i-j| = 1 \\ s_i s_j = s_j s_i & \text{if } |i-j| > 1 \end{array} \right\rangle.$$

The curious reader can consult [28] for results about braid groups, as well as a truck-load of different realizations; one would not be far off by describing B_n as the group of almost surely unplayable guitars with n strings. It is a well-known result that the center C_n of B_n is an infinite cyclic group [28, Theorem 4.2], so B_n is not C^*-simple and does not have unique trace as a discrete group. However, if we define $\mathscr{B}_n = B_n/C_n$ for $n \geq 3$, then it was shown by Dyer and Grossman that $\mathscr{B}_n$ contains the non-abelian free group $\mathbb{F}_{n-1}$ as a characteristic and hence normal subgroup, and that $\mathbb{F}_{n-1}$ has trivial centralizer in $\mathscr{B}_n$ [24, Theorem 15 and Corollary 17]. It follows that $\mathscr{B}_n$ is an ultraweak Powers group.

In fact, ultraweak Powers groups are C^*-simple and have unique trace, a result also due to Bédos (see [5] and [6]). In fact, said author proved something much more general, but even if we might only want to consider reduced group C^*-algebras, we *do* need some results that are quite a bit larger in scope.

Definition 4.1.6. Let Γ be a discrete group, $\mathcal{A}$ be a unital C^*-algebra and $\alpha\colon \Gamma \to \operatorname{Aut}(\mathcal{A})$ (defining $\alpha_s = \alpha(s)$ for $s \in \Gamma$) and $u\colon \Gamma \times \Gamma \to \mathcal{U}(\mathcal{A})$ be maps. We say that (α, u) is a *twisted action* of Γ on $\mathcal{A}$ if it holds for all $r, s, t \in \Gamma$ that

$$\alpha_r \alpha_s = \operatorname{Ad}(u(r,s))\alpha_{rs}, \quad u(r,s)u(rs,t) = \alpha_r(u(s,t))u(r,st), \quad u(s,1) = u(1,s) = 1.$$

(Here $\alpha_g = \alpha(g)$ for all $g \in \Gamma$, and for $v \in \mathcal{U}(\mathcal{A})$, $\operatorname{Ad}(v)$ is the automorphism of $\mathcal{A}$ given by $a \mapsto vav^*$.) The 4-tuple $(\mathcal{A}, \Gamma, \alpha, u)$ is called a *twisted dynamical system.*

Note that if u is trivial in a twisted dynamical system, we just obtain the standard notion of a C^*-dynamical system.

Definition 4.1.7. A *covariant representation* of a twisted dynamical system $(\mathcal{A}, \Gamma, \alpha, u)$ is a triple $(\pi, \rho, \mathcal{H})$ consisting of a Hilbert space $\mathcal{H}$, a representation $\pi\colon \mathcal{A} \to B(\mathcal{H})$ and a map $\rho\colon \Gamma \to \mathcal{U}(\mathcal{H})$ such that

$$\pi(\alpha_s(a)) = \operatorname{Ad}(\rho(s))(\pi(a)), \quad \rho(s)\rho(t) = \pi(u(s,t))\rho(st)$$

for all $s, t \in \Gamma$ and $a \in \mathcal{A}$.

For any twisted dynamical system $(\mathcal{A}, \Gamma, \alpha, u)$, consider the space $C_c(\Gamma, \mathcal{A})$ of finitely supported $\mathcal{A}$-valued functions on $\mathcal{A}$. We then define an (α, u)-twisted convolution product and involution on $C_c(\Gamma, \mathcal{A})$ by

$$(f * g)(t) = \sum_{s \in \Gamma} f(s)\alpha_s(g(s^{-1}t))u(s, s^{-1}t), \quad f^*(t) = u(t, t^{-1})^* \alpha_t(f(t^{-1})^*)$$

for $f, g \in C_c(\Gamma, \mathcal{A})$ and $t \in \Gamma$, yielding a *-algebra structure on $C_c(\Gamma, \mathcal{A})$. If $(\pi, \rho, \mathcal{H})$ is a covariant representation of $(\mathcal{A}, \Gamma, \alpha, u)$, then we can represent $C_c(\Gamma, \mathcal{A})$ on the Hilbert space $\mathcal{H}$ by means of the map

$$(\pi \times \rho)(f) = \sum_{s \in \Gamma} \pi(f(s))\rho(s), \quad f \in C_c(\Gamma, \mathcal{A}).$$

Assume now that we have a faithful non-degenerate representation $\mathcal{A} \subseteq B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, and define a faithful representation of $\mathcal{A}$ on $\mathcal{H} \otimes \ell^2(\Gamma)$ by

$$\pi_\alpha(a)(\xi \otimes \delta_t) = \alpha_{t^{-1}}(a)\xi \otimes \delta_t, \quad a \in \mathcal{A},\ \xi \in \mathcal{H},\ t \in \Gamma.$$

We then define a unitary operator $\lambda_u(s)\colon \Gamma \to \mathcal{U}(\mathcal{H} \otimes \ell^2(\Gamma))$ for all $s \in \Gamma$ by

$$\lambda_u(s)(\xi \otimes \delta_t) = u(t^{-1}s^{-1}, s)\xi \otimes \delta_{st}, \quad \xi \in \mathcal{H},\ t \in \Gamma.$$

As

$$\lambda_u(s)^*(\xi \otimes \delta_t) = u(t^{-1}, s)^* \xi \otimes \delta_{s^{-1}t},$$

it quickly follows that $(\pi_\alpha, \lambda_u, \mathcal{H} \otimes \ell^2(\Gamma))$ is a covariant representation.

Lemma 4.1.8. *For any finite set $F = \{s_1, \dots, s_n\} \subseteq \Gamma$, let $P_F \in B(\ell^2(\Gamma))$ be the projection of $\ell^2(\Gamma)$ onto the linear span of $\{\delta_s \mid s \in F\}$. Then for any $x \in C_c(\Gamma, \mathcal{A})$, we have*

$$(1 \otimes P_F)(\pi_\alpha \times \lambda_u)(x)(1 \otimes P_F) \in M_n(\mathcal{A}).$$

Proof. Assume that $\mathcal{A}$ is represented faithfully on a Hilbert space $\mathcal{H}$, i.e., $\mathcal{A} \subseteq B(\mathcal{H})$. If $x \in C_c(\Gamma, \mathcal{A})$, write

$$y = (\pi_\alpha \times \lambda_u)(x) = \sum_{s\in\Gamma} \pi_\alpha(a_s)\lambda_u(s),$$

where the sum above is finite. Any two elements ξ and η in the range of $1 \otimes P_F$ are of the form $\xi = \sum_{i=1}^n \xi_i \otimes \delta_{s_i}$ and $\eta = \sum_{i=1}^n \eta_i \otimes \delta_{s_i}$ for vectors $\xi_1, \eta_1, \dots, \xi_n, \eta_n \in \mathcal{H}$, and consequently

$$\begin{aligned}
\langle (1 \otimes P_F)y(1 \otimes P_F)\xi, \eta\rangle &= \langle y\xi, \eta\rangle \\
&= \sum_{i,j=1}^n \sum_{s\in\Gamma} \langle \pi_\alpha(a_s)\lambda_u(s)(\xi_i \otimes \delta_{s_i}), \eta_j \otimes \delta_{s_j}\rangle \\
&= \sum_{i,j=1}^n \sum_{s\in\Gamma} \langle \alpha_{s_i^{-1}s^{-1}}(a_s)u(s_i^{-1}s^{-1}, s)(\xi_i \otimes \delta_{ss_i}), \eta_j \otimes \delta_{s_j}\rangle \\
&= \sum_{i,j=1}^n \sum_{\substack{s\in\Gamma \\ ss_i = s_j}} \langle \alpha_{s_i^{-1}s^{-1}}(a_s)u(s_i^{-1}s^{-1}, s)\xi_i, \eta_j\rangle \\
&= \sum_{i,j=1}^n \langle \alpha_{s_j^{-1}}(a_{s_j s_i^{-1}})u(s_j^{-1}, s_j s_i^{-1})\xi_i, \eta_j\rangle.
\end{aligned}$$

Therefore,

$$(1 \otimes P_F)y(1 \otimes P_F) = \Big[\alpha_{s_i^{-1}}(a_{s_i s_j^{-1}})u(s_i^{-1}, s_i s_j^{-1})\Big]_{i,j=1}^n \in M_n(\mathcal{A}),$$

as wanted. □

Lemma 4.1.9. *For all $x \in C_c(\Gamma, \mathcal{A})$, $(\pi_\alpha \times \lambda_u)(x) = 0$ if and only if $x = 0$. Hence the representation*

$$\pi_\alpha \times \lambda_u \colon C_c(\Gamma, \mathcal{A}) \to B(\mathcal{H} \otimes \ell^2(\Gamma))$$

is faithful.

Proof. Writing

$$y = (\pi_\alpha \times \lambda_u)(x) = \sum_{s\in\Gamma} \pi_\alpha(a_s)\lambda_u(s),$$

the assumption that $y = 0$ implies $(1 \otimes P_F)y(1 \otimes P_F) = 0$ for all finite subsets $F \subseteq \Gamma$. For all $s \neq 1$, define $F = \{1, s^{-1}\}$. By Lemma 4.1.8, we then have

$$0 = (1 \otimes P_F)y(1 \otimes P_F) = \begin{bmatrix} a_1 & a_s \\ \alpha_s(a_{s^{-1}})u(s, s^{-1}) & \alpha_s(a_1) \end{bmatrix},$$

since α_1 is the identity map. Thus $a_s = 0$ and $a_1 = 0$. □

Definition 4.1.10. The C^*-algebra obtained by taking the completion of $C_c(\Gamma, \mathcal{A})$ with respect to the norm

$$\|x\| = \|\pi_\alpha \times \lambda_u(x)\|$$

is called the *reduced twisted crossed product* of $\mathcal{A}$ and Γ; it is denoted by $\mathcal{A} \rtimes_{\alpha,r}^u \Gamma$.

Henceforth, we identify $\mathcal{A}$ with its copy $\pi_\alpha(\mathcal{A})$ inside $\mathcal{A} \rtimes_{\alpha,r}^u \Gamma$.

Remark 4.1.11. Note that if $\mathcal{A} = \mathbb{C}$ and (α, u) is the trivial twisted action, then the reduced twisted crossed product $\mathcal{A} \rtimes_{\alpha,r}^u \Gamma$ is just the reduced group C^*-algebra $C_r^*(\Gamma)$. ✺

Proposition 4.1.12. *The reduced twisted crossed product $\mathcal{A} \rtimes_{\alpha,r}^u \Gamma$ does not depend on the choice of faithful representation $\mathcal{A} \subseteq B(\mathcal{H})$.*

Proof. For all finite subsets $F \subseteq \Gamma$, let P_F be the projection defined in Lemma 4.1.8. Then $P_F \to 1_{\ell^2(\Gamma)}$ in the strong operator topology in $B(\ell^2(\Gamma))$, from which it follows that $1 \otimes P_F \to 1_{\mathcal{H}\otimes\ell^2(\Gamma)}$ strongly in $B(\mathcal{H} \otimes \ell^2(\Gamma))$. Therefore, if $x \in C_c(\Gamma, \mathcal{A})$ and $y = (\pi_\alpha \times \lambda_u)(x)$, we have

$$(1 \otimes P_F)y(1 \otimes P_F) \to y$$

strongly and thus

$$\|x\| = \sup_F \|(1 \otimes P_F)y(1 \otimes P_F)\|.$$

By Lemma 4.1.8, each operator $(1 \otimes P_F)y(1 \otimes P_F)$ belongs to $M_F(\mathcal{A})$. Since the C^*-algebra norm of $M_F(\mathcal{A})$ is unique, the norm of $(1 \otimes P_F)y(1 \otimes P_F) \in M_F(\mathcal{A}) \subseteq B(\mathcal{H} \otimes \ell^2(F))$ does not depend on the faithful representation $\mathcal{A} \subseteq B(\mathcal{H})$, so the above equality tells us that the same holds for x. □

The manner in which the reduced twisted crossed product was constructed now reveals some useful structural properties.

Remark 4.1.13. Once again faithfully representing $\mathcal{A}$ on a Hilbert space, the reduced twisted crossed product $\mathcal{A} \rtimes^u_{\alpha,r} \Gamma$ is the C^*-algebra generated by the sets $\pi_\alpha(\mathcal{A})$ and $\lambda_u(\Gamma)$. Moreover, by definition we have that elements of the form

$$\pi_\alpha(a) + \sum_{s\in F} \pi_\alpha(a_s)\lambda_u(s)$$

where $F \subseteq \Gamma \setminus \{1\}$ is a finite subset, $a \in \mathcal{A}$ and $a_s \in \mathcal{A}$ for all $s \in F$ constitute a norm-dense *-subalgebra of $\mathcal{A} \rtimes^u_{\alpha,r} \Gamma$. ✺

Remark 4.1.14. Let $(\mathcal{A}, \Gamma, \alpha, u)$ be a twisted dynamical system, where $\mathcal{A} \subseteq B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, and construct the reduced twisted crossed product $\mathcal{A} \rtimes^u_{\alpha,r} \Gamma \subseteq B(\mathcal{K})$ as before, where $\mathcal{K} = \mathcal{H} \otimes \ell^2(\Gamma)$. Then

$$\pi_\alpha(\alpha_s(a)) = \lambda_u(s)\pi_\alpha(a)\lambda_u(s)^* \tag{4.1.1}$$

for all $s \in \Gamma$ and $a \in \mathcal{A}$. As π_α is also a faithful representation of $\mathcal{A}$, we can identify $\mathcal{A}$ with the C^*-subalgebra $\pi_\alpha(\mathcal{A}) \subseteq B(\mathcal{K})$ and repeat the construction to obtain a copy of $\mathcal{A} \rtimes^u_{\alpha,r} \Gamma$ inside $B(\mathcal{K}\otimes\ell^2(\Gamma))$ and moreover, the action of Γ on $\mathcal{A}$ is unitarily implemented on $\mathcal{K}$ by means of (4.1.1). ✺

Before stating one of the main structural results for twisted dynamical systems, we will define a map $\tilde{u}\colon \Gamma \times \Gamma \to \mathcal{U}(\mathcal{A})$ (where $(\mathcal{A}, \Gamma, \alpha, u)$ is a twisted dynamical system) by

$$\tilde{u}(s,t) = u(s,t)u(sts^{-1}, s)^*, \quad s, t \in \Gamma.$$

It is then easily verified that

$$\lambda_u(s)\lambda_u(t)\lambda_u(s)^* = \pi_\alpha(\tilde{u}(s,t))\lambda_u(sts^{-1}), \quad s, t \in \Gamma.$$

Theorem 4.1.15. *Let $(\mathcal{A}, \Gamma, \alpha, u)$ be a twisted dynamical system, let Λ be a normal subgroup of Γ, let $Q = \Gamma/\Lambda$ and let $j\colon \Gamma \to Q$ denote the canonical epimorphism. Moreover, let (α', u') denote the restriction of (α, u) to Λ. Then for any $s \in \Gamma$, there exists $\gamma_s \in \mathrm{Aut}(\mathcal{A} \rtimes^{u'}_{\alpha',r} \Lambda)$ such that*

$$\gamma_s(\pi_{\alpha'}(a)) = \pi_{\alpha'}(\alpha_s(a)), \quad \gamma_s(\lambda_{u'}(t)) = \pi_{\alpha'}(\tilde{u}(s,t))\lambda_{u'}(sts^{-1}), \quad a \in \mathcal{A},\ t \in \Lambda. \tag{4.1.2}$$

Moreover, if $k\colon \Gamma/\Lambda \to \Gamma$ is a cross-section for j with $k(1) = 1$, define maps

- *$\beta\colon Q \to \mathrm{Aut}(\mathcal{A} \rtimes^{u'}_{\alpha',r} \Lambda)$ by $\beta = \gamma \circ k$,*
- *$m\colon Q \times Q \to \Lambda$ by $m(x,y) = k(x)k(y)k(xy)^{-1}$ for $x, y \in Q$, and*
- *$v\colon Q \times Q \to \mathcal{U}(\mathcal{A} \rtimes^{u'}_{\alpha',r} \Lambda)$ by*

$$v(x,y) = \pi_{\alpha'}(u(k(x), k(y))u(m(x,y), k(xy))^*)\lambda_{u'}(m(x,y)), \quad x, y \in Q.$$

Then (β, v) is a twisted action of Q on $\mathcal{A} \rtimes^{u'}_{\alpha',r} \Lambda$ such that

$$\mathcal{A} \rtimes^u_{\alpha,r} \Gamma \cong (\mathcal{A} \rtimes^{u'}_{\alpha',r} \Lambda) \rtimes^v_{\beta,r} Q.$$

A proof of the above theorem is given in Appendix B. In the sequel, we will just write (α, u) instead of (α', u').

Just like their not-as-twisted siblings, reduced twisted crossed products also admit a conditional expectation onto their C^*-algebra component. We postpone the proof until we have established the theory of so-called regular extensions of von Neumann algebras (see Theorem 4.3.4 and Proposition 4.3.7).

Theorem 4.1.16. *Let $(\mathcal{A}, \Gamma, \alpha, u)$ be a twisted dynamical system. Then there exists a faithful conditional expectation E of norm 1 of $\mathcal{A} \rtimes^u_{\alpha,r} \Gamma$ onto $\mathcal{A}$ (identified with $\pi_\alpha(\mathcal{A})$) such that $E(\lambda_u(s)) = 0$ for all $s \in \Gamma \setminus \{1\}$. Further, if we define $x(s) = E(x\lambda_u(s)^*) \in \mathcal{A}$ for all $x \in \mathcal{A} \rtimes^u_{\alpha,r} \Gamma$ and $s \in \Gamma$, called the* Fourier coefficient of x at s*, then we have the following:*

(i) $E(\lambda_u(s)x\lambda_u(s)^*) = \alpha_s(E(x))$ *for all* $s \in \Gamma$.
(ii) $E(xx^*) = \sum_{s\in\Gamma} x(s)x(s)^*$ *(in the strong operator topology).*

We recall that for a group Γ, an automorphism $\sigma \in \mathrm{Aut}(\Gamma)$ is called *inner* if it is of the form $\sigma(t) = sts^{-1}$ for some $s \in \Gamma$; we will write $\mathrm{Ad}(s)$ for σ in this case. A non-inner automorphism is said to be *outer*. If $\mathcal{A}$ is a unital C^*-algebra, an inner automorphism of $\mathcal{A}$ is one of the form $\mathrm{Ad}(u)$ for $u \in \mathcal{U}(\mathcal{A})$; non-inner automorphisms are also called outer.

Relevant to the next discussion is the notion of free automorphisms of groups.

Definition 4.1.17. Let Λ be a group and let $\sigma \in \mathrm{Aut}(\Lambda)$. If the set $\{\sigma(s)ts^{-1} \,|\, s \in \Lambda\}$ is infinite for all $t \in \Lambda$, we say that σ *acts freely on* Λ.

Definition 4.1.18. Let Γ be a group and let $\Lambda \subseteq \Gamma$ be a normal subgroup. We say that Γ/Λ *acts freely on* Λ if $\mathrm{Ad}(s)$ acts freely on Λ for all $s \in \Gamma \setminus \Lambda$.

We have the following observation concerning outer group automorphisms.

Lemma 4.1.19. *Let Γ be a group and let Λ be a normal icc subgroup of Γ. Then:*

(i) *An automorphism $\sigma \in \mathrm{Aut}(\Lambda)$ is outer if and only if σ is freely acting.*
(ii) *The centralizer of Λ in Γ is trivial if and only if Γ/Λ acts freely on Λ.*

Proof. (i) If σ is inner, then $\{\sigma(s)ts^{-1} \,|\, s \in \Lambda\} = \{t\}$ for some $t \in \Lambda$. Conversely, assume that the subset $D = \{\sigma(s)ts^{-1} \,|\, s \in \Lambda\}$ is finite for some $t \in \Lambda$, note first that $t \in D$. Assuming that $D \neq \{t\}$, take $t' \in D$ with $t' \neq t$; then it is easily shown that $D = \{\sigma(s)t's^{-1} \,|\, s \in \Lambda\}$. Now

$$C = \{st(t')^{-1}s^{-1} \,|\, s \in \Lambda\} = \{\sigma(s)t(t')^{-1}\sigma(s^{-1}) \,|\, s \in \Lambda\} = \{\sigma(s)ts^{-1}(\sigma(s)t's^{-1})^{-1} \,|\, s \in \Lambda\} \subseteq DD^{-1}.$$

Since D is finite, so is C. Because C is a finite conjugacy class in an icc group, we must have $t(t')^{-1} = 1$ or $t = t'$, a contradiction. Hence $D = \{t\}$, so σ is inner.

(ii) If $r \in \Gamma \setminus \Lambda$ and Λ has trivial centralizer in Γ, then suppose that there exists $t \in \Lambda$ such that $rsr^{-1} = tst^{-1}$ for all $s \in \Lambda$. Then $t^{-1}r$ belongs to the centralizer of Λ, so that $r = t \in \Lambda$ – a contradiction. Hence $\mathrm{Ad}(r)$ is outer. Conversely, suppose that there exists $r \in \Gamma \setminus \{1\}$ in the centralizer of Λ. Then $srs^{-1} = r$ for all $t \in \Lambda$, so that $r \notin \Lambda$; otherwise Λ would not be icc. Moreover, $\mathrm{Ad}(r)$ is the identity map on Λ and hence inner. □

An analogue of free action on groups also exists for C^*-algebras.

Definition 4.1.20. Let $\mathcal{A}$ be a C^*-algebra and let $\alpha \in \mathrm{Aut}(\mathcal{A})$. We say that α *acts freely on* $\mathcal{A}$ if 0 is the only element a of $\mathcal{A}$ that satisfies

$$\alpha(x)a = ax$$

for all $x \in \mathcal{A}$.

It is immediate that any freely acting automorphism of a C^*-algebra is outer. For an explanation of why the above property is called free action, we refer to Theorem 4.3.9.

Lemma 4.1.21. *Let $(\mathcal{A}, \Gamma, \alpha, u)$ be a twisted dynamical system, where $\mathcal{A}$ is a C^*-algebra with a faithful α-invariant state ψ. Further, let Λ be a normal subgroup of Γ and let $s_0 \in \Gamma$. If $\mathrm{Ad}(s_0)$ is freely acting, then the automorphism $\gamma_{s_0} \in \mathrm{Aut}(\mathcal{A} \rtimes^u_{\alpha,r} \Lambda)$ of Theorem 4.1.15 is freely acting and hence outer.*

Proof. Note first that since ψ is α-invariant, then for all $s, t \in \Gamma$ and $a \in \mathcal{A}$ we have

$$\psi(u(s,t)au(s,t)^*) = \psi(\alpha_s(\alpha_t(\alpha_{st}^{-1}(a)))) = \psi(a).$$

Furthermore, if we let $(\pi_\psi, \mathcal{H}_\psi, \xi_\psi)$ be the GNS triple associated to ψ, we can identify $\mathcal{A}$ with $\pi_\psi(\mathcal{A})$ so that ψ becomes a vector state and therefore is strongly continuous. To spare the eyes some misery, let $\omega = \mathrm{Ad}(s_0)$ in the following. Assume that $a \in \mathcal{A} \rtimes_{\alpha,r}^u \Lambda$ satisfies $\gamma_{s_0}(x)a = ax$ for all $x \in \mathcal{A} \rtimes_{\alpha,r}^u \Lambda$. Then by (4.1.2), we have

$$\tilde{u}(s_0,t)\lambda_u(\omega(t))a\lambda_u(t)^* = \gamma_{s_0}(\lambda_u(t))a\lambda_u(t)^* = a$$

for all $t \in \Gamma$. We now bring in the Fourier coefficients $a(t) = E(a\lambda_u(t)^*)$ for all $t \in \Lambda$, where E is the canonical conditional expectation of $\mathcal{A} \rtimes_{\alpha,r}^u \Lambda$ onto $\mathcal{A}$. Then for all $g, t \in \Lambda$, note that

$$\begin{aligned}
\tilde{u}(s_0,t)\alpha_{\omega(t)}(a(g)) &= \tilde{u}(s_0,t)\alpha_{\omega(t)}(E(a\lambda_u(g)^*)) \\
&= \tilde{u}(s_0,t)E(\lambda_u(\omega(t))a\lambda_u(g)^*\lambda_u(\omega(t))^*) \\
&= E(\tilde{u}(s_0,t)\lambda_u(\omega(t))a\lambda_u(g)^*\lambda_u(\omega(t))^*) \\
&= E(\tilde{u}(s_0,t)\lambda_u(\omega(t))a\lambda_u(t)^*\lambda_u(t)\lambda_u(g)^*\lambda_u(\omega(t))^*) \\
&= E(a\lambda_u(t)\lambda_u(g)^*\lambda_u(\omega(t))^*).
\end{aligned}$$

Since

$$\begin{aligned}
\lambda_u(\omega(t))\lambda_u(g)\lambda_u(t)^* &= u(\omega(t),g)\lambda_u(\omega(t)g)\lambda_u(t)^* \\
&= u(\omega(t),g)\lambda_u(\omega(t)g)\left[u(t^{-1},t)^*\lambda_u(t^{-1})\right] \\
&= u(\omega(t),g)\left[\lambda_u(\omega(t)g)u(t^{-1},t)^*\lambda_u(\omega(t)g)^*\right]\lambda_u(\omega(t)g)\lambda_u(t^{-1}) \\
&= u(\omega(t),g)\alpha_{\omega(t)g}(u(t^{-1},t))^*\lambda_u(\omega(t)g)\lambda_u(t^{-1}) \\
&= u(\omega(t),g)\alpha_{\omega(t)g}(u(t^{-1},t))^*u(\omega(t)g,t^{-1})\lambda_u(\omega(t)gt^{-1}) \\
&= u(\omega(t),g)u(\omega(t)gt^{-1},t)^*\lambda_u(\omega(t)gt^{-1}).
\end{aligned}$$

If we define $v(t,g) = u(\omega(t),g)u(\omega(t)gt^{-1},t)^*$, we finally see that

$$\tilde{u}(s_0,t)\alpha_{\omega(t)}(a(g)) = E(a\lambda_u(\omega(t)gt^{-1})^*v(t,g)^*) = a(\omega(t)gt^{-1})v(t,g)^*$$

and thus

$$a(\omega(t)gt^{-1}) = \tilde{u}(s_0,t)\alpha_{\omega(t)}(a(g))v(t,g).$$

Fix $g \in \Lambda$ and take an infinite sequence $(t_n)_{n\geq 1}$ of elements in Λ such that $\{s_n = \omega(t_n)gt_n^{-1} \mid n \geq 1\}$ is infinite. Then

$$\begin{aligned}
\psi(a(g_n)a(g_n)^*) &= \psi(\tilde{u}(s_0,t_n)\alpha_{\omega(t_n)}(a(g))v(t_n,g)v(t_n,g)^*\alpha_{\omega(t_n)}(a(g))^*\tilde{u}(s_0,t_n)^*) \\
&= \psi(\alpha_{\omega(t_n)}(a(g))\alpha_{\omega(t_n)}(a(g))^*) \\
&= \psi(a(g)a(g)^*)
\end{aligned}$$

since ψ is α-invariant, and thus

$$\begin{aligned}
\sum_{n=1}^{\infty}\psi(a(g_n)a(g_n)^*) &\leq \sum_{t\in\Lambda}\psi(a(t)a(t)^*) \\
&= \psi\left(\sum_{t\in\Lambda}a(t)a(t)^*\right) \\
&= \psi(E(aa^*)) < \infty,
\end{aligned}$$

by Theorem 4.1.16 (ii), as ψ is also strongly continuous. Since $\psi(a(g_n)a(g_n)^*) = \psi(a(g)a(g)^*)$ for all $n \geq 1$, we must have $\psi(a(g)a(g)^*) = 0$ and hence $a(g) = 0$ by faithfulness of ψ. Because $g \in \Lambda$ was arbitrary, we finally see that $E(aa^*) = 0$ (also by Theorem 4.1.16 (ii)) and hence $a = 0$ by faithfulness of E. We conclude that γ_{s_0} is freely acting. □

In order to piece the main result of this section together, we will need a result by Kishimoto concerning reduced crossed products of simple C^*-algebras, which Bédos then modified to make work for twisted crossed products.

Lemma 4.1.22 (Kishimoto, 1981)**.** *Let a be a positive element of a simple C^*-algebra $\mathcal{A}$, $a_1, \ldots, a_n \in \mathcal{A}$ and $\alpha_1, \ldots, \alpha_n$ be outer automorphisms of $\mathcal{A}$. Then for any $\varepsilon > 0$ there exists a positive $y \in \mathcal{A}$ of norm 1 such that $\|yay\| \geq \|a\| - \varepsilon$ and $\|ya_i\alpha_i(y)\| \leq \varepsilon$ for all $i = 1, \ldots, n$.*

Proof. See [44, Lemma 3.2]. □

Theorem 4.1.23. *Let $(\mathcal{A}, \Gamma, \alpha, u)$ be a twisted dynamical system such that $\mathcal{A}$ is simple and α_s is outer for all $s \neq 1$. Then $\mathcal{A} \rtimes^u_{\alpha,r} \Gamma$ is simple.*

Proof. Let $\mathcal{B} = \mathcal{A} \rtimes^u_{\alpha,r} \Gamma$ and let $E\colon \mathcal{B} \to \mathcal{A}$ be the faithful conditional expectation of Theorem 4.1.16. Let $\mathfrak{J}$ be a proper closed ideal in $\mathcal{B}$. To show that $\mathfrak{J} = \{0\}$, it is enough to show that $E(\mathfrak{J}) = \{0\}$ by faithfulness of E. Now define

$$\|x\|' = \inf\{\|x + j\| \mid j \in \mathfrak{J}\}, \quad x \in \mathcal{B}.$$

Since $\mathcal{A}$ is simple, we have $\mathcal{A} \cap \mathfrak{J} = \{0\}$ (otherwise $\mathcal{A} \subseteq \mathfrak{J}$, implying $\mathfrak{J} = \mathcal{B}$) and hence the restriction of the homomorphism $\mathcal{B} \to \mathcal{B}/\mathfrak{J}$ to $\mathcal{A}$ is injective. Therefore $\|a\|' = \|a\|$ for all $a \in \mathcal{A}$. If we can show that $\|E(x)\| \leq \|x\|'$ for all $x \in \mathcal{B}$, then the fact that $\|j\|' = 0$ for all $j \in \mathfrak{J}$ implies the wanted result.

As elements of the form $x = a + \sum_{s\in F} a_s\lambda_u(s)$ are norm-dense in $\mathcal{B}$, where $a \in \mathcal{A}$, $F \subseteq \Gamma \setminus \{1\}$ is finite and $a_s \in \mathcal{A}$ for $s \in F$, it suffices to check the inequality for such elements. If $a \in \mathcal{A}$ is positive and $\varepsilon > 0$, then since α_s is outer for all $s \in F$, Lemma 4.1.22 yields a positive element $y \in \mathcal{A}$ of norm 1 such that $\|yay\| \geq \|a\| - \varepsilon$ and $\|ya_s\alpha_s(y)\| \leq \varepsilon$ for all $s \in F$. Then

$$\left\| y \left(\sum_{s\in F} a_s\lambda_u(s) \right) y \right\|' \leq \sum_{s\in F} \|ya_s\lambda_u(s)y\lambda_u(s)^*\lambda_u(s)\|' \leq \sum_{s\in F} \|ya_s\alpha_s(y)\|' = \sum_{s\in F} \|ya_s\alpha_s(y)\|,$$

which in turn implies

$$\|E(x)\| = \|a\| \leq \|yay\| + \varepsilon = \|yay\|' + \varepsilon \leq \|yxy\|' + \left\| y \left(\sum_{s\in F} a_s\lambda_u(s) \right) y \right\|' + \varepsilon \leq \|x\|' + (|F| + 1)\varepsilon.$$

As $\varepsilon > 0$ was arbitrary, it follows that $\|E(x)\| \leq \|x\|'$. If a is not positive, note that the element $E(a^*x) = a^*E(x) = a^*a$ is positive, so by what we just found, we have

$$\|E(x)\|^2 = \|a\|^2 = \|a^*a\| = \|E(a^*x)\| \leq \|a^*x\|' \leq \|a\|\|x\|' = \|E(x)\|\|x\|'.$$

This proves the desired inequality. □

Finally we hit upon a true gold egg nugget of C^*-simplicity:

Theorem 4.1.24. *Let $(\mathcal{A}, \Gamma, \alpha, u)$ be a twisted dynamical system, where $\mathcal{A}$ is a C^*-algebra with a faithful α-invariant state ψ. Let Λ be a normal icc subgroup of Γ with trivial centralizer. If $\mathcal{A} \rtimes^u_{\alpha,r} \Lambda$ is simple, then $\mathcal{A} \rtimes^u_{\alpha,r} \Gamma$ is simple. In particular, Γ is C^*-simple whenever Λ is C^*-simple, and ultraweak Powers groups are C^*-simple.*

Proof. Let $Q = \Gamma/\Lambda$, and let $j\colon \Gamma \to Q$ denote the canonical epimorphism. If we take a cross-section $k\colon Q \to \Gamma$ such that $k(1) = 1$, then by Theorem 4.1.15 there exists a twisted action (β, v) of Q on $\mathcal{A} \rtimes^u_{\alpha,r} \Lambda$ such that

$$\mathcal{A} \rtimes^u_{\alpha,r} \Gamma \cong (\mathcal{A} \rtimes^u_{\alpha,r} \Lambda) \rtimes^v_{\beta,r} Q.$$

For each $t \in Q\setminus\{1\}$, the map $\mathrm{Ad}(k(t))$ is an outer automorphism of Λ by Lemma 4.1.19 (ii). By Lemma 4.1.21 and Lemma 4.1.19 (i), $\beta_t = \gamma_{k(t)}$ is an outer automorphism of $(\mathcal{A} \rtimes^u_{\alpha,r} \Lambda) \rtimes^v_{\beta,r} Q \cong \mathcal{A} \rtimes^u_{\alpha,r} \Gamma$ for all $t \in Q \setminus \{1\}$. Therefore simplicity of $\mathcal{A} \rtimes^u_{\alpha,r} \Lambda$ implies simplicity of $\mathcal{A} \rtimes^u_{\alpha,r} \Gamma$ by Theorem 4.1.23. The second statement clearly follows, as all C^*-simple groups are icc by Corollary 1.7.8, and weak Powers groups are C^*-simple by Proposition 3.3.11. □

4.2 Extensions by ultraweak Powers groups

The method applied by Bédos to show C^*-simplicity of ultraweak Powers groups can be in fact also be used to show something a lot stronger, as the next couple of results generalize our previous Dixmier property-related results of the reduced group C^*-algebra of a weak Powers group to twisted reduced crossed products.

Throughout this section, we will *always* let $(\mathcal{A}, \Gamma, \alpha, u)$ be a twisted dynamical system. Letting $\mathcal{H}$ be a Hilbert space on which $\mathcal{A}$ can be faithfully represented, we view $\mathcal{A}$ as a *-subalgebra of

$$\mathcal{B} = \mathcal{A} \rtimes^u_{\alpha,r} \Gamma.$$

Further, let $\mathcal{B}_0$ be the *-subalgebra of $\mathcal{B}$ generated by $\mathcal{A}$ and $\lambda_u(\Gamma)$, so that $\mathcal{B}_0$ is dense in $\mathcal{B}$.

Definition 4.2.1. A linear map $\varphi\colon \mathcal{B} \to \mathcal{B}$ is called a *simple* Γ*-averaging process* if there exist $n \geq 1$ and $s_1, \ldots, s_n \in \Gamma$ such that

$$\varphi(x) = \frac{1}{n}\sum_{i=1}^{n} \lambda_u(s_i)x\lambda_u(s_i)^*, \quad x \in \mathcal{B}.$$

A linear map is called a Γ-*averaging process* if it is a finite composition of simple Γ-averaging processes.

If we let $a \in \mathcal{A}$, we have

$$\lambda_u(s)(a\lambda_u(t))\lambda_u(s)^* = \alpha_s(a)\lambda_u(s)\lambda_u(t)\lambda_u(s)^* = \alpha_s(a)\tilde{u}(s,t)\lambda_u(sts^{-1}). \tag{4.2.1}$$

Consequently, if $x \in \mathcal{B}_0$, then for any simple Γ-averaging process $\varphi\colon \mathcal{B} \to \mathcal{B}$ the element $\varphi(x)$ belongs to $\mathcal{B}_0$. Moreover, we have $E(\varphi(x)) = 0$ if and only if $E(x) = 0$.

The next lemma generalizes Lemma 3.3.13 (for weak Powers groups) to reduced twisted crossed products.

Lemma 4.2.2. *Let $(\mathcal{A}, \Gamma, \alpha, u)$ be as above and let $x \in \mathcal{B}$ be a self-adjoint element satisfying $E(x) = 0$, where E is the conditional expectation of $\mathcal{B}$ onto $\mathcal{A}$. Assume further that Γ is a weak Powers group. Then for any $\varepsilon > 0$ there exists a Γ-averaging process φ on $\mathcal{B}$ such that $\|\varphi(x)\| < \varepsilon$.*

Proof. Suppose first that we have proved the above for self-adjoint $x \in \mathcal{B}_0$ satisfying $E(x) = 0$. Then for any self-adjoint $x \in \mathcal{B}$ with $E(x) = 0$, there exists $x_0 \in \mathcal{B}_0$ such that $\|x - x_0\| < \frac{\varepsilon}{3}$. Applying the result to $x_0 - E(x_0) \in \mathcal{B}_0$, we obtain $n \geq 1$ and $s_1, \ldots, s_n \in \Gamma$ such that if we define $f\colon \mathcal{B} \to \mathcal{B}$ by $f(y) = \frac{1}{n}\sum_{i=1}^{n} \lambda_u(s_i)y\lambda_u(s_i)^*$, we have $\|f(x_0) - f(E(x_0))\| < \frac{\varepsilon}{3}$. We then have

$$\|f(x)\| \leq \|f(x) - f(x_0)\| + \|f(x_0) - f(E(x_0))\| + \|f(E(x_0)) - f(E(x))\| \leq 2\|x - x_0\| + \frac{\varepsilon}{3} < \varepsilon,$$

since both f and E are contractive.

Now let $x \in \mathcal{B}_0$ be self-adjoint and assume that $E(x) = 0$. Then there exist $n \geq 1$, $a_1, \ldots, a_n \in \mathcal{A} \setminus \{0\}$ and $g_1, \ldots, g_n \in \Gamma$ such that by defining

$$x_i = a_i\lambda_u(g_i) + u(g_i^{-1}, g_i)^*\alpha_{g_i^{-1}}(a_i^*)\lambda_u(g_i^{-1})$$

for all $i = 1, \ldots, n$, we have $x = \sum_{i=1}^{n} x_i$. Since $E(x) = 0$, we can further assume that all g_i are not the identity element. Now, for each $i = 1, \ldots, n$ let $N_i \geq 1$ be an integer such that

$$\frac{2}{\sqrt{N_i}}\|x_i\| \leq \frac{\varepsilon}{n}.$$

Since Γ is a weak Powers group, there exists a partition $\Gamma = C_1 \sqcup D_1$ and elements $s_{1,1}, \ldots, s_{1,N_1} \in \Gamma$ such that $g_1C_1 \cap C_1 = g_1^{-1}C_1 \cap C_1 = \emptyset$ and $s_{1,i}D_1 \cap s_{1,j}D_1 = \emptyset$ for all distinct $i, j \in \{1, \ldots, N_1\}$. Let p_1 denote the projection of $\mathcal{H} \otimes \ell^2(\Gamma)$ onto $\mathcal{H} \otimes \ell^2(C_1)$ and define $u_{1,i} = \lambda_u(s_{1,i})$ for all $i = 1, \ldots, N_1$. Then

$$u_{1,i}(1 - p_1)u_{1,i}^*(\xi \otimes \delta_s) = u_{1,i}(1 - p_1)\left(u(s^{-1}, s_{1,i})^*\xi \otimes \delta_{s_{1,i}^{-1}s}\right) = \begin{cases} \xi \otimes \delta_s & \text{if } s_{1,i}^{-1}s \in D_1 \\ 0 & \text{else,} \end{cases}$$

from which it follows that $u_{1,i}(1-p_1)u_{1,i}^*$ is the projection of $\mathcal{H}\otimes\ell^2(\Gamma)$ onto $\mathcal{H}\otimes\ell^2(s_{1,i}D_1)$. Hence the projections $u_{1,i}(1-p_1)u_{1,i}^*$ are pairwise orthogonal, and since

$$p_1 b\lambda_u(s)p_1(\xi\otimes\delta_t) = \begin{cases} \alpha_{t^{-1}}(b)u(t^{-1}s^{-1},s)\xi\otimes\delta_{st} & \text{if } t\in C_1 \text{ and } t\in s^{-1}C_1 \\ 0 & \text{else,} \end{cases} \tag{4.2.2}$$

for all $b\in\mathcal{A}$ and $s\in\Gamma$, it is evident that $p_1x_1p_1 = 0$. Defining $\varphi_1\colon\mathcal{B}\to\mathcal{B}$ by

$$\varphi_1(y) = \frac{1}{N_1}\sum_{i=1}^{N_1}\lambda_u(s_{1,i})y\lambda_u(s_{1,i})^*,$$

then Lemma 3.3.12 tells us that $\|\varphi_1(x_1)\|\le\frac{2}{\sqrt{N_1}}\|x_1\|$.

We proceed inductively from here. Having found elements $s_{j,1},\dots,s_{j,N_j}\in\Gamma$ and an averaging process φ_j for all $1\le j\le k$ where $1\le k\le n-1$, define

$$S_k = \{s_{k,i_k}\cdots s_{1,i_1}\mid i_j = 1,\dots,N_j,\ j=1,\dots,k\}$$

and $F_{k+1} = \langle g_{k+1}\rangle_{S_k}$. Since Γ has the weak Powers property, then from the subset F_{k+1} and the integer N_{k+1} we obtain a partition $\Gamma = C_{k+1}\sqcup D_{k+1}$ and elements $s_{k+1,1},\dots,s_{k+1,N_{k+1}}\in\Gamma$ such that

$$fC_{k+1}\cap C_{k+1} = f^{-1}C_{k+1}\cap C_{k+1} = \emptyset,\quad s_{k+1,i}D_{k+1}\cap s_{k+1,j}D_{k+1} = \emptyset$$

for all distinct $i,j\in\{1,\dots,N_{k+1}\}$. Letting p_{k+1} denote the projection of $\mathcal{H}\otimes\ell^2(\Gamma)$ onto $\mathcal{H}\otimes\ell^2(C_{k+1})$, we see in the same way as above that the projections $\lambda_u(s_{k+1,i})(1-p_{k+1})\lambda_u(s_{k+1,i})^*$ are pairwise orthogonal. Now, as

$$\begin{aligned}(\varphi_k\circ\cdots\circ\varphi_1)(x_{k+1}) &= \frac{1}{N_1\cdots N_k}\sum_{i_1=1}^{N_1}\cdots\sum_{i_k=1}^{N_k}\lambda_u(s_{k,i_k})\cdots\lambda_u(s_{1,i_1})x_{k+1}\lambda_u(s_{1,i_1})^*\cdots\lambda_u(s_{k,i_k})^* \\ &= \sum_{s\in S_k}\left(b_{1,s}\lambda_u(sg_{k+1}s^{-1}) + b_{2,s}\lambda_u(sg_{k+1}^{-1}s^{-1})\right),\end{aligned}$$

for appropriately chosen $b_{1,s},b_{2,s}\in\mathcal{A}$ (seen by applying (4.2.1) a lot of times), it follows as in (4.2.2) that

$$p_{k+1}(\varphi_k\circ\cdots\circ\varphi_1)(x_{k+1})p_{k+1} = 0.$$

Defining a map $\varphi_{k+1}\colon\mathcal{B}\to\mathcal{B}$ by

$$\varphi_{k+1}(y) = \frac{1}{N_{k+1}}\sum_{i=1}^{N_{k+1}}\lambda_u(s_{k+1,i})y\lambda_u(s_{k+1,i})^*,$$

then Lemma 3.3.12 yields

$$\|(\varphi_{k+1}\circ\cdots\circ\varphi_1)(x_{k+1})\|\le\frac{2}{\sqrt{N_{k+1}}}\|(\varphi_k\circ\cdots\circ\varphi_1)(x_{k+1})\|\le\frac{2}{\sqrt{N_{k+1}}}\|x_{k+1}\|.$$

Having obtained simple Γ-averaging processes $\varphi_1,\dots,\varphi_n$ in this manner, we now find that

$$\begin{aligned}\|(\varphi_n\circ\cdots\circ\varphi_1)(x)\| &\le\sum_{i=1}^n\|(\varphi_n\circ\cdots\circ\varphi_1)(x_i)\| \\ &\le\sum_{i=1}^n\|(\varphi_i\circ\cdots\circ\varphi_1)(x_i)\| \\ &\le\sum_{i=1}^n\frac{2}{\sqrt{N_i}}\|x_i\|<\varepsilon,\end{aligned}$$

and the proof is complete. □

Having proved these preliminary results, we now turn to the question of whether weak Powers groups have nice properties with respect to twisted crossed products.

Definition 4.2.3. Let $(\mathcal{A},\Gamma,\alpha,u)$ be a twisted dynamical system. We say that the twisted action (α,u) is *minimal* if there are no proper closed two-sided ideals in $\mathcal{A}$ that are invariant under α_s for all $s\in\Gamma$.

If $\mathcal{A}$ is simple, then the action (α,u) is automatically minimal.

Lemma 4.2.4. *Suppose that the twisted action (α,u) on $\mathcal{A}$ is minimal, and let $a\in\mathcal{A}\setminus\{0\}$ be positive. Then there are $n\geq 1$, $a_1,\dots,a_n\in\mathcal{A}$ and $s_1,\dots,s_n\in\Gamma$ such that*

$$\sum_{i=1}^{n} a_i\alpha_{s_i}(a)a_i^*\geq 1_{\mathcal{A}}.$$

Proof. Let $\mathfrak{I}$ be the two-sided ideal generated by $\{\alpha_s(a)\mid s\in\Gamma\}$; note that we do not assume that $\mathfrak{I}$ is closed. Then for all $n\geq 1$, $x_1,y_1,\dots,x_n,y_n\in\mathcal{A}$ and $s,s_1,\dots,s_n\in\Gamma$, define $x=\sum_{i=1}^n x_i\alpha_{s_i}(a)y_i$ and note that

$$\alpha_s(x)=\sum_{i=1}^{n}\alpha_s(x_i)\alpha_s(\alpha_{s_i}(a))\alpha_s(y_i)=\sum_{i=1}^{n}\left[\alpha_s(x_i)u(s,s_i)\right]\alpha_{ss_i}(a)\left[u(s,s_i)^*\alpha_s(y_i)\right]\in\mathfrak{I},$$

so that $\overline{\mathfrak{I}}$ is a non-zero, closed two-sided ideal that is invariant under all α_s for $s\in\Gamma$. Since (α,u) is assumed to be minimal, we must have $\overline{\mathfrak{I}}=\mathcal{A}$ and hence $\mathfrak{I}=\mathcal{A}$ by Lemma A.2.1. Therefore there must exist $n\geq 1$, $b_1,c_1,\dots,b_n,c_n\in\mathcal{A}$ and $s_1,\dots,s_n\in\Gamma$ such that

$$\sum_{i=1}^{n} b_i\alpha_{s_i}(a)c_i=\frac{1}{2}1_{\mathcal{A}}.$$

If we define $a_i=b_i+c_i^*$ for all $i=1,\dots,n$, then

$$\sum_{i=1}^{n} a_i\alpha_{s_i}(a)a_i^*=\sum_{i=1}^{n} b_i\alpha_{s_i}(a)b_i^*+\sum_{i=1}^{n} c_i\alpha_{s_i}(a)c_i^*+1_{\mathcal{A}}\geq 1_{\mathcal{A}},$$

as wanted. □

Theorem 4.2.5. *If Γ is a weak Powers group and the twisted action (α,u) is minimal on $\mathcal{A}$, then $\mathcal{B}=\mathcal{A}\rtimes_{\alpha,r}^{u}\Gamma$ is simple.*

Proof. Let $\mathfrak{I}$ be a non-zero closed ideal of $\mathcal{B}$ and let $x\in\mathfrak{I}$ be a non-zero element. Then $y=x^*x\in\mathfrak{I}\setminus\{0\}$ is positive. The canonical conditional expectation E of $\mathcal{B}$ onto $\mathcal{A}$ is faithful and positive, so $E(y)$ is a non-zero positive element of $\mathcal{A}$. By Lemma 4.2.4, we have $n\geq 1$, $a_1,\dots,a_n\in\mathcal{A}$ and $s_1,\dots,s_n\in\Gamma$ such that

$$\sum_{i=1}^{n} a_i\alpha_{s_i}(E(y))a_i^*\geq 1_{\mathcal{A}}.$$

Note that

$$\begin{aligned}E\left(\sum_{i=1}^{n} a_i\lambda_u(s_i)y\lambda_u(s_i)^*a_i^*\right)&=\sum_{i=1}^{n} a_iE(\lambda_u(s_i)y\lambda_u(s_i)^*)a_i^*\\&=\sum_{i=1}^{n} a_i\alpha_{s_i}(E(y))a_i^*\geq 1_{\mathcal{A}}\end{aligned}$$

by Theorem 4.1.16 (i). Define $y_1=\sum_{i=1}^n a_i\lambda_u(s_i)y\lambda_u(s_i)^*a_i^*$. By Lemma 4.2.2, there now exists a Γ-averaging process $\varphi\colon\mathcal{B}\to\mathcal{B}$ such that

$$\|\varphi(y_1)-\varphi(E(y_1))\|=\|\varphi(y_1-E(y_1))\|<\frac{1}{2}.$$

As $\varphi(E(y_1))\geq\varphi(1_{\mathcal{A}})=1_{\mathcal{A}}$ and the elements $\varphi(y_1)$ and $\varphi(E(y_1))$ are positive, the above inequality implies that

$$\varphi(y_1)-1_{\mathcal{A}}\geq\varphi(y_1)-\varphi(E(y_1))\geq-\frac{1}{2}1_{\mathcal{A}}$$

and hence $\varphi(y_1)\geq\frac{1}{2}1_{\mathcal{A}}$. Therefore $\varphi(y_1)$ is invertible. Evidently $y_1\in\mathfrak{I}$ and $\varphi(y_1)\in\mathfrak{I}$, so $\mathfrak{I}$ contains an invertible element and thus must be all of $\mathcal{B}$. Therefore $\mathcal{B}$ is simple. □

We next turn to the question of uniqueness of trace.

Lemma 4.2.6. *If $\mathcal{A}$ has an α-invariant trace ψ, then $\tau = \psi \circ E$ is a trace on $\mathcal{B}$.*

Proof. Note first that τ is obviously a state on $\mathcal{B}$. For $x, y \in \mathcal{B}_0$, write

$$x = a + \sum_{i=1}^{n} a_i \lambda_u(s_i), \quad y = b + \sum_{i=1}^{n} \lambda_u(s_i)^* b_i,$$

where $a, b, a_1, b_1, \ldots, a_n, b_n \in \mathcal{A}$ and the elements $s_1, \ldots, s_n \in \Gamma \setminus \{1\}$ are distinct. Note that

$$\begin{aligned}
\tau(xy) &= \psi(E(xy)) \\
&= \psi(ab) + \sum_{i,j=1}^{n} \psi\left(a_i E(\lambda_u(s_i)\lambda_u(s_j)^*) b_j\right) \\
&= \psi(ab) + \sum_{i,j=1}^{n} \psi\left(a_i u(s_i, s_j^{-1}) E(\lambda_u(s_i s_j^{-1})) u(s_j, s_j^{-1})^* b_j\right) \\
&= \psi(ab) + \sum_{i=1}^{n} \psi\left(a_i b_i\right)
\end{aligned}$$

and that by Γ-invariance of ψ, we also have

$$\begin{aligned}
\tau(yx) &= \psi(E(yx)) \\
&= \psi(ba) + \sum_{i,j=1}^{n} \psi\left(\alpha_{s_i}(E(\lambda_u(s_i)^* b_i a_j \lambda_u(s_j)))\right) \\
&= \psi(ba) + \sum_{i,j=1}^{n} \psi\left(b_i a_j E(\lambda_u(s_j)\lambda_u(s_i)^*)\right) \\
&= \psi(ba) + \sum_{i,j=1}^{n} \psi\left(b_i a_j u(s_j, s_i^{-1}) E(\lambda_u(s_j s_i^{-1})) u(s_i, s_i^{-1})^*\right) \\
&= \psi(ba) + \sum_{i=1}^{n} \psi\left(b_i a_i\right).
\end{aligned}$$

Therefore $\tau(xy) = \tau(yx)$, since ψ is a trace. A standard density and continuity argument yields that τ is tracial on all of $\mathcal{B}$. □

Theorem 4.2.7. *If Γ is a weak Powers group and τ is a trace on $\mathcal{B}$, then $\tau' = \tau|_{\mathcal{A}}$ is an α-invariant trace on $\mathcal{A}$, i.e., $\tau' = \tau' \circ \alpha_s$ for all $s \in \Gamma$. Moreover, if E denotes the conditional expectation of $\mathcal{B}$ onto $\mathcal{A}$, then $\tau = \tau' \circ E$. In particular, if $\mathcal{A}$ has a unique α-invariant trace, then $\mathcal{B}$ has a unique trace.*

Proof. If $\varphi\colon \mathcal{B} \to \mathcal{B}$ is a simple Γ-averaging process, then clearly $\tau(\varphi(x)) = \tau(x)$ for all $x \in \mathcal{B}$. Hence this also holds for all Γ-averaging processes. Thus for all self-adjoint $x \in \mathcal{B}$, we have

$$|\tau(x - E(x))| = |\tau(\varphi(x - E(x)))| \leq \|\varphi(x - E(x))\|$$

for all averaging processes $\varphi\colon \mathcal{B} \to \mathcal{B}$, and Lemma 4.2.2 tells us that $\tau(x) = \tau(E(x))$. This clearly implies $\tau = \tau \circ E = \tau' \circ E$. Finally, τ' is α-invariant, since Theorem 4.1.16 (ii) implies

$$\tau'(\alpha_s(a)) = \tau'(E(\alpha_s(a))) = \tau(E(\alpha_s(a))) = \tau(\lambda_u(s) a \lambda_u(s)^*) = \tau(a) = \tau'(a)$$

for all $a \in \mathcal{A}$ and $s \in \Gamma$.

The last statement follows from the above considerations along with Lemma 4.2.6. □

What we have shown now is that when constructed by means of weak Powers groups, twisted reduced crossed products preserve simplicity and uniqueness of trace of the original C^*-algebra. If we now combine these results with the main structure theorem of the previous section (Theorem 4.1.15), we get an abundance of neat permanence results in return.

Corollary 4.2.8 (Bédos)**.** *Let $(\mathcal{A}, \Gamma, \alpha, u)$ be a twisted dynamical system and let Λ be a normal subgroup of Γ. Further, assume that Γ/Λ is a weak Powers group. Then the following holds:*

(i) *If $\mathcal{A} \rtimes^u_{\alpha,r} \Lambda$ is simple (resp. has unique trace), then $\mathcal{A} \rtimes^u_{\alpha,r} \Gamma$ is simple (resp. has unique trace).*
(ii) *If $C^*_r(\Lambda)$ is simple (resp. has unique trace), then $C^*_r(\Gamma)$ is simple (resp. has unique trace).*

Proof. It is clear that (ii) follows from (i), and (i) itself follows from Theorems 4.2.5 and 4.2.7 applied to the twisted dynamical system $(\mathcal{A} \rtimes^u_{\alpha,r} \Lambda, Q, \beta, v)$ of Theorem 4.1.15, where $Q = \Gamma/\Lambda$. Note here that with respect to this particular system, a unique trace on $\mathcal{A} \rtimes^u_{\alpha,r} \Lambda$ is necessarily β-invariant (so that Theorem 4.2.7 does apply). □

Corollary 4.2.9. *Let $(\mathcal{A}, \Gamma, \alpha, u)$ be a twisted dynamical system, where $\mathcal{A}$ is a simple C^*-algebra with a faithful α-invariant state and Γ is an ultraweak Powers group. Then $\mathcal{A} \rtimes^u_{\alpha,r} \Gamma$ is simple.*

Proof. Let $\Lambda \subseteq \Gamma$ be a normal weak Powers subgroup with trivial centralizer. Then $\mathcal{A} \rtimes^u_{\alpha,r} \Lambda$ is simple by Theorem 4.2.5, so that simplicity of $\mathcal{A} \rtimes^u_{\alpha,r} \Gamma$ follows from Theorem 4.1.24. □

Corollary 4.2.10. *Let $(\mathcal{A}, \Gamma, \alpha, u)$ be a twisted dynamical system, where $\mathcal{A}$ is a simple C^*-algebra with a faithful α-invariant state ψ. Suppose further that there exists a short exact sequence*

$$1 \longrightarrow \Lambda \longrightarrow \Gamma \longrightarrow Q \longrightarrow 1,$$

where Λ and Q are ultraweak Powers groups. Then $\mathcal{A} \rtimes^u_{\alpha,r} \Gamma$ is simple. In particular, Γ is C^*-simple whenever such a short exact sequence exists.

Proof. We may assume that Λ is a normal subgroup of Γ and that $Q = \Gamma/\Lambda$. By Theorem 4.1.15, there exists a twisted action (β, v) of Q on $\mathcal{A} \rtimes^u_{\alpha,r} \Lambda$ such that

$$\mathcal{A} \rtimes^u_{\alpha,r} \Gamma \cong (\mathcal{A} \rtimes^u_{\alpha,r} \Lambda) \rtimes^v_{\beta,r} Q.$$

By Corollary 4.2.9, $\mathcal{B} = \mathcal{A} \rtimes^u_{\alpha,r} \Lambda$ is simple.

Letting E denote the canonical conditional expectation of $\mathcal{B}$ onto $\mathcal{A}$, we now claim that $\tilde{\psi} = \psi \circ E$ is a faithful β-invariant state on $\mathcal{B}$. That $\tilde{\psi}$ is a faithful state is immediate, so we focus on β-invariance. Let γ and k be as in Theorem 4.1.15, and let $y \in Q$. We then need to show that $\tilde{\psi} \circ \beta_y = \tilde{\psi}$, and by continuity it suffices to check the equality for elements of the form $x_0 = a_0 + \sum_{s \in F} a_s \lambda_u(s)$ where $F \subseteq \Gamma \setminus \{1\}$ is a finite subset, $a_0 \in \mathcal{A}$ and $a_s \in \mathcal{A}$ for all $s \in F$. Because

$$\begin{aligned}
\tilde{\psi}(\beta_y(x_0)) &= \psi\left(E\left(\gamma_{k(y)}(a_0) + \sum_{s \in F} \gamma_{k(y)}(a_s)\gamma_{k(y)}(\lambda_u(s))\right)\right) \\
&= \psi\left(\alpha_{k(y)}(a_0) + \sum_{s \in F} \alpha_{k(y)}(a_s)\tilde{u}(k(y), s)E(\lambda_u(k(y)sk(y)^{-1}))\right) \\
&= \psi(\alpha_{k(y)}(a_0)) \\
&= \psi(a_0) = \tilde{\psi}(x_0),
\end{aligned}$$

our claim does hold, and it now follows from Corollary 4.2.9 applied to $(\mathcal{B}, Q, \beta, v)$ that $\mathcal{A} \rtimes^u_{\alpha,r} \Gamma$ is simple. □

Corollary 4.2.11. *Let*

$$1 \longrightarrow \Gamma' \longrightarrow \Gamma \longrightarrow \Gamma'' \longrightarrow 1$$

be a short exact sequence of groups, where Γ' is C^*-simple and Γ'' is an ultraweak Powers group. Then Γ is C^*-simple.

Proof. Once again, we can assume that Γ' is a normal subgroup of Γ and that $\Gamma'' = \Gamma/\Gamma'$. By Theorem 4.1.15, there exists a twisted action (β, v) of Γ'' on $C^*_r(\Gamma')$ such that

$$C^*(\Gamma) \cong C^*_r(\Gamma') \rtimes^v_{\beta,r} \Gamma''.$$

We now claim that the faithful trace τ of $C_r^*(\Gamma')$ is β-invariant. Indeed, let γ and k be as in Theorem 4.1.15 and note that for all $x \in \Gamma''$ and $s \in \Gamma'$,

$$\tau(\beta_x(\lambda(s))) = \tau(\gamma_{k(x)}(\lambda(s))) = \tau(\lambda(k(x)sk(x)^{-1})) = \langle \delta_{k(x)sk(x)^{-1}}, \delta_1 \rangle = \begin{cases} 1 & \text{if } s = 1 \\ 0 & \text{else,} \end{cases}$$

or simply $\tau(\beta_x(\lambda(s))) = \tau(\lambda(s))$. By continuity, τ is β-invariant as claimed, and hence it follows from Corollary 4.2.9 that $C_r^*(\Gamma)$ is simple. $\square$

One might now ask: what happened to uniqueness of trace in the last three results? The answer is of course that the zenith of the previous section (Theorem 4.1.24) "only" uncovered simplicity of the reduced twisted crossed products, and that in itself was quite non-trivial: we needed an extraneous result by Kishimoto to make it all come together. If we want to settle the issue of uniqueness of trace, there is one obvious thing to try out: pass to von Neumann algebras.

4.3 Regular extensions of von Neumann algebras by discrete groups

We now introduce a von Neumann algebra version of the reduced twisted crossed product, and it should not be surprising that many of the results for reduced twisted crossed products carry over. Suppose we are given a Hilbert space $\mathcal{H}$ and a twisted dynamical system $(\mathscr{M}, \Gamma, \alpha, u)$ where $\mathscr{M} \subseteq B(\mathcal{H})$ is a von Neumann algebra, and let $(\pi_\alpha, \lambda_u, \mathcal{H} \otimes \ell^2(\Gamma))$ be the covariant representation that we constructed when we were defining the reduced twisted crossed product. Then π_α is a normal representation of $\mathscr{M}$ (following from the fact that α_s is normal for all $s \in \Gamma$).

Definition 4.3.1. The von Neumann algebra $\mathscr{M} \times_{(\alpha,u)} \Gamma$ generated by the subsets $\pi_\alpha(\mathscr{M})$ and $\lambda_u(\Gamma)$ of $B(\mathcal{H} \otimes \ell^2(\Gamma))$ is called the *regular extension of* $\mathscr{M}$ *by* Γ.

By the von Neumann density theorem, $\mathscr{M} \times_{(\alpha,u)} \Gamma$ is then the closure of the *-subalgebra

$$\left\{ \sum_{s \in F} \pi_\alpha(x_s)\lambda_u(s) \,\middle|\, F \subseteq \Gamma \text{ finite}, \, x_s \in \mathscr{M} \right\}$$

in any of the standard operator topologies.

Though it may seem not to be the case, the algebraic structure of $\mathscr{M} \times_{(\alpha,u)} \Gamma$ in fact does not depend on the Hilbert space $\mathcal{H}$. First and foremost, observe that

$$\pi_\alpha(x) = \sum_{t \in \Gamma} \alpha_{t^{-1}}(x) \otimes P_t \quad \text{and} \quad \lambda_u(s) = \sum_{t \in \Gamma} u(t^{-1}s^{-1}, s) \otimes \lambda(s)P_t$$

for all $x \in \mathscr{M}$ and $s \in \Gamma$, where P_t denotes the projection of $\ell^2(\Gamma)$ onto $\mathbb{C}\delta_t$ for all $t \in \Gamma$, λ is the standard left-regular representation of Γ on $\ell^2(\Gamma)$ and the sums converge in the strong operator topology. It then follows that

$$\mathscr{M} \times_{(\alpha,u)} \Gamma = \{\pi_\alpha(x), \, \lambda_u(s) \,|\, x \in \mathscr{M}, \; s \in \Gamma\}'' \subseteq \mathscr{M} \mathbin{\overline{\otimes}} B(\ell^2(\Gamma)),$$

where $\mathscr{M} \mathbin{\overline{\otimes}} B(\ell^2(\Gamma))$ denotes the von Neumann algebra tensor product of $\mathscr{M}$ and $B(\ell^2(\Gamma))$ (see [15, Section 1.3] for details on this type of tensor product).

Proposition 4.3.2. *Let $(\mathscr{M}, \Gamma, \alpha, u)$ and $(\mathscr{N}, \Gamma, \beta, v)$ be twisted dynamical systems, where $\mathscr{M} \subseteq B(\mathcal{H})$ and $\mathscr{N} \subseteq B(\mathcal{K})$ are von Neumann algebras. Suppose that there exists an isomorphism $\varphi\colon \mathscr{M} \to \mathscr{N}$ such that*

$$\varphi \circ \alpha_s = \beta_s \circ \varphi, \quad \varphi(u(s,t)) = v(s,t), \quad s, t \in \Gamma.$$

Then there is an isomorphism $\tilde{\varphi}\colon \mathscr{M} \times_{(\alpha,u)} \Gamma \to \mathscr{N} \times_{(\beta,v)} \Gamma$ satisfying

$$\pi_\beta \circ \varphi = \tilde{\varphi} \circ \pi_\alpha, \quad \tilde{\varphi} \circ \lambda_u = \lambda_v.$$

Proof. By [69, Corollary IV.5.3], there exists an isomorphism $\Omega\colon \mathscr{M} \mathbin{\overline{\otimes}} B(\ell^2(\Gamma)) \to \mathscr{N} \mathbin{\overline{\otimes}} B(\ell^2(\Gamma))$ such that

$$\Omega(x_1 \otimes x_2) = \varphi(x_1) \otimes x_2, \quad x_1 \in \mathscr{M}, \; x_2 \in B(\ell^2(\Gamma)).$$

Note that Ω is normal and hence SOT-to-SOT continuous on bounded sets, yielding

$$\Omega(\pi_\alpha(x)) = \sum_{t\in\Gamma} \varphi(\alpha_{t^{-1}}(x)) \otimes P_t = \sum_{t\in\Gamma} \beta_{t^{-1}}(\varphi(x)) \otimes P_t = \pi_\beta(\varphi(x))$$

and $\Omega(\lambda_u(s)) = \lambda_v(s)$ similarly. It then follows that the image of $\mathscr{M} \times_{(\alpha,u)} \Gamma$ under Ω is indeed $\mathscr{N} \times_{(\beta,v)} \Gamma$, so the proof is complete if we let $\tilde{\varphi}$ denote the restriction of Ω to $\mathscr{M} \times_{(\alpha,u)} \Gamma$. □

We then have the following beautiful result: a von Neumann algebraic version of Theorem 4.1.15.

Theorem 4.3.3 (Bédos). *Let $(\mathscr{M}, \Gamma, \alpha, u)$ be a twisted dynamical system where $\mathscr{M} \subseteq B(\mathcal{H})$ is a von Neumann algebra. Further, let Λ be a normal subgroup of Γ, define $Q = \Gamma/\Lambda$ and let $j\colon \Gamma \to Q$ denote the canonical epimorphism. If (α', u') is the restriction of (α, u) to Λ, then for all $s \in \Gamma$ there exists $\gamma_s \in \mathrm{Aut}(\mathscr{M} \times_{(\alpha',u')} \Lambda)$ such that*

$$\gamma_s(\pi_{\alpha'}(x)) = \pi_{\alpha'}(\alpha_s(x)), \quad \gamma_s(\lambda_{u'}(t)) = \pi_{\alpha'}(\tilde{u}(s,t))\lambda_{u'}(sts^{-1}), \quad x \in \mathscr{M},\ t \in \Lambda.$$

Moreover, if $k\colon \Gamma/\Lambda \to \Gamma$ is a cross-section for j with $k(1) = 1$, define maps

$$\beta\colon Q \to \mathrm{Aut}(\mathscr{M} \times_{(\alpha',u')} \Lambda), \quad m\colon Q \times Q \to \Lambda, \quad v\colon Q \times Q \to \mathcal{U}(\mathscr{M} \times_{(\alpha',u')} \Lambda)$$

as in Theorem 4.1.15. Then (β, v) is a twisted action of Q on $\mathscr{M} \times_{(\alpha',u')} \Lambda$ such that

$$\mathscr{M} \times_{(\alpha,u)} \Gamma \cong (\mathscr{M} \times_{(\alpha',u')} \Lambda) \times_{(\beta,v)} Q.$$

Proof. Almost all of the considerations and computations in the proof of Theorem 4.1.15 (given in Appendix B) carry over verbatim. In preparation for finding γ_s in that proof we were able to assume the existence of a map $a\colon \Gamma \to \mathcal{U}(\mathcal{H})$ implementing the action of Γ on $\mathscr{M}$ by unitary operators on $\mathcal{H}$ by means of Remark 4.1.14, but we may not be as lucky here. To ensure that we can still find automorphisms γ_s with the wanted properties, we do as follows. Define a twisted action $(\overline{\alpha}, \overline{u})$ of Γ on the von Neumann algebra $\pi_\alpha(\mathscr{M})$ by $\overline{\alpha}_s = \pi_\alpha \alpha_s \pi_\alpha^{-1}$ and $\overline{u} = \pi_\alpha \circ u$. Then $\overline{\alpha}_s(x) = \lambda_u(s) x \lambda_u(s)^*$ for all $x \in \pi_\alpha(\mathscr{M})$. Letting $(\overline{\alpha}', \overline{u}')$ denote the restriction of ω to Λ, the (relevant part of the) proof of Theorem 4.1.15 now applies and we obtain an automorphism $\overline{\gamma}_s \in \mathrm{Aut}(\pi_\alpha(\mathscr{M}) \times_{(\overline{\alpha}',\overline{u}')} \Lambda)$ satisfying

$$\overline{\gamma}_s(\pi_{\overline{\alpha}'}(x)) = \pi_{\overline{\alpha}'}(\overline{\alpha}_s(x)), \quad \overline{\gamma}_s(\lambda_{\overline{u}'}(t)) = \pi_{\overline{\alpha}'}(\tilde{\overline{u}}(s,t))\lambda_{\overline{u}'}(sts^{-1}), \quad x \in \pi_\alpha(\mathscr{M}),\ t \in \Lambda.$$

Now Proposition 4.3.2 yields an isomorphism $\varphi\colon \pi_\alpha(\mathscr{M}) \times_{(\overline{\alpha}',\overline{u}')} \Lambda \to \mathscr{M} \times_{(\alpha',u')} \Lambda$ such that

$$\gamma_s = \varphi \overline{\gamma}_s \varphi^{-1}$$

has the desired properties. □

Theorem 4.3.4. *Let $(\mathscr{M}, \Gamma, \alpha, u)$ be a twisted dynamical system, where $\mathscr{M} \subseteq B(\mathcal{H})$ is a von Neumann algebra. Then there exists a faithful normal conditional expectation E of norm 1 of $\mathscr{M} \times_{(\alpha,u)} \Gamma$ onto $\mathscr{M}$ (identified with $\pi_\alpha(\mathscr{M})$) such that $E(\lambda_u(s)) = 0$ for all $s \in \Gamma \setminus \{1\}$. Further, we have*

$$E(\lambda_u(s) x \lambda_u(s)^*) = \alpha_s(E(x))$$

for all $x \in \mathscr{M} \times_{(\alpha,u)} \Gamma$ and $s \in \Gamma$.

Proof. For all $s \in \Gamma$, let P_s denote the projection of $\mathcal{K} = \mathcal{H} \otimes \ell^2(\Gamma)$ onto the closed subspace $\mathcal{H} \otimes \mathbb{C}\delta_s$. We now define

$$F(x) = \sum_{s\in\Gamma} P_s x P_s, \quad x \in B(\mathcal{K}),$$

the series convergent in the strong operator topology. Then F is evidently linear and positive. Setting $\mathscr{N} = \{P_s \mid s \in \Gamma\}'$, then clearly $F(x) \in \mathscr{N}$ for all $x \in B(\mathcal{K})$ and $F(x) = x$ for all $x \in \mathscr{N}$, as $\sum_{s\in\Gamma} P_s = 1$. For all $\xi \in \mathcal{K}$ and $x \in B(\mathcal{K})$, we finally have

$$\|F(x)\xi\|^2 = \sum_{s\in\Gamma} \|P_s x P_s \xi\|^2 \le \sum_{s\in\Gamma} \|x\|^2 \|P_s\xi\|^2 = \|x\|^2 \sum_{s\in\Gamma} \|P_s\xi\|^2 = \|x\|^2\|\xi\|^2,$$

so that F is contractive. By Tomiyama's theorem [14, Theorem 1.5.10], F is a conditional expectation of $B(\mathcal{K})$ onto $\mathscr{N}$. Finally, if $F(x^*x) = 0$ for some $x \in B(\mathcal{K})$ then $(xP_s)^* x P_s = 0$ and hence $xP_s = 0$ for all $s \in \Gamma$. This implies that $x = 0$, so that F is faithful.

To see that F is normal, let $\omega \in B(\mathcal{H})_*$ and let $S \in B(\mathcal{H})$ be a trace class operator such that $\omega(x) = \mathrm{tr}(Sx)$, where $\mathrm{tr}(\cdot) = \sum_{\xi \in \mathcal{E}} \langle \cdot\, \xi, \xi \rangle$ for some orthonormal basis $\mathcal{E}$ for $\mathcal{K}$ (cf. [15, Section B.1]). Letting $\mathcal{E}_0$ be a fixed orthonormal basis for $\mathcal{H}$, then

$$\mathcal{E} = \{\xi \otimes \delta_s \,|\, \xi \in \mathcal{E}_0,\ s \in \Gamma\}$$

is an orthonormal basis for $\mathcal{K}$. If $x \in B(\mathcal{K})$ is trace class, we then have

$$\mathrm{tr}(F(x)) = \sum_{\substack{\xi \in \mathcal{E}_0 \\ s \in \Gamma}} \sum_{t \in \Gamma} \langle P_t x P_t (\xi \otimes \delta_s), \xi \otimes \delta_s \rangle = \sum_{\substack{\xi \in \mathcal{E}_0 \\ s \in \Gamma}} \langle x(\xi \otimes \delta_s), \xi \otimes \delta_s \rangle = \mathrm{tr}(x).$$

Hence for any $x \in B(\mathcal{K})$ we have

$$\omega(F(x)) = \mathrm{tr}(SF(x)) = \mathrm{tr}(F(SF(x))) = \mathrm{tr}(F(S)F(x)) = \mathrm{tr}(F(F(S)x)) = \mathrm{tr}(F(S)x).$$

If $(x_i)_{i \in I}$ is a bounded, increasing net of self-adjoint operators in $B(\mathcal{H})$ converging strongly to $x \in B(\mathcal{H})$ we then have

$$\omega(F(x_i)) = \mathrm{tr}(F(S)x_i) \to \mathrm{tr}(F(S)x) = \omega(F(x)),$$

since $\mathrm{tr}(F(S)\,\cdot)$ is a normal linear functional. Hence F is normal.

For all $s, t \in \Gamma$ with $t \neq 1$, we have $P_s \lambda_u(t) P_s = 0$ and that $xP_s = P_s x$ for all $x \in \mathscr{M}$ and $s \in \Gamma$. Therefore, if E denotes the restriction of F to $\mathscr{M} \times_{(\alpha,u)} \Gamma$ then E annihilates all operators of the form $\lambda_u(t)$ for $t \neq 1$. As $\mathscr{M} \subseteq \mathscr{N}$ and $\|E\| = \|E(1)\| = 1$, it is evident that the map E is the desired conditional expectation. To prove the final statement, recall that the *-algebra of operators of the form

$$y = x_0 + \sum_{s \in F} x_s \lambda_u(s)$$

is ultraweakly dense in $\mathscr{M} \times_{(\alpha,u)} \Gamma$, where $x_0 \in \mathscr{M}$, $F \subseteq \Gamma \setminus \{1\}$ is a finite subset and $x_s \in \mathscr{M}$ for all $s \in F$. If $y \in \mathscr{M} \rtimes^u_{\alpha,r} \Gamma$ is of the above form, we have

$$\begin{aligned} E\left(\lambda_u(g) y \lambda_u(g)^*\right) &= E(\lambda_u(g) x_0 \lambda_u(g)^*) + \sum_{s \in F} \alpha_g(x_s) \tilde{u}(g,s) E(\lambda_u(gsg^{-1})) \\ &= \alpha_g(E(y)), \end{aligned}$$

so the equality holds on an ultraweakly dense subalgebra and therefore on all of $\mathscr{M} \times_{(\alpha,u)} \Gamma$, as α_s and E are normal for all $g \in \Gamma$. □

As before, the conditional expectation of Theorem 4.3.4 is usually referred to as the canonical one, which we will do here.

Definition 4.3.5. Let E denote the canonical conditional expectation of $\mathscr{M} \times_{(\alpha,u)} \Gamma$ onto $\mathscr{M}$. For any $s \in \Gamma$ and $x \in \mathscr{M} \times_{(\alpha,u)} \Gamma$, the operator

$$x(s) = E(x\lambda_u(s)^*) \in \mathscr{M}$$

is called the *Fourier coefficient of x at s*.

Remark 4.3.6. Of course, with the mention of Fourier coefficients comes the hope that one might express x as a sum involving the Fourier coefficients, i.e.,

$$x = \sum_{s \in \Gamma} x(s) \lambda_u(s),$$

once again identifying $\mathscr{M}$ with $\pi_\alpha(\mathscr{M})$. As we shall see, this is true in some sense, but not in a mode of convergence that is immediately familiar. In a paper by Mercer [46] it is proved that even when the twisted action is trivial, the above sum might not even converge *weakly* to x, and hence not in any of the common operator topologies.

To work around this, let $(P_s)_{s \in \Gamma}$ be the family of projections from the proof of Theorem 4.3.4 and let $x \in \mathscr{M} \times_{(\alpha,u)} \Gamma$. Then for all $s, t \in \Gamma$, it is easily seen that

$$\lambda_u(t)^* P_s \lambda_u(t) = P_{t^{-1}s},$$

from which it follows that

$$P_s x(t)\lambda_u(t) = P_s x P_{t^{-1}s} \quad \text{and} \quad x(s)\lambda_u(s)P_t = P_{st} x P_t.$$

If we now define $x_F = \sum_{s\in F} x(s)\lambda_u(s)$ for all finite subsets $F \subseteq \Gamma$, then it is easily shown as in [46, Lemma 2] that

(i) $E(x_F^* x_F) = E(x_F^* x) = E(x^* x_F) = E(x^*(\sum_{s\in F} P_s)x)$.
(ii) $E(x_F x_F^*) = E(x_F x^*) = E(x x_F^*) = E(x(\sum_{s\in F} P_{s^{-1}})x^*)$.

One way of realizing this is to define operators $Q_g\colon \mathcal{H}\otimes \ell^2(\Gamma) \to \mathcal{H}$ for all $g \in \Gamma$ by $Q_g(\xi\otimes\delta_t) = \delta_{g,t}\xi$ and note that

$$Q_g^* Q_g = P_g, \quad Q_g P_s = \delta_{g,s} Q_g, \quad s \in \Gamma.$$

It can then be checked that $E(x) = Q_1 x Q_1^*$ for all $x \in \mathscr{M} \times_{(\alpha,u)} \Gamma$.

We now define a locally convex topology on $\mathscr{M} \times_{(\alpha,u)} \Gamma$, calling it the $\mathscr{M}$-topology, by means of the semi-norms

$$x \mapsto \omega(E(x^*x))^{1/2},$$

where ω runs over the normal states of $\mathscr{M}$. As E is faithful, the $\mathscr{M}$-topology is Hausdorff, and because both $\sum_{s\in F} P_{s^{-1}}$ and $\sum_{s\in F} P_s$ converge ultraweakly to the identity, it is easily seen that $x_F \to x$ and $x_F^* \to x^*$ in the $\mathscr{M}$-topology. This mode of convergence also ensures that x is uniquely determined by its Fourier coefficients. ✺

Proposition 4.3.7. *Let $x, y \in \mathscr{M} \times_{(\alpha,u)} \Gamma$ and $s \in \Gamma$. Then*

(i) $(xy)(s) = \sum_{t\in\Gamma} x(t)\alpha_t(y(t^{-1}s))u(t, t^{-1}s)$,
(ii) $x^*(s) = u(s, s^{-1})^*\alpha_s(x(s^{-1}))^* = \alpha_s(x(s^{-1})u(s^{-1}, s))^*$ *and*
(iii) $E(xx^*) = \sum_{t\in\Gamma} x(t)x(t)^*$,

the sums converging in the strong operator topology in $\mathscr{M}$.

Proof. Letting $(Q_y)_{g\in\Gamma}$ and $(P_s)_{s\in\Gamma}$ denote the families of operators from the above remark, we note that $Q_1\lambda_u(t) = Q_{t^{-1}}$ for all $t \in \Gamma$. By Theorem 4.3.4, we have

$$\begin{aligned}\alpha_t(y(t^{-1}s)) &= E(\lambda_u(t)y\lambda_u(t^{-1}s)^*\lambda_u(t)^*)\\ &= E(\lambda_u(t)y\lambda_u(s)^*)u(t, t^{-1}s)^*\\ &= Q_1\lambda_u(t)y\lambda_u(s)^*Q_1^*u(t, t^{-1}s)^*\\ &= Q_{t^{-1}}yQ_{s^{-1}}^*u(t, t^{-1}s)^*.\end{aligned}$$

Therefore

$$Q_{t^{-1}}y\lambda_u(s)^*Q_1^* = Q_{t^{-1}}yQ_{s^{-1}}^* = \alpha_t(y(t^{-1}s))u(t, t^{-1}s)$$

for all $t \in \Gamma$, so that

$$(xy)(s) = Q_1xy\lambda_u(s)Q_1^* = \sum_{t\in\Gamma}(Q_1xQ_{t^{-1}}^*)(Q_{t^{-1}}y\lambda_u(s)^*Q_1^*) = \sum_{t\in\Gamma} x(t)\alpha_t(y(t^{-1}s))u(t, t^{-1}s).$$

Further, since $x(t) = Q_1xQ_{t^{-1}}^*$ for all $t \in \Gamma$, we have

$$x(s) = (Q_{s^{-1}}^*xQ_1)^* = (\alpha_s(y(s^{-1}))u(s, s^{-1}))^* = u(s, s^{-1})^*\alpha_s(x(s^{-1}))^*,$$

proving (ii), since $\alpha_s(u(s^{-1}, s)) = u(s, s^{-1})$. Finally,

$$E(xx^*) = (xx^*)(1) = \sum_{t\in\Gamma} x(t)\alpha_t(\alpha_{t^{-1}}(x(t)u(t, t^{-1})))^*u(t, t^{-1}) = \sum_{t\in\Gamma} x(t)x(t)^*$$

by applying (i) and (ii). □

We are now able to prove Theorem 4.1.16, concerning the canonical conditional expectation of reduced twisted crossed products.

Proof of Theorem 4.1.16. Represent $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ such that the action of Γ on $\mathcal{A}$ can be implemented by a map $a\colon \Gamma \to \mathcal{U}(\mathcal{H})$ as in the proof of Theorem 4.1.15. If we define $\mathscr{M} = \mathcal{A}''$ and $\tilde{\alpha}_s = \mathrm{Ad}(a(s))$, then $(\tilde{\alpha}, u)$ becomes a twisted action of Γ on $\mathscr{M}$. Then

$$\mathcal{A} \rtimes^u_{\alpha,r} \Gamma \subseteq \mathscr{M} \times_{(\tilde{\alpha},u)} \Gamma \subseteq B(\mathcal{H} \otimes \ell^2(\Gamma)).$$

Identifying $\pi_\alpha(\mathcal{A}) = \pi_{\tilde{\alpha}}(\mathcal{A})$ and $\mathcal{A}$, then the canonical conditional expectation

$$\tilde{E}\colon \mathscr{M} \times_{(\tilde{\alpha},u)} \Gamma \to \mathscr{M}$$

maps $\mathcal{A} \rtimes^u_{\alpha,r} \Gamma$ onto $\mathcal{A}$. Consequently, the restriction E of $\tilde{E}$ to $\mathcal{A} \rtimes^u_{\alpha,r} \Gamma$ is the wanted conditional expectation, and Theorem 4.3.4 and Proposition 4.3.7 together yield the wanted properties of E. □

The next theorem proves that the existence of the canonical conditional expectation of a regular extension characterizes the regular extensions in a certain sense.

Theorem 4.3.8 (cf. [17]). *Let $(\mathscr{M}, \Gamma, \alpha, u)$ be a twisted dynamical system where $\mathscr{M} \subseteq B(\mathcal{H})$ is a von Neumann algebra with a faithful normal state ω. Let $\mathscr{N}$ be a von Neumann algebra such that there exist*

- *an injective unital normal *-homomorphism $\iota\colon \mathscr{M} \to \mathscr{N}$,*
- *a faithful normal conditional expectation $F\colon \mathscr{N} \to \iota(\mathscr{M})$ and*
- *a map $v\colon \Gamma \to \mathcal{U}(\mathscr{N})$,*

with the following properties:

(i) *$\mathscr{N}$ is generated by $\iota(\mathscr{M})$ and $v(\Gamma)$.*
(ii) *$\iota(\alpha_s(x)) = v(s)\iota(x)v(s)^*$ for all $s \in \Gamma$ and $x \in \mathscr{M}$.*
(iii) *$v(s)v(t) = \iota(u(s,t))v(st)$ for all $s, t \in \Gamma$.*
(iv) *$F(v(s)) = 0$ for all $s \in \Gamma \setminus \{1\}$.*

Then there exists a *-isomorphism $\varphi\colon \mathscr{N} \to \mathscr{M} \times_{(\alpha,u)} \Gamma$ such that

$$\varphi(\iota(x)) = \pi_\alpha(x), \quad \varphi(v(s)) = \lambda_u(s), \quad \varphi(F(y)) = \pi_\alpha(E(\varphi(y)))$$

for all $x \in \mathscr{M}$, $s \in \Gamma$ and $y \in \mathscr{N}$, where E denotes the canonical conditional expectation of $\mathscr{M} \times_{(\alpha,u)} \Gamma$ onto $\mathscr{M}$.

Proof. We adapt the proof of [67, Proposition 22.2]. Letting $\mathscr{N}_0$ be the subset of $\mathscr{N}$ consisting of operators of the form $\sum_{s\in F} \iota(x_s)v(s)$ where $F \subseteq \Gamma$ is finite and $x_s \in \mathscr{M}$ for all $s \in F$, then properties (ii) and (iii) tell us $\mathscr{N}_0$ is a unital *-algebra and that $\mathscr{N}_0$ is strongly dense in $\mathscr{N}$. Likewise, if we define $\mathscr{M}_0$ to be the subset of $\mathscr{M} \times_{(\alpha,u)} \Gamma$ consisting of the operators $\sum_{s\in F} \pi_\alpha(x_s)\lambda_u(s)$, then $\mathscr{M}_0$ is also a strongly dense unital *-subalgebra of $\mathscr{M} \times_{(\alpha,u)} \Gamma$. Note now that we can define a faithful normal state ψ on $\mathscr{N}$ by

$$\psi = \omega \circ \iota^{-1} \circ F.$$

Let $(\pi_1, \mathcal{H}_1, \xi_1)$ and $(\pi_2, \mathcal{H}_2, \xi_2)$ be the GNS triples associated to $\omega \circ E$ and ψ, respectively. Then for all $s \in \Gamma$ and $x \in \mathscr{M}$, we have

$$\langle \pi_2(\iota(x)v(s))\xi_2, \xi_2\rangle = \psi(\iota(x)v(s)) = (\omega \circ E)(\pi_\alpha(x)\lambda_u(s)) = \langle \pi_1(\pi_\alpha(x)\lambda_u(s))\xi_1, \xi_1\rangle.$$

Because ξ_1 is cyclic for $\pi_1(\mathscr{M} \times_{(\alpha,u)} \Gamma)$ and ξ_2 is cyclic for $\pi_2(\mathscr{N})$, it follows that ξ_1 is in fact cyclic for $\pi_1(\mathscr{M}_0)$ and that ξ_2 is cyclic for $\pi_2(\mathscr{N}_0)$ by strong operator density. It follows from Lemma 4.1.9 as well as properties (ii) and (iii) that we then have a surjective *-homomorphism $\mathscr{M}_0 \to \mathscr{N}_0$ given by

$$\sum_{s\in F} \pi_\alpha(x_s)\lambda_u(s) \mapsto \sum_{s\in F} \iota(x_s)v(s).$$

To see that this map is injective, note that if $x = 0$ in $\mathscr{N}_0$ and we write $x = \sum_{s\in F} \iota(x_s)v(s)$, then

$$0 = F(xv(s)^*) = \iota(x_s)$$

and hence $x_s = 0$ for all $s \in F$. By [23, Lemma I.4.1.3] there exists a unitary operator $U \colon \mathcal{H}_1 \to \mathcal{H}_2$ such that $U\pi_1(\mathscr{M}_0)U^* = \pi_2(\mathscr{N}_0)$ and

$$\pi_2\left(\sum_{s\in F}\iota(x_s)v(s)\right) = U\pi_1\left(\sum_{s\in F}\pi_\alpha(x_s)\lambda_u(s)\right)U^*$$

for all finite subsets $F \subseteq \Gamma$, where $x_s \in \mathscr{M}$ for all $s \in F$. By normality of π_1 and π_2 [15, Proposition 2.50], it is quickly verified that $U\pi_1(\mathscr{M} \times_{(\alpha,u)} \Gamma)U^* = \pi_2(\mathscr{N})$, so we can define a *-isomorphism $\varphi \colon \mathscr{N} \to \mathscr{M} \times_{(\alpha,u)} \Gamma$ by

$$\varphi(x) = \pi_1^{-1}(U^*\pi_2(x)U), \quad x \in \mathscr{N}.$$

The desired properties of φ are then easily established. □

Finally, the next theorem is utterly essential – really, the original article [41] is a goldmine of pretty results. For any element $x \in \mathscr{M}$ where $\mathscr{M}$ is a von Neumann algebra, the central support of x is denoted by $c(x)$.

Theorem 4.3.9 (Kallman)**.** *Let $\mathscr{M} \subseteq B(\mathcal{H})$ be a von Neumann algebra and let $\alpha \in \mathrm{Aut}(\mathscr{M})$. Then α is outer if and only if it satisfies:*

$$\text{If } m \in \mathscr{M} \text{ satisfies } \alpha(x)m = mx \text{ for all } x \in \mathscr{M}, \text{ then } c(m) < 1_{\mathscr{M}}. \qquad (\dagger)$$

In particular, an automorphism of a factor is outer if and only if it is freely acting.

Proof. Suppose that α is inner. Then there exists a unitary $u \in \mathscr{M}$ such that $\alpha(x)u = ux$, but $c(u) = 1_{\mathscr{M}}$, so that (†) does not hold. Conversely, suppose that there exists $m \in \mathscr{M}$ such that $\alpha(x)m = mx$ for all $x \in \mathscr{M}$ and $c(m) = 1_{\mathscr{M}}$. Then for all unitaries $v \in \mathcal{U}(\mathscr{M})$, we have

$$v^*m^*mv = m^*\alpha(v^*)\alpha(v)m = m^*m \quad \text{and} \quad \alpha(v)mm^*\alpha(v^*) = mvv^*m^* = mm^*,$$

proving that $m^*m, mm^* \in \mathcal{Z}(\mathscr{M})$ since any element in $\mathscr{M}$ is a linear combination of four unitaries. Recall now that the *right support* $s(x) \in \mathscr{M}$ of an operator $x \in \mathscr{M}$ is the projection such that $1 - s(x)$ has range $\ker x \subseteq \mathcal{H}$. We now have

$$s(m) = s(m^*m) = c(m^*m) = c(m) = 1;$$

indeed, the second equality follows from [39, Proposition 5.5.2] and the fact that m^*m is central and positive, while the first and third are a consequence of the fact that $\ker(x) = \ker(x^*x)$ for all $x \in \mathscr{M}$. Similarly, the fact that $c(m^*) = 1$ implies $s(m^*) = 1$. If we now let $m = u|m|$ be the polar decomposition of m, then $u^*u = s(m) = 1$ and $uu^* = s(m^*) = 1$ (cf. [11, p. 23]), so that u is a unitary! Now let $x \in \mathscr{M}$. Because $|m| \in \mathcal{Z}(\mathscr{M})$, we have

$$\alpha(x)u|m| = \alpha(x)m = mx = u|m|x = ux|m|.$$

If $y \in \mathscr{M}$, then the range of the right support $s(y^*)$ is the closure of the range of y. On the grounds that $s(|m|) = s(m^*m) = 1$, we conclude that $|m|$ has dense range in $\mathcal{H}$ and thus

$$\alpha(x)u = ux.$$

Since x was arbitrary, α is inner, completing the proof. □

Kallman's theorem is actually the explanation for why certain automorphisms of both groups and C^*-algebras are called freely acting. Indeed, if Γ is a non-trivial discrete icc group, then $L(\Gamma)$ is a II_1-factor, and an automorphism of Γ (resp. $L(\Gamma)$) is outer if and only if it is freely acting by Lemma 4.1.19 (i) (resp. Kallman's theorem).

Kallman's theorem also has an interesting consequence for regular extensions.

Theorem 4.3.10 (Choda)**.** *Let $(\mathscr{M}, \Gamma, \alpha, u)$ be a twisted dynamical system where $\mathscr{M}$ is a von Neumann algebra and α_s is freely acting for all $s \in \Gamma \setminus \{1\}$. Then*

$$\pi_\alpha(\mathscr{M})' \cap (\mathscr{M} \times_{(\alpha,u)} \Gamma) \subseteq \mathcal{Z}(\pi_\alpha(\mathscr{M})).$$

Proof. Let E denote the canonical conditional expectation of $\mathscr{M} \times_{(\alpha,u)} \Gamma$ onto $\pi_\alpha(\mathscr{M})$. If an operator $x \in \mathscr{M} \times_{(\alpha,u)} \Gamma$ commutes with all elements of $\pi_\alpha(\mathscr{M})$, then for all $y \in \mathscr{M}$ and $s \in \Gamma$ we find that

$$\pi_\alpha(x(s)^* y) = E(\lambda_u(s) x^* \pi_\alpha(y)) = E(\lambda_u(s)\pi_\alpha(y)\lambda_u(s)^*\lambda_u(s)x^*) = \pi_\alpha(\alpha_s(y)x(s)^*).$$

If $s \in \Gamma \setminus \{1\}$, then α_s is freely acting and therefore $x(s) = 0$. By Remark 4.3.5, it now follows that $x = \pi_\alpha(x(1)) \in \pi_\alpha(\mathscr{M})$. □

Corollary 4.3.11. *Let $(\mathscr{M}, \Gamma, \alpha, u)$ be a twisted dynamical system, where $\mathscr{M}$ is a factor and α_s is outer for all $s \in \Gamma \setminus \{1\}$. Then*

$$\pi_\alpha(\mathscr{M})' \cap (\mathscr{M} \times_{(\alpha,u)} \Gamma) \subseteq \mathbb{C}1.$$

In particular, $\mathscr{M} \times_{(\alpha,u)} \Gamma$ is a factor.

Proof. This follows directly from Theorems 4.3.9 and 4.3.10. □

4.4 Uniqueness of trace for reduced twisted crossed products

We now turn back to the issue of uniqueness of trace for reduced twisted crossed products – our goal is to obtain the same stream of corollaries concerning C^*-simplicity and ultraweak Powers groups as obtained two sections ago, but for the unique trace property. The author wishes to thank Erik Bédos for swiftly answering our questions about the content in [6] and for providing us with [7].

The following results are needed in the impending exposition; we state the first without proof.

Proposition 4.4.1 (cf. [40, 7.6.7]). *For $i = 1, 2$, let $\mathscr{M}_i$ be a von Neumann algebra with a faithful normal state ω_i and let $\mathcal{A}_i$ be a weakly dense *-subalgebra of $\mathscr{M}_i$. Suppose further that there exists a *-isomorphism $\varphi\colon \mathcal{A}_1 \to \mathcal{A}_2$ such that $\omega_1(x) = \omega_2(\varphi(x))$ for all $x \in \mathcal{A}_1$. Then φ extends to a *-isomorphism $\theta\colon \mathscr{M}_1 \to \mathscr{M}_2$ satisfying $\omega_1(x) = \omega_2(\theta(x))$ for all $x \in \mathscr{M}_1$.*

Proposition 4.4.2. *Let $\mathcal{A}$ be a unital C^*-subalgebra of $B(\mathcal{H})$ with $1_{\mathcal{A}} = 1_{\mathcal{H}}$ and let $\mathscr{M} = \mathcal{A}''$. Assume that there exists a unit vector $\xi \in \mathcal{H}$ that is a cyclic trace vector for $\mathcal{A}$. Then ξ is a cyclic trace vector for $\mathscr{M}$ and the linear functional $x \mapsto \langle x\xi, \xi\rangle$ is a faithful normal trace on $\mathscr{M}$. Consequently, $\mathscr{M}$ is finite.*

Proof. By a weak operator density argument, ξ is a cyclic trace vector for $\mathscr{M}$. By [40, Lemma 7.2.14], ξ_σ is also cyclic for $\mathscr{M}'$ and hence separating for $\mathscr{M}'' = \mathscr{M}$. The rest of the desired result now follows immediately. □

In the sequel, we will always let $(\mathcal{A}, \Gamma, \alpha, u)$ be a twisted dynamical system. Assume that $\mathcal{A}$ possesses a faithful α-invariant trace σ, and let $(\pi_\sigma, \mathcal{H}_\sigma, \xi_\sigma)$ be the GNS triple associated to σ. We will henceforth identify $\mathcal{A}$ with the unital C^*-subalgebra $\pi_\sigma(\mathcal{A})$ of $B(\mathcal{H}_\sigma)$. Now define $\mathscr{M} = \mathcal{A}'' \subseteq B(\mathcal{H}_\sigma)$. For all $s \in \Gamma$, we have

$$\|\alpha_s(a)\xi_\sigma\|^2 = \sigma(\alpha_s(a^*a)) = \sigma(a^*a) = \|a\xi_\sigma\|^2$$

for all $a \in \mathcal{A}$. This allows us to define a unitary operator $U_{\alpha_s} \in B(\mathcal{H}_\sigma)$ by

$$U_{\alpha_s}(a\xi_\sigma) = \alpha_s(a)\xi_\sigma, \quad a \in \mathcal{A}.$$

It is easy to check that $U_{\alpha_s}^* = U_{\alpha_s^{-1}}$ and that $\mathrm{Ad}(U_{\alpha_s})|_{\mathcal{A}} = \alpha_s$ on $\mathcal{A}$, and consequently we can define

$$\tilde{\alpha}_s = \mathrm{Ad}(U_{\alpha_s})|_{\mathscr{M}}$$

so that $\tilde{\alpha}_s \in \mathrm{Aut}(\mathscr{M})$. Thus we obtain a twisted action $(\tilde{\alpha}, u)$ of Γ on $\mathscr{M}$.

Definition 4.4.3. We say that the twisted action (α, u) is *σ-outer* (resp. *σ-freely acting*) whenever $\tilde{\alpha}_s$ is outer (resp. freely acting) for all $s \in \Gamma \setminus \{1\}$, the σ referring to the faithful state on $\mathcal{A}$.

Define

$$\tilde{\sigma}(x) = \langle x\xi_\sigma, \xi_\sigma\rangle \tag{4.4.1}$$

for all $x \in \mathscr{M}$. Then $\tilde{\sigma}$ is a faithful normal trace on $\mathscr{M}$ and $\mathscr{M}$ is finite. Furthermore, note that $\tilde{\sigma}$ is $\tilde{\alpha}$-invariant by α-invariance of σ.

Let $\mathscr{N} = \mathscr{M} \times_{(\tilde{\alpha},u)} \Gamma$ in the following.

Lemma 4.4.4. *If $\mathcal{A}$ has a unique faithful α-invariant trace σ and (α, u) is σ-outer, then $\mathscr{N}$ is a finite factor with unique faithful normal trace $\tilde{\sigma} \circ \tilde{E}$, where $\tilde{E}$ is the canonical faithful conditional expectation of $\mathscr{N}$ onto $\mathscr{M}$.*

Proof. Note first that $\tilde{\sigma}$ (as defined in (4.4.1)) is the unique normal trace of $\mathscr{M}$ by uniqueness of σ, so that $\mathscr{M}$ is a factor by Lemma 2.7. Therefore $\mathscr{N}$ is a factor too, by Corollary 4.3.11. As $\mathscr{N}$ has a faithful normal trace, $\mathscr{N}$ is also finite, so $\tilde{\sigma} \circ \tilde{E}$ is the unique trace on $\mathscr{N}$. □

Theorem 4.4.5 (Bédos)**.** *If $\mathcal{A}$ has a unique faithful α-invariant trace σ and (α, u) is σ-outer, then $\mathcal{B} = \mathcal{A} \rtimes^u_{\alpha,r} \Gamma$ has unique trace given by $\tau = \sigma \circ E$, where E is the canonical conditional expectation of $\mathcal{B}$ onto $\mathcal{A}$.*

Proof. Since σ is α-invariant, $\tau = \sigma \circ E$ is indeed a trace on $\mathcal{B}$ by Lemma 4.2.6. If $(\pi_\sigma, \mathcal{H}_\sigma, \xi_\sigma)$ is the GNS triple associated to σ and $\mathcal{A}$ is identified with $\pi_\sigma(\mathcal{A})$. Considering the covariant representation $(\pi_{\tilde{\alpha}}, \lambda_u, \mathcal{H}_\sigma \otimes \ell^2(\Gamma))$ of $(\mathscr{M}, \Gamma, \tilde{\alpha}, u)$, we find that the reduced twisted crossed product $\mathcal{B}$ is the C^*-subalgebra of $B(\mathcal{H}_\sigma \otimes \ell^2(\Gamma))$ generated by $\pi_{\tilde{\alpha}}(\mathcal{A})$ and $\lambda_u(\Gamma)$. It is then evident that $\mathcal{B}''$ equals $\mathscr{N}$, which is a finite factor by the previous lemma and has unique faithful normal trace $\tilde{\tau} = \tilde{\sigma} \circ \tilde{E}$, where $\tilde{E}$ denotes the canonical conditional expectation of $\mathscr{N}$ onto $\mathscr{M}$. Henceforth, we will identify $\mathscr{M}$ and $\pi_{\tilde{\alpha}}(\mathscr{M})$, so that $\mathcal{A}$ and $\pi_{\tilde{\alpha}}(\mathcal{A})$ are also identified.

Now let ω be a trace on $\mathcal{B}$ and let $(\pi_\omega, \mathcal{H}_\omega, \xi_\omega)$ denote the GNS triple associated to ω. We want to prove that $\omega = \tau$, and this will be accomplished by first showing that the von Neumann algebras $\pi_\omega(\mathcal{B})''$ and $\mathcal{B}''$ are isomorphic in a way that befits our needs. As above, define a positive linear functional $\tilde{\omega}$ on $\pi_\omega(\mathcal{B})''$ by

$$\tilde{\omega}(x) = \langle x\xi_\omega, \xi_\omega \rangle, \quad x \in \pi_\omega(\mathcal{B})''.$$

By Proposition 4.4.2, $\tilde{\omega}$ is a faithful normal trace on $\pi_\omega(\mathcal{B})''$. Hence if we define $\widehat{\omega} = \tilde{\omega}|_{\pi_\omega(\mathcal{A})''}$, then $\widehat{\omega}$ is a faithful normal trace on $\pi_\omega(\mathcal{A})''$. Also, since $\omega|_\mathcal{A} = \sigma$ by uniqueness of σ, faithfulness of σ implies that π_ω defines a *-isomorphism of $\mathcal{A}$ onto $\pi_\omega(\mathcal{A})$. Moreover, we have

$$\tilde{\sigma}(a) = \sigma(a) = \omega(a) = \widehat{\omega}(\pi_\omega(a)), \quad a \in \mathcal{A}.$$

By Proposition 4.4.1, we now obtain a *-isomorphism $\theta\colon \mathscr{M} \to \pi_\omega(\mathcal{A})''$ satisfying $\theta|_\mathcal{A} = \pi_\omega$ and $\tilde{\sigma} = \widehat{\omega} \circ \theta$. For $s \in \Gamma$, define

$$v(s) = \pi_\omega(\lambda_u(s)) \in \mathcal{U}(\pi_\omega(\mathcal{B})''), \quad \beta_s = \theta\tilde{\alpha}_s\theta^{-1} \in \mathrm{Aut}(\pi_\omega(\mathcal{A})'').$$

Then

$$v(s)\pi_\omega(a)v(s)^* = \pi_\omega(\lambda_u(s)a\lambda_u(s)^*) = \pi_\omega(\alpha_s(a)) = \theta(\tilde{\alpha}_s(a)) = \beta_s(\theta(a)) = \beta_s(\pi_\omega(a))$$

for all $s \in \Gamma$ and $a \in \mathcal{A}$ (recall that $\mathcal{A}$ and $\pi_\alpha(\mathcal{A})$ are identified). By ultraweak continuity, it follows that $\beta_s = \mathrm{Ad}(v(s))$ on $\pi_\omega(\mathcal{A})''$ and also that $\theta(\tilde{\alpha}_s(x)) = \beta_s(\theta(x)) = v(s)\theta(x)v(s)^*$ for all $s \in \Gamma$ and $x \in \mathscr{M}$.

As $\tilde{\alpha}_s$ is outer for all $s \in \Gamma \setminus \{1\}$ and $\mathscr{M}$ is a factor, each $\tilde{\alpha}_s$ is freely acting on $\mathscr{M}$ for all $s \in \Gamma \setminus \{1\}$ by Theorem 4.3.9. By our definition of β_s, each β_s is then freely acting on $\pi_\omega(\mathcal{A})''$ for all $s \in \Gamma \setminus \{1\}$. Since $\pi_\omega(\mathcal{B})''$ has a faithful normal trace $\tilde{\omega}$, [14, Lemma 1.5.11] yields a faithful normal conditional expectation F of $\pi_\omega(\mathcal{B})''$ onto $\pi_\omega(\mathcal{A})''$ such that $\widehat{\omega} \circ F = \tilde{\omega} \circ F = \tilde{\omega}$. For all $s \in \Gamma$ and $x \in \pi_\omega(\mathcal{A})''$ we have $\beta_s(x)v(s) = v(s)x$, and therefore

$$\beta_s(x)F(v(s)) = F(v(s))x.$$

Hence for all $s \in \Gamma \setminus \{1\}$, we have $F(v(s)) = 0$ by the free action of β_s.

Note finally that because $\pi_\omega(\mathcal{B})$ is generated by $\pi_\omega(\mathcal{A})$ and $v(\Gamma) = \pi_\omega(\lambda_u(\Gamma))$, $\pi_\omega(\mathcal{B})''$ is generated by $\theta(\mathscr{M}) = \pi_\omega(\mathcal{A})''$ and $v(\Gamma)$. By Theorem 4.3.8, there now exists a *-isomorphism

$$\varphi\colon \pi_\omega(\mathcal{B})'' \to \mathscr{M} \times_{(\tilde{\alpha},u)} \Gamma = \mathcal{B}''$$

such that

$$\varphi(v(s)) = \lambda_u(s), \quad \varphi(\theta(x)) = x$$

for all $s \in \Gamma$ and $x \in \mathscr{M}$. Since $\mathcal{B}''$ is a finite factor, $\pi_\omega(\mathcal{B})''$ is a finite factor as well and therefore it has unique trace. This in turn implies that $\tilde{\omega} = \tilde{\tau} \circ \varphi$, and thus

$$\omega(a\lambda_u(s)) = \tilde{\omega}(\pi_\omega(a\lambda_u(s))) = \tilde{\tau}(\varphi(\theta(a))\varphi(v(s))) = \tilde{\tau}(a\lambda_u(s)) = \tau(a\lambda_u(s))$$

for all $a \in \mathcal{A}$ and $s \in \Gamma$. As ω and τ are continuous in norm, it follows that $\omega = \tau$. □

We then prove the analogue of Theorem 4.1.24 for uniqueness of trace.

Theorem 4.4.6. *Let $(\mathcal{A}, \Gamma, \alpha, u)$ be a twisted dynamical system, where $\mathcal{A}$ is a C^*-algebra with a faithful α-invariant trace φ. Moreover, suppose that $\Lambda \subseteq \Gamma$ is a normal subgroup and that Γ/Λ acts freely on Λ. If $\mathcal{B}_\Lambda = \mathcal{A} \rtimes^u_{\alpha,r} \Lambda$ has a unique trace σ, then $\mathcal{B}_\Gamma = \mathcal{A} \rtimes^u_{\alpha,r} \Gamma$ has unique trace.*

Proof. First, let $(\pi_\varphi, \mathcal{H}_\varphi, \xi_\varphi)$ be the GNS triple associated to φ, identify $\mathcal{A}$ with $\pi_\varphi(\mathcal{A})$ and define $\mathscr{M} = \mathcal{A}'' \subseteq B(\mathcal{H}_\varphi)$. Since φ is α-invariant, the remarks before Theorem 4.4.5 yield extensions $\tilde{\alpha}_s$ of the automorphisms α_s to $\mathscr{M}$ for all $s \in \Gamma$.

Now let $(\pi_\sigma, \mathcal{H}_\sigma, \xi_\sigma)$ be the GNS triple associated to the unique trace σ of $\mathcal{B}_\Lambda$. Letting E denote the canonical conditional expectation of $\mathcal{B}_\Lambda$ onto $\mathcal{A}$, uniqueness of σ yields $\sigma = \varphi \circ E$ by Theorem 4.2.7. Defining

$$\eta = \xi_\varphi \otimes \delta_1 \in \mathcal{H}_\varphi \otimes \ell^2(\Lambda),$$

then η is a cyclic unit vector for $\mathcal{B}_\Lambda$ in $\mathcal{H}_\varphi \otimes \ell^2(\Lambda)$. Indeed, if $\mathcal{K}$ denotes the closure of the subspace $\mathcal{B}_\Lambda\eta$ of $\mathcal{H}_\varphi \otimes \ell^2(\Lambda)$, then for all $a \in \mathcal{A}$ and $s \in \Lambda$ we have

$$a\xi_\varphi \otimes \delta_s = \lambda_u(s)\pi_\alpha(u(s^{-1}, s)^*a)\eta \in \mathcal{K},$$

so that $\xi \otimes \delta_s \in \mathcal{K}$ for all $\xi \in \mathcal{H}_\varphi$ and $s \in \Lambda$ by ξ_φ being cyclic. Hence $\mathcal{K} = \mathcal{H}_\varphi \otimes \ell^2(\Lambda)$. Further, it is easily verified that

$$\langle \pi_\sigma(x)\xi_\sigma, \xi_\sigma \rangle = \sigma(x) = \langle E(x)\xi_\varphi, \xi_\varphi \rangle = \langle x\eta, \eta \rangle$$

for all $x \in \mathcal{B}_\Lambda$. Therefore [23, Lemma I.4.1.3] yields a unitary operator $U\colon \mathcal{H}_\varphi \otimes \ell^2(\Lambda) \to \mathcal{H}_\sigma$ such that $UxU^* = \pi_\sigma(x)$ for all $x \in \mathcal{B}_\Lambda$.

Let $Q = \Gamma/\Lambda$. By Theorems 4.1.15 and 4.3.3, there exist twisted actions (β, v) resp. (χ, v) of Q on $\mathcal{B}_\Lambda$ resp. $\mathscr{M} \times_{(\tilde{\alpha},u)} \Lambda$ such that

$$\mathcal{B}_\Gamma \cong \mathcal{B}_\Lambda \rtimes^v_{\beta,r} Q \quad \text{and} \quad \mathscr{M} \times_{(\tilde{\alpha},u)} \Gamma \cong (\mathscr{M} \times_{(\tilde{\alpha},u)} \Lambda) \times_{(\chi,v)} Q.$$

Recall that β and χ are defined as follows. By the same theorems, for each $s \in \Gamma$ there exists $\gamma_s \in \operatorname{Aut}(\mathcal{B}_\Lambda)$ (resp. $\tilde{\gamma}_s \in \operatorname{Aut}(\mathscr{M} \times_{(\tilde{\alpha},u)} \Lambda)$ such that

(i) $\gamma_s(\pi_\alpha(a)) = \pi_\alpha(\alpha_s(a))$ for all $a \in \mathcal{A}$ (resp. $\tilde{\gamma}_s(\pi_{\tilde{\alpha}}(x)) = \pi_{\tilde{\alpha}}(\tilde{\alpha}_s(x))$ for all $x \in \mathscr{M}$) and
(ii) $\gamma_s(\lambda_u(t)) = \tilde{\gamma}_s(\lambda_u(t)) = \pi_\alpha(\tilde{u}(s,t))\lambda_u(sts^{-1})$ for all $s, t \in \Gamma$.

Since $\pi_\alpha(a) = \pi_{\tilde{\alpha}}(a)$ for all $a \in \mathcal{A}$, it is clear that $\tilde{\gamma}_s = \gamma_s$ on $\mathcal{B}_\Lambda$. Now for some cross-section $k\colon Q \to \Gamma$ for the quotient map $j\colon \Gamma \to Q$ with $k(1) = 1$, we have

$$\beta_y = \gamma_{k(y)} \quad \text{and} \quad \chi_y = \tilde{\gamma}_{k(y)}$$

for all $y \in Q$. It is now our intention to apply Theorem 4.4.5 to the twisted dynamical system $(\mathcal{B}_\Lambda, Q, \beta, v)$, and hence we need to consider the automorphisms $\tilde{\beta}_y \in \operatorname{Aut}(\pi_\sigma(\mathcal{B}_\Lambda)'')$ for $y \in Q$, as defined before that theorem. If $y \in Q$, note that for all $x \in \mathcal{B}_\Lambda$ we have

$$\tilde{\beta}_y(UxU^*) = \tilde{\beta}_y(\pi_\sigma(x)) = \pi_\sigma(\beta_y(x)) = U\chi_y(x)U^*.$$

Since $U\mathcal{B}_\Lambda''U^* = \pi_\sigma(\mathcal{B}_\Lambda)''$, ultraweak continuity of χ_y and $\tilde{\beta}_y$ yields

$$U^*\tilde{\beta}_y(x)U = \chi_y(U^*xU) = \tilde{\gamma}_{k(y)}(U^*xU), \quad x \in \pi_\sigma(\mathcal{B}_\Lambda)''.$$

Hence to prove that $\tilde{\beta}_y$ is outer for all $y \in Q \setminus \{1\}$, it suffices to prove that $\tilde{\gamma}_s$ is outer for all $s \in \Gamma \setminus \Lambda$.

Let $s_0 \in \Gamma \setminus \Lambda$. Since Γ/Λ acts freely on Λ, the set $\{\mathrm{Ad}(s_0)(s)ts^{-1} \mid s \in \Lambda\}$ is infinite for all $t \in \Lambda$. Now the argument of Lemma 4.1.21 can essentially be repeated to prove that $\tilde{\gamma}_{s_0}$ is in fact freely acting; these are the modifications. By defining

$$\psi(x) = \langle x\xi_\varphi, \xi_\varphi \rangle, \quad x \in \mathscr{M},$$

then ψ is a faithful strongly continuous state on $\mathscr{M}$ (the argument used to show that $\tilde{\sigma}$ was faithful in the proof of Theorem 4.4.5 also applies here). Moreover, ψ is $\tilde{\alpha}$-invariant, since φ is α-invariant and both ψ and all $\tilde{\alpha}_s$ are normal. We then consider the twisted dynamical system $(\mathscr{M}, \Gamma, \tilde{\alpha}, u)$, replacing $\mathcal{A} \rtimes^u_{\alpha,r} \Gamma$ by $\mathscr{M} \times_{(\tilde{\alpha},u)} \Gamma$, γ_{s_0} by $\tilde{\gamma}_{s_0}$ and E by the canonical conditional expectation of $\mathscr{M} \times_{(\tilde{\alpha},u)} \Gamma$ onto $\mathscr{M}$ in the proof of Lemma 4.1.21. Finally, by referring to Proposition 4.3.7 instead of Theorem 4.1.16 throughout, we find that $\tilde{\gamma}_{s_0}$ is outer. □

We can then prove the analogues of Corollaries 4.2.9, 4.2.10 and 4.2.11 in the same manner as their proofs.

Corollary 4.4.7. *Let Γ be a group and let $\Lambda \subseteq \Gamma$ be a normal subgroup with trivial centralizer in Γ. If $C^*_r(\Lambda)$ has unique trace, then $C^*_r(\Gamma)$ has unique trace. In particular, ultraweak Powers groups have unique trace.*

Proof. Because $C^*_r(\Lambda)$ has unique trace, the group Λ is icc, and hence Γ/Λ acts freely on Λ. Defining $\mathcal{A} = \mathbb{C}$ and letting (α, u) be the trivial action, the result now follows from Theorem 4.4.6. □

Corollary 4.4.8. *Let $(\mathcal{A}, \Gamma, \alpha, u)$ be a twisted dynamical system, where $\mathcal{A}$ is a unital C^*-algebra with a unique trace that is faithful and Γ is an ultraweak Powers group. Then $\mathcal{A} \rtimes^u_{\alpha,r} \Gamma$ has unique trace.*

Proof. Since the trace on $\mathcal{A}$ is unique, it is α-invariant. As Γ contains a normal weak Powers subgroup Λ with trivial centralizer in Γ, then $\mathcal{A} \rtimes^u_{\alpha,r} \Lambda$ has unique trace by Theorem 4.2.7. By Theorem 4.4.6, $\mathcal{A} \rtimes^u_{\alpha,r} \Gamma$ has unique trace. □

Corollary 4.4.9. *Let*

$$1 \longrightarrow \Gamma' \longrightarrow \Gamma \longrightarrow \Gamma'' \longrightarrow 1$$

be a short exact sequence of groups, where $C^*_r(\Gamma')$ has unique trace and Γ'' is an ultraweak Powers group. Then $C^*_r(\Gamma)$ has unique trace.

Proof. The proof is the same as for Corollary 4.2.11, the reference to Corollary 4.2.9 replaced by one to Corollary 4.4.8. □

Chapter 5

Permanence properties of C^*-simplicity and unique trace

As we so far have demonstrated, showing that a group is either C^*-simple or has the unique trace property can often be done by showing that a group has certain combinatorial traits (e.g., the Powers property), for which we have also shown various permanence properties. What we haven't considered yet is whether C^*-simplicity or the unique trace property themselves have permanence properties, and this is exactly what this chapter is devoted to. As before, we will only consider permanence properties for discrete groups.

5.1 Direct products and automorphism groups

We saw in Chapter 3 that weak Powers groups and PH groups are stable under forming direct products. An indication that this is true (if one is to view C^*-simplicity and the most important property of such groups) comes in the form of the following theorem:

Theorem 5.1.1. *If the reduced group C^*-algebras of two discrete groups Γ_1 and Γ_2 are simple (resp. have unique trace), then the reduced group C^*-algebra of $\Gamma_1 \times \Gamma_2$ is simple (resp. has unique trace).*

The proof is in two parts.

Lemma 5.1.2. *For discrete groups Γ_1 and Γ_2, we have*

$$C_r^*(\Gamma_1 \times \Gamma_2) = C_r^*(\Gamma_1) \otimes_{\min} C_r^*(\Gamma_2).$$

Proof. Let λ_1 and λ_2 denote the left-regular representations of Γ_1 and Γ_2 respectively, and let λ be the left-regular representation of $\Gamma_1 \times \Gamma_2$. The minimal tensor product $C_r^*(\Gamma_1) \otimes_{\min} C_r^*(\Gamma_2)$ is the norm closure of the linear span of the subset

$$\{S \otimes T \mid S \in C_r^*(\Gamma_1),\ T \in C_r^*(\Gamma_2)\} \subseteq B(\ell^2(\Gamma_1) \otimes \ell^2(\Gamma_2)).$$

We identify the Hilbert spaces $\ell^2(\Gamma_1 \times \Gamma_2)$ and $\ell^2(\Gamma_1) \otimes \ell^2(\Gamma_2)$ by means of the unitary operator U that satisfies

$$U(\delta_{(s,t)}) = \delta_s \otimes \delta_t, \quad s \in \Gamma_1,\ t \in \Gamma_2.$$

Under this identification, it is easy to see that $\lambda(s,t) = \lambda_1(s) \otimes \lambda_2(t)$ for all $s \in \Gamma_1$ and $t \in \Gamma_2$, from which it follows that the reduced group C^*-algebra $C_r^*(\Gamma_1 \times \Gamma_2)$ is contained in the minimal tensor product of $C_r^*(\Gamma_1)$ and $C_r^*(\Gamma_2)$. As the complex group rings clearly satisfy $\mathbb{C}\Gamma_1 \odot \mathbb{C}\Gamma_2 \subseteq \mathbb{C}(\Gamma_1 \times \Gamma_2)$, a density argument shows the other inclusion. □

Thus the question of showing that direct products are C^*-simple or have the unique trace property boils down to showing that simplicity and unique trace are preserved when taking the minimal tensor products. The first part of the following theorem is due to Takesaki [68]; the second is a byproduct of Takesaki's famous theorem stating that the minimal norm on C^*-algebraic tensor products is the smallest possible C^*-norm.

Theorem 5.1.3 (Takesaki)**.** *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be C^*-algebras and let $\mathcal{A} = \mathcal{A}_1 \otimes_{\min} \mathcal{A}_2$.*

(i) *If $\mathcal{A}_1$ and $\mathcal{A}_2$ are simple, then $\mathcal{A}$ is simple.*
(ii) *If $\mathcal{A}_1$ and $\mathcal{A}_2$ are unital with unique trace, then $\mathcal{A}$ has a unique trace.*

Proof. (i) Let $\mathfrak{I}$ be a proper closed ideal in $\mathcal{A}$, let $\pi\colon \mathcal{A} \to \mathcal{A}/\mathfrak{I}$ be the quotient homomorphism and let $\sigma\colon \mathcal{A}/\mathfrak{I} \to B(\mathcal{H})$ be an irreducible representation (by [22, Proposition 2.7.1] these always exist). Then by [14, Theorem 3.2.6] we have non-degenerate representations σ_1 and σ_2 of $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively, such that the ranges of σ_1 and σ_2 commute and

$$\sigma_1(x)\sigma_2(y) = \sigma(\pi(x \otimes y)), \quad x \in \mathcal{A}_1,\ y \in \mathcal{A}_2.$$

As σ_1 and σ_2 have commuting ranges and $\sigma(\pi(\mathcal{A}))'' = B(\mathcal{H})$, we have

$$B(\mathcal{H}) = \sigma(\mathcal{A}/\mathfrak{I})'' = (\sigma_2(\mathcal{A}_2) \cup \sigma_1(\mathcal{A}_1))'' \subseteq (\sigma_1(\mathcal{A}_1)' \cup \sigma_1(\mathcal{A}_1))''$$

and hence $\mathbb{C}1_{\mathcal{H}} = \sigma_1(\mathcal{A}_1)'' \cap \sigma_1(\mathcal{A}_1)'$, so that $\sigma_1(\mathcal{A}_1)''$ is a factor. Similarly we see that $\sigma_2(\mathcal{A}_2)''$ is a factor. Since $\mathcal{A}_1$ and $\mathcal{A}_2$ are simple, σ_1 and σ_2 are injective. If $x \in \mathfrak{I}$ is of the form $x = \sum_{i=1}^n x_i \otimes y_i$ for $x_1, \dots, x_n \in \mathcal{A}_1$ and $y_1, \dots, y_n \in \mathcal{A}_2$, note that

$$\sum_{i=1}^n \sigma_1(x_i)\sigma_2(y_i) = \sum_{i=1}^n \sigma(\pi(x_i \otimes y_i)) = \sigma(\pi(x)) = 0.$$

By [15, Lemma 5.22] there exists a complex matrix $(\lambda_{ij})_{i,j=1}^n \in M_n(\mathbb{C})$ such that

$$\sum_{i=1}^n \lambda_{ij}\sigma_1(x_i) = 0, \quad \sum_{j=1}^n \lambda_{ij}\sigma_2(y_j) = \sigma_2(y_i).$$

Injectivity of σ_1 and σ_2 implies $\sum_{i=1}^n \lambda_{ij}x_i = 0$ and $\sum_{j=1}^n \lambda_{ij}y_j = y_i$, so that

$$x = \sum_{i=1}^n x_i \otimes y_i = \sum_{i=1}^n \sum_{j=1}^n \lambda_{ij}x_i \otimes y_j = 0.$$

Hence $\mathfrak{I} \cap (\mathcal{A}_1 \odot \mathcal{A}_2) = \{0\}$. We can now define a C^*-norm $\|\cdot\|_\alpha$ on $\mathcal{A}_1 \odot \mathcal{A}_2$ by

$$\|x\|_\alpha = \|\pi(x)\|, \quad x \in \mathcal{A}_1 \odot \mathcal{A}_2.$$

Since the minimal norm is the smallest C^*-norm and π is a contraction, we now have

$$\|x\|_{\min} \le \|x\|_\alpha = \|\pi(x)\| \le \|x\|_{\min}$$

for all $x \in \mathcal{A}_1 \odot \mathcal{A}_2$, so that π preserves norms on $\mathcal{A}_1 \odot \mathcal{A}_2$ (equipped with the minimal norm) and is therefore an isometry on all of $\mathcal{A}$. Therefore $\mathfrak{I} = \{0\}$, so $\mathcal{A}$ is simple.

(ii) Suppose that $\mathcal{A}_1$ and $\mathcal{A}_2$ have unique traces τ_1 and τ_2 respectively. If τ is a trace on $\mathcal{A}$, then we must have

$$\tau_1(x) = \tau(x \otimes 1), \quad \tau_2(y) = \tau(1 \otimes y), \quad x \in \mathcal{A}_1,\ y \in \mathcal{A}_2,$$

so that $\tau(\sum_{i=1}^n x_i \otimes y_i) = \sum_{i=1}^n \tau_1(x_i)\tau_2(y_i)$ for all $x_1, \dots, x_n \in \mathcal{A}_1$ and $y_1, \dots, y_n \in \mathcal{A}_2$. Hence all traces on $\mathcal{A}$ are equal on a dense subset, and thus on all of $\mathcal{A}$. The existence of a trace on $\mathcal{A}$ follows from [14, Proposition 3.4.7]. □

The next two results are consequences of the theorems of Bédos from the previous chapter and the above theorem.

Proposition 5.1.4. *Let Γ be a discrete group. If Γ is C^*-simple (resp. has unique trace), then* $\mathrm{Aut}(\Gamma)$ *is C^*-simple (resp. has unique trace) as a discrete group.*

Proof. Since Γ is centerless by Corollary 1.7.8 and Proposition 1.8.6, we can embed Γ into $\mathrm{Aut}(\Gamma)$ by mapping each element $s \in \Gamma$ to the inner automorphism $\sigma_s \in \mathrm{Aut}(\Gamma)$ given by $\sigma_s(t) = sts^{-1}$. For all $\gamma \in \mathrm{Aut}(\Gamma)$ and $s \in \Gamma$ we then have $\gamma\sigma_s = \sigma_{\gamma(s)}\gamma$, proving that Γ is a normal icc subgroup of $\mathrm{Aut}(\Gamma)$. Finally, if γ belongs to the centralizer of Γ in $\mathrm{Aut}(\Gamma)$, then for all $s, t \in \Gamma$ we have

$$s\gamma(t)s^{-1} = \gamma(s)\gamma(t)\gamma(s)^{-1}.$$

Since γ is a bijection, we see that $s^{-1}\gamma(s)$ belongs to the center of Γ. Therefore γ must be the identity map, so Γ has trivial centralizer in $\mathrm{Aut}(\Gamma)$. The desired result now follows from Theorem 4.1.24 and Corollary 4.4.7. □

Proposition 5.1.5. *Let Γ be a discrete group. If Γ is C^*-simple (resp. has unique trace), then its holomorph* $\mathrm{Hol}(\Gamma) = \Gamma \rtimes \mathrm{Aut}(\Gamma)$ *induced by the identity map on* $\mathrm{Aut}(\Gamma)$ *is C^*-simple (resp. has unique trace) as a discrete group.*

Proof. Letting $\Gamma' \subseteq \mathrm{Aut}(\Gamma)$ be the normal subgroup of inner automorphisms on Γ as in the proof of Proposition 5.1.4, we can view $\Gamma \rtimes \Gamma'$ as a subgroup of $\mathrm{Hol}(\Gamma)$. As

$$(r, \sigma_s)(t, \gamma) = (r\sigma_s(t), \sigma_s\gamma) = (r\sigma_s(t), \gamma\sigma_{\gamma^{-1}(s)}) = (t, \gamma)(\gamma^{-1}(t^{-1}r\sigma_s(t)), \sigma_{\gamma^{-1}(s)}) \subseteq (t, \gamma)(\Gamma \rtimes \Gamma')$$

for all $r, s, t \in \Gamma$ and $\gamma \in \mathrm{Aut}(\Gamma)$, $\Gamma \rtimes \Gamma'$ is a normal subgroup, and if (t, γ) belongs to the centralizer of $\Gamma \rtimes \Gamma'$, then the above calculations yield

$$(\gamma^{-1}(t^{-1}r\sigma_s(t)), \sigma_{\gamma^{-1}(s)}) = (r, \sigma_s)$$

for all $r, s \in \Gamma$. Therefore γ is the identity map, and by putting $s = t$, we see that t belongs to the center of Γ and therefore must be 1. Finally, note that $\Gamma \rtimes \Gamma'$ is isomorphic to $\Gamma \times \Gamma$ by the map $(s, \sigma_t) \mapsto (st, t)$. Since $\Gamma \times \Gamma$ is C^*-simple (resp. has unique trace) by Theorem 5.1.1 whenever Γ is C^*-simple (resp. has unique trace), the wanted result follows. □

5.2 Inductive limits

We first recall the definition of an inductive limit of a family of discrete groups. Assume that $(\Gamma_i)_{i\in I}$ is a family of discrete groups indexed by a directed set I and that we have homomorphisms $\varphi_{ij} \colon \Gamma_i \to \Gamma_j$ for all $i, j \in I$ with $i \leq j$ such that

(i) φ_{ii} is the identity of Γ_i, and
(ii) $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ for all $i \leq j \leq k$.

The pair $(\Gamma_i, \varphi_{ij})_{i\in I}$ is then called an *inductive system.* We then define an equivalence relation $\sim$ on the disjoint union $\coprod_{i\in I} \Gamma_i$ by writing $s_i \sim s_j$ for $s_i \in \Gamma_i$ and $s_j \in \Gamma_j$ if there is a $k \in I$ such that $\varphi_{ik}(s_i) = \varphi_{jk}(s_j)$. The set of equivalence classes $\Gamma = (\coprod_{i\in I} \Gamma_i)/\sim$ is then turned into a group by defining $[s_i][s_j] = [\varphi_{ik}(s_i)\varphi_{jk}(s_j)]$ for $s_i \in \Gamma_i$ and $s_j \in \Gamma_j$, where $k \in I$ satisfies $i, j \leq k$. The group Γ is then called the *inductive limit* of the system (Γ_i, φ_{ij}), and we write

$$\Gamma = \varinjlim(\Gamma_i, \varphi_{ij}).$$

Further, we can define canonical homomorphisms $\varphi_i \colon \Gamma_i \to \Gamma$ by $\varphi_i(s_i) = [s_i]$ for $i \in I$ and $s_i \in \Gamma_i$, so that $\Gamma = \bigcup_{i\in I} \varphi_i(\Gamma_i)$.

If we are to say anything about the connection between the reduced group C^*-algebras of two discrete groups Γ_1 and Γ_2 whenever there exists a homomorphism $\varphi \colon \Gamma_1 \to \Gamma_2$, it is a good idea to look at Remark 1.7.9: there exists a homomorphism $\tilde{\varphi} \colon C^*_r(\Gamma_1) \to C^*_r(\Gamma_2)$ satisfying $\tilde{\varphi}(\lambda_{\Gamma_1}(s)) = \lambda_{\Gamma_2}(\varphi(s))$ for all $s \in \Gamma_1$ if and only if $\ker \varphi$ is amenable. We then have the following result.

Proposition 5.2.1. *Let (Γ_i, φ_{ij}) be an inductive system of discrete groups such that each φ_{ij} has amenable kernel, and let $\Gamma = \varinjlim(\Gamma_i, \varphi_{ij})$. Then:*

(i) *If Γ_i is C^*-simple for all $i \in I$, then Γ is C^*-simple.*
(ii) *If Γ_i has unique trace for all $i \in I$, then Γ has unique trace.*

Proof. Let $\varphi_i \colon \Gamma_i \to \Gamma$ be the canonical homomorphism for each $i \in I$. Regardless of the initial assumptions of (i) and (ii) about the Γ_i's, the amenable radical of each Γ_i must be $\{1\}$ by Theorem 1.7.5 and Proposition 1.8.5. Hence all homomorphisms φ_{ij} are injective, so that each φ_i is also injective. By Proposition 1.3.11, for all $i \in I$ we then have injective *-homomorphisms $\tilde{\varphi}_i \colon C^*_r(\Gamma_i) \to C^*_r(\Gamma)$ satisfying $\tilde{\varphi}_i(\lambda_{\Gamma_i}(s)) = \lambda_\Gamma(\varphi_i(s))$ for $s \in \Gamma_i$. Since $\Gamma = \bigcup_{i\in I} \varphi_i(\Gamma_i)$, we find that

$$C^*_r(\Gamma) = \overline{\bigcup_{i\in I} \tilde{\varphi}_i(C^*_r(\Gamma_i))}.$$

If $\mathfrak{I}$ is a non-zero, closed ideal of $C^*_r(\Gamma_i)$, then $\mathfrak{I} \cap \tilde{\varphi}_i(C^*_r(\Gamma_i))$ is an ideal of $\tilde{\varphi}_i(C^*_r(\Gamma_i))$ for all $i \in I$, C^*-simplicity of all Γ_i implies that $\tilde{\varphi}_i(C^*_r(\Gamma_i)) \subseteq \mathfrak{I}$ for all $i \in I$. Since $\mathfrak{I}$ is closed, we find that

$C_r^*(\Gamma) \subseteq \mathfrak{I}$, so that $C_r^*(\Gamma)$ is simple, proving (i). If τ is a trace on $C_r^*(\Gamma_i)$ and $s \in \Gamma \setminus \{1\}$, then there exists $s_i \in \Gamma_i \setminus \{1\}$ for some $\varphi_i(s_i) = s$. If all Γ_i have unique trace, then $\tau \circ \tilde{\varphi}_i$ is the canonical trace on $C_r^*(\Gamma_i)$. Therefore

$$\tau(\lambda_\Gamma(s)) = \tau(\lambda_\Gamma(\varphi_i(s_i))) = (\tau \circ \tilde{\varphi})(\lambda_{\Gamma_i}(s_i)) = 0,$$

so τ must be the canonical trace on $C_r^*(\Gamma)$. □

5.3 Finite index subgroups

In order to study permanence properties for C^*-simplicity and unique traces with respect to subgroups of finite index, it will come in handy to introduce a similar notion of index for the group von Neumann algebras. Recall first that if Γ is a discrete non-trivial group, then $L(\Gamma)$ is a II_1-factor if and only if Γ is icc. Since C^*-simplicity and unique trace both imply that the group in question is icc, we will only focus on the theory of indices for finite factors and subfactors.

Definition 5.3.1. If $\mathscr{M} \subseteq B(\mathcal{H})$ is a finite factor and $\mathscr{M}'$ is finite, we define the *coupling constant* $\dim_{\mathscr{M}}(\mathcal{H})$ of $\mathscr{M}$ by

$$\dim_{\mathscr{M}}(\mathcal{H}) = \frac{\tau_{\mathscr{M}}(p_\xi^{\mathscr{M}'})}{\tau_{\mathscr{M}'}(p_\xi^{\mathscr{M}})},$$

where ξ is a non-zero vector in $\mathcal{H}$, $\tau_{\mathscr{M}}$ resp. $\tau_{\mathscr{M}'}$ are the unique normal traces on $\mathscr{M}$ resp. $\mathscr{M}'$, and $p_\xi^{\mathscr{M}}$ resp. $p_\xi^{\mathscr{M}'}$ are the projections of $\mathcal{H}$ onto $\overline{\mathscr{M}\xi}$ resp. $\overline{\mathscr{M}'\xi}$.

By [23, Propositions I.1.4.4 and III.6.1.1], $\dim_{\mathscr{M}}(\mathcal{H})$ is well-defined and independent of the choice of ξ. Moreover, for all projections $p \in \mathscr{M}'$ it holds that

$$\dim_{\mathscr{M}_p}(p\mathcal{H}) = \dim_{\mathscr{M}}(\mathcal{H})\tau_{\mathscr{M}'}(p). \tag{5.3.1}$$

Using the structure of normal surjective homomorphisms of von Neumann algebras, one is in fact able to prove the following:

Proposition 5.3.2. *Let $\mathscr{M}$ be a finite factor and let $\mathscr{N} \subseteq \mathscr{M}$ be a subfactor. If $\mathscr{N}'$ is finite, then the number*

$$\frac{\dim_{\mathscr{N}}(\mathcal{H})}{\dim_{\mathscr{M}}(\mathcal{H})}$$

is independent of $\mathcal{H}$.

Proof. See [38, Proposition 2.1.7]. □

This cleans our conscience, allowing us to define the main object of study:

Definition 5.3.3. For all subfactors $\mathscr{N}$ of $\mathscr{M}$ with $\mathscr{N}'$ finite, the *global index of $\mathscr{N}$ in $\mathscr{M}$* is the number $[\mathscr{M} : \mathscr{N}]$ given by

$$[\mathscr{M} : \mathscr{N}] = \frac{\dim_{\mathscr{N}}(\mathcal{H})}{\dim_{\mathscr{M}}(\mathcal{H})},$$

where $\mathcal{H}$ is a Hilbert space on which $\mathscr{M}$ is represented.

Before going on to study properties of the global index, we will first take a look at how indices can be determined over group von Neumann algebras; this in itself requires some preliminaries. Henceforth, unless denoted otherwise, the unique trace of a finite factor $\mathscr{M}$ will be denoted by $\tau_{\mathscr{M}}$.

Remark 5.3.4. Let Γ be a discrete group and let Λ be a subgroup. Then the canonical embedding $J\colon C_r^*(\Lambda) \to C_r^*(\Gamma)$ satisfies

$$\tau_\Gamma(J(\lambda_\Lambda(s))) = \tau_\Lambda(\lambda_\Lambda(s)), \quad s \in \Lambda,$$

where τ_Λ and τ_Γ denote the canonical faithful traces on $L(\Lambda)$ and $L(\Gamma)$, respectively. By Proposition 4.4.1, we obtain a normal, injective *-homomorphism $\theta\colon L(\Lambda) \to L(\Gamma)$ satisfying $\theta(\lambda_\Lambda(s)) = \lambda_\Gamma(s)$ for all $s \in \Lambda$. Hence we can view $L(\Lambda)$ as a von Neumann subalgebra of $L(\Gamma)$, which we will do in the sequel, and we will also denote the canonical trace on $L(\Lambda)$ resp. $L(\Gamma)$ by τ_Λ resp. τ_Γ, as we just did. Also, whenever we consider $L(\Lambda)'$, we will view it as a subalgebra of $B(\ell^2(\Gamma))$ (*not* $B(\ell^2(\Lambda))$). ✺

Remark 5.3.5. It has not been necessary until now, but it is worthwhile to introduce the *right-regular representation* $\rho_G\colon G \to \mathcal{U}(L^2(G))$ for locally compact groups G with a fixed Haar measure, given by

$$\rho_G(s)f = f.s, \quad s \in G,\ f \in L^2(G).$$

Considering a discrete group Γ, then ρ_Γ uniquely satisfies

$$\rho_\Gamma(s)\delta_t = \delta_{ts^{-1}}, \quad s,t \in \Gamma.$$

Moreover, the von Neumann algebra $\rho_\Gamma(\Gamma)''$ is denoted by $R(\Gamma)$, and it has the following properties:

(i) $L(\Gamma)' = R(\Gamma)$.
(ii) The linear functional $x \mapsto \tau_\Gamma(x) := \langle x\delta_1, \delta_1\rangle$ is a faithful trace on $R(\Gamma)$.
(iii) If Γ is non-trivial, then $R(\Gamma)$ is a II_1-factor if and only if Γ is icc.

For proofs of these statements, the reader can consult [39, Section 6.7]. ✺

One would expect for group von Neumann algebras that the global index has something to do with the usual index of subgroups in groups, and that isn't far off – at all. Note that if Γ is an icc group and Λ is a subgroup of finite index, then Λ is also icc. Indeed, let T be a finite transversal for Λ in Γ and suppose that Λ contains a finite conjugacy class $F \neq \{1\}$. Then for all $s \in F$, we have

$$\{wsw^{-1} \,|\, w \in \Gamma\} = \bigcup_{t\in T}\{wsw^{-1} \,|\, w \in t\Lambda\} = \bigcup_{t\in T} tFt^{-1},$$

which is a finite set.

Proposition 5.3.6. *Let Γ be a discrete non-trivial icc group and let Λ be a non-trivial subgroup of Γ of finite index. Then $L(\Lambda)' \subseteq B(\ell^2(\Gamma))$ is a finite factor and*

$$[L(\Gamma) : L(\Lambda)] = [\Gamma : \Lambda].$$

Proof. If we take a right transversal T for Λ in Γ that contains 1, let $V_s = \overline{L(\Lambda)\delta_s} \subseteq \ell^2(\Gamma)$ and let p_s denote the projection onto V_s for all $s \in T$. Note that V_s is the subspace generated by all vectors

$$\sum_{t\in F}\alpha_t\lambda_\Gamma(t)\delta_s = \sum_{t\in F}\alpha_t\delta_{ts} \in \ell^2(\Gamma),$$

where $F \subseteq \Lambda$ is a finite subset and $\alpha_t \in \mathbb{C}$ for $t \in F$, and that $V_s = \ell^2(\Lambda s)$. Moreover, we have $\sum_{s\in T} p_s = 1$ and $p_s \sim p_1$ for all $s \in T$, since $\rho_\Gamma(s)^*p_s\rho_\Gamma(s) = p_1$. It is now easy to see that for all $s \in T$ that

$$L(\Lambda)'_{p_s} = \{\lambda_\Gamma(t)|_{\ell^2(\Lambda s)} \,|\, t \in \Lambda\}'$$

which is isomorphic to $R(\Lambda) \subseteq B(\ell^2(\Lambda))$. Therefore $L(\Lambda)'$ is isomorphic to $\bigoplus_{s\in T} R(\Lambda)$ and is therefore finite, so that it has a unique trace $\tau_{L(\Lambda)'}$.

Now, note that

$$1 = \tau_{L(\Lambda)'}(1) = \sum_{s\in T}\tau_{L(\Lambda)'}(p_s) = \sum_{s\in T}\tau_{L(\Lambda)'}(p_1) = [\Gamma : \Lambda]\tau_{L(\Lambda)'}(p_1). \tag{5.3.2}$$

We then consider the von Neumann algebra $L(\Lambda)_{p_1}$ acting on $p_1(\ell^2(\Gamma)) = \ell^2(\Lambda)$. It is easy to see, for instance by means of [15, Proposition 2.17], that

$$L(\Lambda)_{p_1} = \{\lambda_\Gamma(s)|_{\ell^2(\Lambda)} \,|\, s \in \Lambda\}'' = \{\lambda_\Lambda(s) \,|\, s \in \Lambda\}'' = L(\Lambda) \subseteq B(\ell^2(\Lambda)).$$

We have $\dim_{L(\Lambda)}(\ell^2(\Lambda)) = 1$, since $\delta_1 \in \ell^2(\Lambda)$ is a cyclic vector for both $L(\Lambda)$ and $R(\Lambda)$; similarly we have $\dim_{L(\Gamma)}(\ell^2(\Gamma)) = 1$. Consequently, by (5.3.1) it follows that

$$[\Gamma : \Lambda] = \dim_{L(\Lambda)}(\ell^2(\Gamma)) = \frac{\dim_{L(\Lambda)}(\ell^2(\Gamma))}{\dim_{L(\Gamma)}(\ell^2(\Gamma))} = [L(\Gamma) : L(\Lambda)],$$

completing the proof. □

Our next goal is to translate the work of [38] into results about group von Neumann algebras; more precisely, we will find another II_1-factor inside $B(\ell^2(\Gamma))$ for an icc group Λ that has the same relationship to $L(\Gamma)$ as $L(\Gamma)$ has to $L(\Lambda)$, where Λ is a subgroup of finite index.

Assume first that Γ and Λ are both icc groups, and that Λ is a subgroup of Γ. Then there exists a unique τ-preserving conditional expectation $\mathscr{E}\colon L(\Gamma) \to L(\Lambda)$ (cf. [14, Lemma 1.5.11]), and $\mathscr{E}$ is also faithful and normal. If p denotes the orthogonal projection of $\ell^2(\Gamma)$ onto $\ell^2(\Lambda)$ and $x \in L(\Gamma)$, then for all $y \in L(\Gamma)$ we have

$$\langle \mathscr{E}(x)\delta_1, y\delta_1\rangle = \tau(y^* E(x)) = \tau(E(y^*x)) = \tau(y^*x) = \langle x\delta_1, y\delta_1\rangle.$$

As $\overline{L(\Lambda)\delta_1} = \ell^2(\Lambda)$ in $\ell^2(\Gamma)$, it follows that $p(x\delta_1) = \mathscr{E}(x)\delta_1$; since δ_1 is a separating vector, it follows for all $s \in \Gamma$ that $\mathscr{E}(\lambda_\Gamma(s)) = 1_\Lambda(s)\lambda_\Gamma(s)$.

Lemma 5.3.7. *The projection p has the following properties:*

(i) *For all $x \in L(\Gamma)$, $pxp = \mathscr{E}(x)p$.*
(ii) *If $x \in L(\Gamma)$, then $x \in L(\Lambda)$ if and only if $px = xp$.*
(iii) *We have $L(\Lambda)' = (L(\Gamma)' \cup \{p\})''$ in $B(\ell^2(\Gamma))$.*

Proof. (i) For all $y \in \mathscr{M}$, we have

$$pxp(y\delta_1) = px\mathscr{E}(y)\delta_1 = \mathscr{E}(x\mathscr{E}(y))\delta_1 = \mathscr{E}(x)\mathscr{E}(y)\delta_1 = \mathscr{E}(x)p(y\delta_1).$$

As the set of all $y\delta_1$ is dense in $\ell^2(\Gamma)$, the equality follows.

(ii) If $x \in L(\Lambda)$, then $pxy\delta_1 = \mathscr{E}(xy)\delta_1 = x\mathscr{E}(y)\delta_1 = xpy\delta_1$, by which a density argument yields $px = xp$. If $x \in L(\Gamma)$ commutes with p, then $\mathscr{E}(x)\delta_1 = px\delta_1 = xp\delta_1 = x\delta_1$, so since δ_1 is separating, we have $x = \mathscr{E}(x) \in L(\Lambda)$.

(iii) By (ii), we have $(L(\Gamma)' \cup \{p\})' = L(\Gamma) \cap \{p\}' = L(\Lambda)$. □

Definition 5.3.8. The projection $p \in B(\ell^2(\Gamma))$ of the above discussion is called the *Jones projection.* The von Neumann algebra

$$\langle L(\Gamma), p\rangle = (L(\Gamma) \cup \{p\})''$$

is called the *basic construction* (cf. [38, p. 8]).

We now investigate the most immediately invigorating properties of the basic construction.

Lemma 5.3.9. *The basic construction $\mathscr{M} = \langle L(\Gamma), p\rangle$ enjoys the following properties:*

(i) *The set consisting of all operators of the form $x_0 + \sum_{i=1}^n x_i p y_i$ where $x_0, x_1, y_1, \ldots, x_n, y_n \in L(\Gamma)$ constitutes a strongly dense *-subalgebra of $\mathscr{M}$.*
(ii) *The map $x \mapsto xp$ is an isomorphism of $L(\Lambda)$ onto $p\mathscr{M}p$.*
(iii) *$\mathscr{M}$ is a factor.*
(iv) *If $[\Gamma : \Lambda] < \infty$, then $\mathscr{M}$ is a II_1-factor satisfying $[\mathscr{M} : L(\Gamma)] = [\Gamma : \Lambda]$. If $\tau_{\mathscr{M}}$ denotes the unique trace of $\mathscr{M}$, then $\tau_{\mathscr{M}}$ satisfies $\tau_{\mathscr{M}}|_{L(\Gamma)} = \tau_\Gamma$ and*

$$\tau_{\mathscr{M}}(px) = [\Gamma\colon \Lambda]^{-1}\tau_\Gamma(x), \quad x \in L(\Gamma).$$

Moreover, if $\mathscr{E}_1$ denotes the canonical $\tau_{\mathscr{M}}$-preserving conditional expectation of $\mathscr{M}$ onto $L(\Gamma)$, then $\mathscr{E}_1(p) = [\Gamma\colon \Lambda]^{-1}1$.

Proof. Property (i) is easily checked by using the properties of p from Lemma 5.3.7. To see that (ii) holds, note first that $L(\Lambda)p = pL(\Lambda)p \subseteq p\mathscr{M}p$ and that

$$p(x_0 + x_1py_1)p = px_0p + px_1py_1p = \mathscr{E}(x_0)p + \mathscr{E}(x_1)\mathscr{E}(y_1)p \in L(\Lambda)p$$

for all $x_0, x_1, y_1 \in L(\Gamma)$ by Lemma 5.3.7 (i). Since $L(\Lambda)p$ is strongly closed, it follows from (i) that $p\mathscr{M}p = L(\Lambda)p$. Finally, note that $xp\delta_1 = \mathscr{E}(x)\delta_1 = x\delta_1$ for $x \in L(\Lambda)$. Hence if $xp = 0$, we must have $x\delta_1 = 0$ and therefore $x = 0$, so the map $x \mapsto xp$ is injective and clearly a homomorphism. For (iii), assume that $x \in \mathscr{M} \cap \mathscr{M}'$. Then $pxp = xp$ commutes with everything in $p\mathscr{M}p$, so that $x \in \mathbb{C}1$ by (ii).

Now we assume that $[\Gamma : \Lambda] < \infty$. Then by Proposition 5.3.6, $L(\Lambda)'$ is finite. If we define $J\colon \ell^2(\Gamma) \to \ell^2(\Gamma)$ by

$$J\left(\sum_{s\in\Gamma} \beta_s \delta_s\right) = \sum_{s\in\Gamma} \overline{\beta_{s^{-1}}}\delta_s,$$

then J is a conjugate linear surjective isometry satisfying $J^2 = 1$ and $Jx\delta_1 = x^*\delta_1$ for all $x \in L(\Gamma)$ – it is in fact the *modular conjugation operator* for $L(\Gamma)$ by [23, Lemma I.4.1.3] and [15, Lemma 5.27], so that $JL(\Gamma)J = L(\Gamma)'$. As

$$(JpJ)x\delta_1 = Jpx^*\delta_1 = J\mathscr{E}(x)^*\delta_1 = \mathscr{E}(x)\delta_1 = px\delta_1$$

for all $x \in L(\Gamma)$, it follows that $JpJ = p$. Therefore

$$J\mathscr{M}J = J(L(\Gamma) \cup \{p\})''J = (L(\Gamma)' \cup \{p\})'' = L(\Lambda)',$$

from which it follows that $\mathscr{M}$ is finite. Since $p\mathscr{M}p$ is a II_1-factor, $\mathscr{M}$ itself must be a II_1-factor.

It is easy to see that $J\mathscr{M}'J \subseteq (J\mathscr{M}J)'$; moreover, if $x \in (J\mathscr{M}J)'$, then for all $y \in \mathscr{M}$, we have

$$(JxJ)y = Jx(JyJ)J = J(JyJ)xJ = y(JxJ),$$

so that $JxJ \in \mathscr{M}'$. Therefore $J\mathscr{M}'J = (J\mathscr{M}J)' = L(\Lambda)$. As both $\mathscr{M}$ and $\mathscr{M}'$ are finite factors, their traces are unique and hence must be given by

$$\tau_{\mathscr{M}}(x) = \tau_{L(\Lambda)'}(JxJ), \quad \tau_{\mathscr{M}'}(y) = \tau_{L(\Lambda)}(JyJ), \quad x \in \mathscr{M},\ y \in \mathscr{M}'.$$

If $p_{\mathscr{M}'}$ resp. $p_{\mathscr{M}}$ denote the projections of $\ell^2(\Gamma)$ onto $\overline{\mathscr{M}'\delta_1}$ resp. $\overline{\mathscr{M}\delta_1}$, then as $\overline{\mathscr{M}'\delta_1} = \ell^2(\Lambda)$ and $\overline{\mathscr{M}\delta_1} = \ell^2(\Gamma)$ we have $p_{\mathscr{M}'} = p$ and $p_{\mathscr{M}} = 1$. Thus by (5.3.2), it follows that

$$[\mathscr{M} : L(\Gamma)] = \frac{\dim_{L(\Gamma)}(\ell^2(\Gamma))}{\dim_{\mathscr{M}}(\ell^2(\Gamma))} = \frac{\tau_{\mathscr{M}'}(p_{\mathscr{M}})}{\tau_{\mathscr{M}}(p_{\mathscr{M}'})} = \frac{\tau_{L(\Lambda)}(1)}{\tau_{L(\Lambda)'}(p)} = [\Gamma\colon \Lambda]$$

and $\tau_{\mathscr{M}}(p) = [\Gamma\colon \Lambda]^{-1}$. Finally, it is clear that $\tau_{\mathscr{M}}|_{L(\Gamma)} = \tau_\Gamma$, since $L(\Gamma)$ has unique trace. Moreover, if we define $\tau'(x) = [\Gamma\colon \Lambda]\tau_{\mathscr{M}}(px)$ for $x \in L(\Lambda)$, then it is easy to see by Lemma 5.3.7 (ii) that τ' is a trace on $L(\Lambda)$. Therefore $\tau' = \tau_\Lambda = \tau_\Gamma|_{L(\Lambda)}$ by $L(\Lambda)$ having unique trace, so that for all $x \in L(\Gamma)$, we have

$$\tau_{\mathscr{M}}(px) = \tau_{\mathscr{M}}(pxp) = \tau_{\mathscr{M}}(p\mathscr{E}(x)) = [\Gamma\colon \Lambda]^{-1}\tau_\Gamma(\mathscr{E}(x)) = [\Gamma\colon \Lambda]^{-1}\tau_\Gamma(x).$$

Finally, this implies that

$$\tau_{\mathscr{M}}(x^*(\mathscr{E}_1(p) - [\Gamma : \Lambda]^{-1})x) = 0$$

for all $x \in L(\Gamma)$. Thus

$$\langle(\mathscr{E}_1(p) - [\Gamma : \Lambda]^{-1})\xi, \xi\rangle = 0$$

for all $\xi \in \ell^2(\Gamma)$, so because $\mathscr{E}_1(p) - [\Gamma : \Lambda]^{-1}1$ is self-adjoint, we must have $\mathscr{E}_1(p) = [\Gamma : \Lambda]^{-1}1$. □

In order to turn this information about the group von Neumann algebras into information about the corresponding C^*-algebras, we turn to a notion due to Pimsner and Popa:

Definition 5.3.10. Let $\mathcal{A}$ be a unital C^*-algebra and let $\mathcal{B} \subseteq \mathcal{A}$ be a unital C^*-subalgebra with $1_{\mathcal{B}} = 1_{\mathcal{A}}$ for which there exists a conditional expectation $E\colon \mathcal{A} \to \mathcal{B}$. Then the *index* of E is given by

$$\mathrm{Ind}(E) = (\sup\{c \geq 0 \,|\, E(x) \geq cx \text{ for all } x \in \mathcal{A}_+\})^{-1} \in (0, \infty].$$

Note that $\mathrm{Ind}(E) \geq 1$ since $E(1_{\mathcal{A}}) = 1_{\mathcal{A}}$. By the definition, we have $\mathrm{Ind}(E) < \infty$ if and only if there exists a $c > 0$ such that $E(x) \geq cx$ for all $x \in \mathcal{A}_+$, which in turn implies that $E(x) \geq \mathrm{Ind}(E)^{-1}x$ for all $x \in \mathcal{A}_+$. The study of finite index conditional expectations has been the focal point of quite a few articles by Popa, one of which ([57]) will be of particular interest in the following discussion.

If we are only to consider II_1-factors, there exists a quite non-trivial result, originally proved by Pimsner and Popa in [55, Theorem 2.2], that comes to our aid; we shall only need the part of the theorem that is the easiest to prove. Recall again that if $\mathscr{M}$ is a von Neumann algebra with a faithful normal trace and $\mathscr{N} \subseteq \mathscr{M}$ is a von Neumann subalgebra, then there is a unique trace-preserving conditional expectation $E\colon \mathscr{M} \to \mathscr{N}$ (cf. [14, Lemma 1.5.11]) that is also faithful and normal.

Theorem 5.3.11 (The Pimsner-Popa inequality). *If $\mathscr{N} \subseteq \mathscr{M}$ is an inclusion of* II_1*-factors such that $\mathscr{N}'$ is finite, and E is the unique trace-preserving conditional expectation $\mathscr{M} \to \mathscr{N}$, then* $\mathrm{Ind}(E) = [\mathscr{M} : \mathscr{N}]$.

Our next stop is a weaker version of [57, Lemma 4.4], adjusted to fit in with our future purposes.

Lemma 5.3.12. *Let $\mathcal{B} \subseteq \mathcal{A}$ be an inclusion of unital C^*-algebras with $1_\mathcal{B} = 1_\mathcal{A}$. Assume further that there exists a unital faithful representation $\pi\colon \mathcal{A} \to B(\mathcal{H})$ and let $\mathscr{M} = \pi(\mathcal{A})''$ and $\mathscr{N} = \pi(\mathcal{B})''$. If $\varphi_0, \varphi_1 \in \mathcal{A}^*$ and $\omega \in \mathscr{M}_*$ are positive and*

$$\varphi_0 \leq \varphi_1 = \omega \circ \pi,$$

then there exists a normal positive linear functional $\psi \in \mathscr{M}_*$ such that $\varphi_0 = \psi \circ \pi$.

Proof. There exists a surjective, normal *-homomorphism $\theta\colon \mathcal{A}^{**} \to \mathscr{M}$ such that $\theta|_\mathcal{A} = \pi$ (if we agree to consider $\mathcal{A}$ as a subalgebra of its enveloping von Neumann algebra, cf. [15, p. 63]). Let $\tilde{\varphi}_i$ denote the canonical normal positive extension of φ_0 to $\mathcal{A}^{**}$ for $i = 0, 1$, and define $\tilde{\omega} = \omega \circ \theta_1$. If $x \in \mathcal{A}^{**}$, then by taking a net $(x_i)_{i \in I}$ in $\mathcal{A}$ such that $x_i \to x$ ultraweakly, we find

$$\tilde{\varphi}_1(x) = \lim_i \varphi_1(x_i) = \lim_i \omega(\pi(x_i)) = \lim_i \omega(\theta_1(x_i)) = \tilde{\omega}(x).$$

As we now have $\tilde{\varphi}_0 \leq \tilde{\omega}$, it follows from Sakai's Radon-Nikodym theorem [65, Theorem 1.24.3] that there exists a positive element $t \in \mathcal{A}^{**}$ with $t_0 \leq 1$ such that $\tilde{\varphi}_0(x) = \tilde{\omega}(txt) = \omega(\theta_1(t)\theta_1(x)\theta_1(t))$ for all $x \in \mathcal{A}^{**}$. Defining $m = \theta_1(t) \in \mathscr{M}$ and $\psi(x) = \omega(mxm)$ for all $x \in \mathscr{M}$ now does the trick. □

Proposition 5.3.13. *Let $\mathcal{A}$, $\mathcal{B}$, π, $\mathscr{M}$ and $\mathscr{N}$ be as in Lemma 5.3.12. Assume that there exist conditional expectations $E\colon \mathcal{A} \to \mathcal{B}$and $\mathscr{E}\colon \mathscr{M} \to \mathscr{N}$ such that $\mathscr{E} \circ \pi = \pi \circ E$, with E of finite index and $\mathscr{E}$ normal. If ϕ is a state on $\mathcal{A}$ and there exists a normal state $\omega \in \mathscr{N}_*$ such that $\phi|_\mathcal{B} = \omega \circ \pi|_\mathcal{B}$, then there exists a normal state $\psi \in \mathscr{M}_*$ such that $\phi = \psi \circ \pi$.*

Proof. Define $c = (\mathrm{Ind}(E))^{-1}$. Then all $a \in \mathcal{A}$, we have

$$\phi(a) \leq c^{-1}\phi(E(a)) = c^{-1}\omega(\pi(E(a))) = c^{-1}\omega(\mathscr{E}(\pi(a))),$$

so since ω and $\mathscr{E}$ are normal, then by Lemma 5.3.12 there exists a positive linear functional $\psi \in \mathscr{M}_*$ such that $\phi = \psi \circ \pi$. Since $\psi(1) = \phi(1) = 1$, ψ is a state. □

Theorem 5.3.14. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-subalgebras of $B(\mathcal{H})$ with $\mathcal{B} \subseteq \mathcal{A}$ and $1_\mathcal{B} = 1_\mathcal{A} = 1_\mathcal{H}$, such that there exists a conditional expectation $E\colon \mathcal{A} \to \mathcal{B}$ of finite index. Assume further that*

- ➢ *$\mathcal{A}$ has a faithful state φ such that $\varphi = \varphi \circ E$;*
- ➢ *$\mathscr{M} = \mathcal{A}''$ and $\mathscr{N} = \mathcal{B}''$ are finite factors and the unique normal trace $\tau_\mathscr{M}$ on $\mathscr{M}$ satisfies $\tau_\mathscr{M}|_\mathcal{B} = \varphi|_\mathcal{B}$;*
- ➢ *there exists a normal conditional expectation $\mathscr{E}\colon \mathscr{M} \to \mathscr{N}$ such that $\mathscr{E}|_\mathcal{A} = E$.*

Then the following holds:

(i) *If $\mathcal{B}$ is simple, then $\mathcal{A}$ is simple.*
(ii) *If $\mathcal{B}$ has unique trace, then $\mathcal{A}$ has unique trace.*

Proof. (i) Suppose that $\mathcal{A}$ is not simple, let $\mathfrak{I}$ be a non-trivial proper closed ideal of $\mathcal{A}$ and let $\pi\colon \mathcal{A} \to \mathcal{A}/\mathfrak{I}$ denote the quotient map. Then $\pi|_\mathcal{B}$ is injective, so by the Hahn-Banach theorem, there exists a state φ' on $\mathcal{A}/\mathfrak{I}$ such that $\varphi'(\pi(b)) = \varphi(b)$ for all $b \in \mathcal{B}$ [15, Lemma 2.42]. Since $\tau_\mathscr{M}|_\mathcal{B} = \varphi' \circ \pi|_\mathcal{B}$, Proposition 5.3.13 yields a normal state $\psi \in \mathscr{M}_*$ such that $\psi|_\mathcal{A} = \varphi' \circ \pi$. Since the weak operator closure $\overline{\mathfrak{I}}$ of $\mathfrak{I}$ is an ideal of $\mathscr{M}$, we must have $\overline{\mathfrak{I}} = \mathscr{M}$ since $\mathscr{M}$ is simple by Corollary 2.12. Hence there exists a bounded net $(x_i)_{i \in I}$ of operators in $\mathfrak{I}$ such that $x_i \to 1_\mathscr{M}$ weakly and hence ultraweakly, in which case

$$1 = \psi(1_\mathscr{M}) = \lim_i \psi(x_i) = \lim_i \varphi'(\pi(x_i)) = 0,$$

a contradiction.

(ii) Suppose that $\mathcal{B}$ has unique trace $\tau_\mathcal{B}$ and let ψ be a trace on $\mathcal{A}$. Then $\psi|_\mathcal{B} = \tau_\mathcal{B} = \tau_\mathscr{N}|_\mathcal{B}$, so by Proposition 5.3.13 there is a normal state $\omega \in \mathscr{M}_*$ satisfying $\omega|_\mathcal{A} = \psi$. By strong density of $\mathcal{A}$ in $\mathscr{M}$ and Kaplansky's density theorem, ω must be a trace on $\mathscr{M}$, so $\omega = \tau_\mathscr{M}$ and therefore $\psi = \tau_\mathscr{M}|_\mathcal{A}$. □

Lemma 5.3.15. *If $\mathcal{A}$ is a simple unital C^*-algebra, then $p\mathcal{A}p$ is a simple unital C^*-algebra for all non-zero projections $p \in \mathcal{A}$.*

Proof. Since

$$\mathfrak{I}_{\mathcal{A}}(a) = \left\{ \sum_{i=1}^{n} x_i a y_i \,|\, n \geq 1,\ x_1, y_1, \ldots, x_n, y_n \in \mathcal{A} \right\}$$

is a non-zero algebraic ideal in $\mathcal{A}$ for all non-zero $a \in \mathcal{A}$ and $\mathcal{A}$ is simple, it follows from Lemma A.2.1 that $\mathfrak{I}_{\mathcal{A}}(a) = \mathcal{A}$ for all non-zero $a \in \mathcal{A}$. Hence if $p \in \mathcal{A}$ is a non-zero projection and $\mathfrak{I} \subseteq p\mathcal{A}p$ is a non-zero ideal, then take $a \in \mathfrak{I}_{p\mathcal{A}p} \setminus \{0\}$. Then $p \in \mathcal{A} = \mathfrak{I}_{\mathcal{A}}(a)$, so that there exist $x_1, y_1, \ldots, x_n, y_n \in \mathcal{A}$ such that $p = \sum_{i=1}^{n} x_i a y_i$. But since $a = pap$ in $\mathcal{A}$, we then have $p = \sum_{i=1}^{n} (p x_i p) a (p y_i p)$, so that $\mathfrak{I} \supseteq \mathfrak{I}_{p\mathcal{A}p}(a) = p\mathcal{A}p$. Hence $p\mathcal{A}p$ is simple. $\square$

Theorem 5.3.16. *Let Γ be a discrete group and let $\Lambda \subseteq \Gamma$ be a subgroup of finite index. Then the following holds:*

(i) *If Γ is C^*-simple, then Λ is C^*-simple.*
(ii) *If Λ is C^*-simple, then Γ is C^*-simple if and only if Γ is icc.*
(iii) *If Γ has unique trace, then Λ has unique trace.*
(iv) *If Λ has unique trace, then Γ has unique trace if and only if Γ is icc.*

Proof. We omit the proof of (iii). If $\Lambda = \{1\}$, then Γ is C^*-simple (or has unique trace) if and only if $\Gamma = \{1\}$, so we can assume that Λ is non-trivial. The "only if" implications of (ii) and (iv) are clear, so assume that Γ is icc. Then Λ is also icc, so that both $L(\Gamma)$ and $L(\Lambda)$ are both II_1-factors. Note now that

$$[L(\Gamma) : L(\Lambda)] = [\Gamma : \Lambda] < \infty$$

by Proposition 5.3.6. Letting $E\colon C_r^*(\Gamma) \to C_r^*(\Lambda)$ and $\mathscr{E}\colon L(\Gamma) \to L(\Lambda)$ denote the canonical τ_Γ-preserving conditional expectations, it then follows that $\mathscr{E}$ and hence also E has finite index by the Pimsner-Popa inequality. As $\mathscr{E}|_{C_r^*(\Gamma)} = E$ and τ_Γ is faithful, (ii) and (iv) now follow from Theorem 5.3.14.

To prove (i), assume that $C_r^*(\Gamma)$ is simple. Denote by $\mathcal{A}$ the C^*-algebra generated by $C_r^*(\Gamma)$ and the Jones projection $p \in B(\ell^2(\Gamma))$, let $\mathscr{M}$ be the basic construction $\langle L(\Gamma), p \rangle$ and let $\mathscr{E}_1$ be the canonical trace-preserving conditional expectation $\mathscr{M} \to L(\Gamma)$ (cf. Lemma 5.3.9). By the properties of p from Lemma 5.3.7, the set of operators of the form

$$x_0 + \sum_{i=1}^{n} x_i p y_i$$

for $x_0, x_1, y_1, \ldots, x_n, y_n \in C_r^*(\Gamma)$ is easily seen to be a norm-dense *-subalgebra of $\mathcal{A}$. Since $\mathscr{E}_1(x) = x$ for all $x \in C_r^*(\Gamma)$ and $\mathscr{E}_1(p) = [\Gamma : \Lambda]^{-1} 1$, it follows that $\mathscr{E}_1(\mathcal{A}) = C_r^*(\Gamma)$. Thus if we write $E_1 = \mathscr{E}_1|_{\mathcal{A}}$, then E_1 is a conditional expectation of index $[\Gamma\colon \Lambda] < \infty$ by Lemma 5.3.9. As $\mathscr{M} = \mathcal{A}''$, then by letting $\varphi = \tau_\Gamma \circ E_1$, it now follows from Theorem 5.3.14 that $\mathcal{A}$ is simple. Finally, note by Lemma 5.3.9 (ii) that

$$C_r^*(\Lambda) \cong C_r^*(\Lambda) p = p\mathcal{A}p;$$

therefore $C_r^*(\Lambda)$ is simple by Lemma 5.3.15. $\square$

We finally consider an application of the above result.

Example 5.3.17. We define the projective general linear group

$$P = \mathrm{PGL}(2, \mathbb{R}) = \mathrm{GL}(2, \mathbb{R}) / \mathrm{ZGL}(2, \mathbb{R})$$

where $\mathrm{ZGL}(2, \mathbb{R})$ is the center of $\mathrm{GL}(2, \mathbb{R})$ consisting of all invertible real scalar matrices (see Lemma 6.1.1). Then $\mathrm{PSL}(2, \mathbb{R})$ is a subgroup of P of index 2, as it is the kernel of the homomorphism $\mathrm{PGL}(2, \mathbb{R}) \to \{1, -1\}$ mapping the class of $A \in \mathrm{GL}(2, \mathbb{R})$ to the sign of $\det(A)$. Moreover, P is icc: let $A \in P$ be represented by the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $a, b, c, d \in \mathbb{R}$. If $c \neq 0$, then the lower left entry of

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} a + cx & -cx^2 + (d-a)x + b \\ c & d - cx \end{pmatrix}$$

is $c \neq 0$ for all $x \in \mathbb{R}$, so that we can obtain infinitely many distinct elements of P on the right hand side by varying x. If $c = 0$, then A is conjugate in P to an element with representative

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix},$$

and since

$$\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} a & bx + (1-a)y \\ 0 & 1 \end{pmatrix}$$

for all $x, y \in \mathbb{R}$, we obtain an infinite amount of elements of P on the right hand side by varying x and y, as long as $a \neq 1$ and $b \neq 0$. As P is an icc group containing $\mathrm{PSL}(2, \mathbb{R})$ as a subgroup of index 2, Theorem 5.3.16 yields that P is C^*-simple with unique trace.

CHAPTER 6

C^*-SIMPLICITY AND UNIQUENESS OF TRACE OF SUBGROUPS OF $\mathrm{PSL}(n, \mathbb{R})$

We finally turn to another example of C^*-simple groups with unique trace. Bekka, Cowling and de la Harpe proved in [8] and [9] that certain subgroups of $\mathrm{PSL}(n, \mathbb{R})$, including $\mathrm{PSL}(n, \mathbb{R})$ itself, indeed have these properties. It is unknown whether *all* of these are Powers groups or even weak Powers groups, but it is nonetheless clear that the proof is still based on finding certain Powers-like properties of these groups. Moreover, the proof serves as a great example of why considering actions of a group can be immensely useful for determining properties of the group itself. In this case, we will consider the canonical action of $\mathrm{PSL}(n, \mathbb{R})$ on the projective space $\mathbb{P}^{n-1}(\mathbb{R})$.

Throughout this chapter, let $n \geq 2$ be a fixed integer and let $\mathbb{K}$ be a field of characteristic zero.

6.1 Projective special linear groups

Denoting the centers of $\mathrm{GL}(n, \mathbb{K})$ and $\mathrm{SL}(n, \mathbb{K})$ by $\mathrm{ZGL}(n, \mathbb{K})$ and $\mathrm{ZSL}(n, \mathbb{K})$ respectively, we define the *projective special linear group* $\mathrm{PSL}(n, \mathbb{K})$ by

$$\mathrm{PSL}(n, \mathbb{K}) = \mathrm{SL}(n, \mathbb{K})/\mathrm{ZSL}(n, \mathbb{K}).$$

For any unital subring $R \subseteq \mathbb{K}$, we then define

$$\mathrm{PSL}(n, R) = \mathrm{SL}(n, R)/(\mathrm{SL}(n, R) \cap \mathrm{ZSL}(n, \mathbb{K})).$$

More often than not, we will view $\mathrm{PSL}(n, R)$ as the image of $\mathrm{SL}(n, R) \subseteq \mathrm{SL}(n, \mathbb{K})$ under the quotient homomorphism $\mathrm{SL}(n, \mathbb{K}) \to \mathrm{PSL}(n, \mathbb{K})$.

Lemma 6.1.1. *We have* $\mathrm{ZGL}(n, \mathbb{K}) = \{x1 \mid x \in \mathbb{K}^\times\}$.

Proof. If $A \in \mathrm{ZGL}(n, \mathbb{K})$, then A commutes with every matrix of the form $\mathrm{diag}(x, 1, \dots, 1)$ for $x \in \mathbb{K}^\times$. If $A = (a_{ij})_{i,j=1}^n$, we therefore have $a_{i1}x = a_{i1}$ and $xa_{1j} = a_{1j}$ for all $i, j \neq 1$ and $x \in \mathbb{K}^\times$, implying $a_{i1} = a_{1j} = 0$ for all $i, j \neq 1$. Since A also commutes with $\mathrm{diag}(1, x, 1, \dots, 1)$, we also have $a_{i2}x = a_{i2}$ and $xa_{2j} = a_{2j}$ for all $i, j \neq 2$ and $x \in \mathbb{K}^\times$, again implying $a_{i2} = a_{2j} = 0$ for all $i, j \neq 2$. Continuing this way, we see that A must be a diagonal matrix. Finally, since A also commutes with all permutation matrices, the entries must all be the equal, so the statement follows. □

In particular, all scalar matrices of determinant 1 are contained in $\mathrm{ZSL}(n, \mathbb{K})$. In fact, the reverse inclusion holds as well:

$$\mathrm{ZSL}(n, \mathbb{K}) = \{x1 \mid x \in \mathbb{K}^\times,\ x^n = 1\}. \tag{6.1.1}$$

To see this, we define the *elementary matrices* $E_{ij}(x)$ for $i, j \in \{1, \dots, n\}$ and $x \in \mathbb{K}$ by letting its entries be the same as those of the identity matrix, and letting the entry at place (i, j) be x. Then $E_{ij}(x) \in \mathrm{SL}(n, \mathbb{K})$ for all distinct i, j. Note now that left multiplication by $E_{ij}(x)$ alters the i'th row by adding x times the j'th row, and that right multiplication by $E_{ij}(x)$ alters the j'th column by adding x times the i'th column. Thus if $A = (a_{ij})_{i,j=1}^n \in \mathrm{SL}(n, \mathbb{K})$ commutes with $E_{ij}(1)$ where $i, j \in \{1, \dots, n\}$ are distinct, then

- in place (i, i) we have the equality $a_{ii} + a_{ji} = a_{ii}$, so that $a_{ji} = 0$, and
- in place (i, j) we have $a_{ij} + a_{jj} = a_{ij} + a_{ii}$ and $a_{jj} = a_{ii}$.

This yields (6.1.1).

In the following discussion, we will say that non-zero vectors $x, y \in \mathbb{K}^n$ are equivalent (and write $x \sim y$) if there exists $\lambda \in \mathbb{K}$ such that $x = \lambda y$. We then define the *projective space* $\mathbb{P}^{n-1}(\mathbb{K})$ to be the set of equivalence classes of $\mathbb{K}^n \setminus \{0\}$ by the equivalence relation $\sim$. Once a basis $(x_i)_{i=1}^n$ of $\mathbb{K}^n$ is chosen, the image of $x = \sum_{i=1}^n \alpha_i x_i \in \mathbb{K}^n \setminus \{0\}$ under the quotient map $\mathbb{K}^n \setminus \{0\} \to \mathbb{P}^{n-1}(\mathbb{K})$ will be denoted by $[\alpha_1 : \cdots : \alpha_n]$, the so-called *homogeneous coordinates*. The projective spaces $\mathbb{P}^{n-1}(\mathbb{R})$ resp. $\mathbb{P}^{n-1}(\mathbb{C})$ are called *real* resp. *complex projective space*.

We define a canonical action of $\mathrm{PSL}(n,\mathbb{K})$ on $\mathbb{P}^{n-1}(\mathbb{K})$ in the obvious way: if $x \in \mathbb{P}^{n-1}(\mathbb{K})$ is a one-dimensional subspace of $\mathbb{K}^n$ and $s \in \mathrm{PSL}(n,\mathbb{K})$, we let sx denote the representative of the subspace Ax, where $A \in \mathrm{SL}(n,\mathbb{K})$ is a representative of s. It is then easy to see that this action is well-defined.

Lemma 6.1.2. *The action of $G = \mathrm{PSL}(n,\mathbb{K})$ on $\Omega = \mathbb{P}^{n-1}(\mathbb{K})$ is faithful.*

Proof. Assume that $A \in \mathrm{SL}(n,\mathbb{K})$ stabilizes the subspaces $\mathbb{K}e_i$ for $i = 1, \dots, n$, where $e_1, \dots, e_n$ denote the canonical basis vectors of $\mathbb{K}^n$. Then $Ae_i = \lambda_i e_i$ for some $\lambda_i \in \mathbb{K}^\times$ for all $i = 1, \dots, n$, so that $A = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$. Since

$$\lambda_i e_i + \lambda_j e_j = A(e_i + e_j) \in \mathbb{K}(e_i + e_j),$$

we must have $\lambda_i = \lambda_j$ for all distinct $i, j = 1, \dots, n$, so $A = \lambda 1$ for some $\lambda \in \mathbb{K}^\times$. It follows that A has image 1 in $\mathrm{PSL}(n,\mathbb{K})$ under the quotient map. □

For our later study of the projective special linear group, we will need a stronger variant of transitivity of the canonical action.

Definition 6.1.3. Let G be a group acting on a set X. We say that the action of G of X is *doubly transitive* if it holds for all $x_1, x_2, y_1, y_2 \in X$ with $x_1 \neq x_2$ and $y_1 \neq y_2$ that there exists $s \in G$ such that $sx_1 = y_1$ and $sx_2 = y_2$.

For the rest of this section, let $G = \mathrm{PSL}(n,\mathbb{K})$ and $\Omega = \mathbb{P}^{n-1}(\mathbb{K})$.

Lemma 6.1.4. *The canonical action of G on Ω is doubly transitive.*

Proof. If $x_1, x_2 \in \Omega$ are distinct elements, take non-zero vectors $y_1 \in x_1$ and $y_2 \in x_2$. Then y_1 and y_2 are linearly independent and the subset $\{y_1, y_2\}$ extends to a basis for $\mathbb{K}^n$. Letting $e_1, \dots, e_n$ be the canonical vector basis of $\mathbb{K}^n$, we thus obtain a matrix $A \in \mathrm{GL}(n,\mathbb{K})$ such that $Ae_1 = y_1$ and $Ae_2 = y_2$. If $\lambda = \det(A)$, then by defining

$$B = A\,\mathrm{diag}(\lambda^{-1}, 1, \dots, 1) \in \mathrm{SL}(n,\mathbb{K}),$$

we see that $B(\mathbb{K}e_i) = A(\mathbb{K}e_i) = \mathbb{K}y_i = x_i$ for $i = 1, 2$. This yields the wanted result. □

A nice consequence is the following:

Corollary 6.1.5. *Distinct points in Ω have distinct stabilizers in G.*

Proof. If $x, y, z \in \Omega$ are distinct, then double transitivity of the G-action on Ω yields $s \in G$ such that $sx = x$ and $sy = z \neq y$. Hence $\mathrm{Stab}(x) \neq \mathrm{Stab}(y)$. □

Finally, let $\mathbb{K}$ be either of the fields $\mathbb{R}$ or $\mathbb{C}$. The standard topology of G is then obtained by giving $\mathrm{SL}(n,\mathbb{K})$ the usual Euclidean topology as a subspace of $M_n(\mathbb{K})$ and endowing G with the quotient topology. We now claim that the canonical action of G on Ω is continuous in the sense that the map $(s, x) \mapsto sx$ of $G \times \Omega$ into Ω is continuous. Indeed, let $\pi\colon \mathrm{SL}(n,\mathbb{K}) \to G$ and $j\colon \mathbb{K}^n \setminus \{0\} \to \Omega$ be the canonical quotient maps and consider the diagram

$$\begin{array}{ccccc} \mathrm{SL}(n,\mathbb{K}) \times \mathbb{K}^n \setminus \{0\} & \xrightarrow{g} & \mathbb{K}^n \setminus \{0\} & \xrightarrow{j} & \Omega \\ \Big\downarrow{\scriptstyle \pi \times j} & & & & \\ G \times \Omega & & & & \end{array}$$

where g maps (A, x) to Ax. As $\mathbb{K}^n \setminus \{0\}$ and G are locally compact, it follows from the Whitehead theorem [49, 29.11] that $\pi \times j$ is a quotient map (it is in fact open, as both π and j are open). The continuous map $j \circ g$ then induces a continuous map $G \times \Omega \to \Omega$ which is in fact the action of G on Ω.

6.2 The Zariski topology on PSL$(n, \mathbb{K})$

In order to fully understand the main result of the next section, we need to consider a topology usually defined on what is known as *affine space*, usually described as "vector spaces where the origin is forgotten". For our purposes, we will not be as general and will just consider vector spaces of the form $\mathbb{K}^k$ for $k \geq 1$.

For any $k \geq 1$ and any subset of k-variable polynomials $S \subseteq \mathbb{K}[x_1, \ldots, x_k]$, the set

$$\mathcal{V}(S) = \{\alpha \in \mathbb{K}^k \mid f(\alpha) = 0 \text{ for all } f \in S\}$$

is called the *zero set* of S. Subsets of $\mathbb{K}^k$ of this form are called *algebraic sets*. It can be easily verified that

(i) $\mathbb{K}^k = \mathcal{V}(\{0\})$ and $\emptyset = \mathcal{V}(\{1\})$;
(ii) if $S_1 \subseteq S_2 \subseteq \mathbb{K}[x_1, \ldots, x_k]$, then $\mathcal{V}(S_2) \subseteq \mathcal{V}(S_1)$;
(iii) for any family $(S_i)_{i \in I}$ of subsets of $\mathbb{K}[x_1, \ldots, x_k]$, then $\bigcap_{i \in I} \mathcal{V}(S_i) = \mathcal{V}(\bigcup_{i \in I} S_i)$, and
(iv) for any subsets $S_1, S_2 \subseteq \mathbb{K}[x_1, \ldots, x_k]$, we have $\mathcal{V}(S_1) \cup \mathcal{V}(S_2) = \mathcal{V}(S_1 S_2)$.

The *Zariski topology* on $\mathbb{K}^k$ can then be defined in terms of algebraic sets, simply by saying that a subset $V \subseteq \mathbb{K}^k$ is *Zariski-open* if its complement is an algebraic set.

If we now identify the vector space of matrices $M_n(\mathbb{K})$ with $\mathbb{K}^{n^2}$, note that GL$(n, \mathbb{K})$ is Zariski-open and that SL$(n, \mathbb{K})$ is Zariski-closed in $M_n(\mathbb{K})$, since the determinant is an n^2-variable polynomial.

Definition 6.2.1. The *Zariski topology* on PSL$(n, \mathbb{K})$ is the quotient topology on PSL$(n, \mathbb{K})$ induced by the Zariski topology on SL$(n, \mathbb{K})$.

To put it lightly, the Zariski topology is quite weak, and we will illuminate this fact straight away by considering a particular kind of Zariski-closed subset of $\mathbb{K}^k$.

Definition 6.2.2. Let $k \geq 1$. A subset X of $\mathbb{K}^k$ is said to be *reducible* if there exist non-empty proper (relatively) Zariski-closed subsets X_1, X_2 of X such that $X = X_1 \cup X_2$. Otherwise, X is said to be *irreducible.*

Remark 6.2.3. It is easily checked that a subset X of $\mathbb{K}^k$ is irreducible if and only if it holds for all non-empty Zariski-closed subsets $X_1, X_2 \subseteq X$ such that $X \subseteq X_1 \cup X_2$ that either $X \subseteq X_1$ or $X \subseteq X_2$. ✺

Proposition 6.2.4. *Let $k \geq 1$ and let $X \subseteq \mathbb{K}^k$ be Zariski-closed. Then the following are equivalent:*

(i) *X is irreducible.*
(ii) *Any two non-empty Zariski-open subsets of X have non-empty intersection.*
(iii) *Any non-empty Zariski-open subset $A \subseteq X$ is Zariski-dense in X.*

Proof. To see that (i) implies (ii), let $A_1, A_2 \subseteq X$ be non-empty Zariski-open subsets and define $X_i = X \setminus A_i$ for $i = 1, 2$. If it were true that A_1 and A_2 had empty intersection, then the assumption that X is irreducible would yield that $X \subseteq X_i$ for some i, contradicting that A_i is non-empty. Hence (ii) follows. Conversely, if $X = X_1 \cup X_2$ for non-empty proper Zariski-closed subsets $X_1, X_2 \subseteq X$, then the non-empty Zariski-open subsets $X \setminus X_1$ and $X \setminus X_2$ of X have empty intersection. Hence (i) and (ii) are equivalent. It is clear that (ii) and (iii) are equivalent. □

Remark 6.2.5. Let us show that $\mathbb{K}^k$ itself is irreducible. We need the following lemma:

Lemma 6.2.6. *For all $k \geq 1$, then if $f \in \mathbb{K}[x_1, \ldots, x_k]$ vanishes on all of $\mathbb{K}^k$, f must be the zero polynomial.*

Proof. For the case $k = 1$, it is well-known that a non-zero polynomial $f \in \mathbb{K}[x]$ has only finitely many zeros; indeed, the exact number does not exceed the degree of f and $\mathbb{K}$ is infinite. Assume that it holds for $k = n - 1$ for some $n > 1$, and let $f \in \mathbb{K}[x_1, \ldots, x_n]$ be a non-zero polynomial. We can write $f = a_r x_n^r + \ldots + a_1 x_n + a_0$ for some $r \geq 0$, where $a_0, \ldots, a_r \in \mathbb{K}[x_1, \ldots, x_{n-1}]$ and $a_r \neq 0$. By our hypothesis, there exists $\alpha \in \mathbb{K}^{n-1}$ such that $a_r(\alpha) \neq 0$. If we consider the polynomial $g \in \mathbb{K}[x]$ given by $g(x) = f(\alpha, x)$, then g has only finitely many zeroes in $\mathbb{K}$. As $\mathbb{K}$ is infinite, there exists $\beta \in \mathbb{K}$ such that $g(\beta) \neq 0$, so f does not vanish on $(\alpha, \beta) \in \mathbb{K}^n$. This completes the proof. ✺

Now let $A_1, A_2 \subseteq \mathbb{K}^k$ be non-empty Zariski-open subsets and write $\mathbb{K}^k \setminus A_i = \mathcal{V}(S_i)$ for some $S_i \subseteq \mathbb{K}[x_1, \dots, x_k]$, where $i = 1, 2$. Then $\mathbb{K}^k \setminus (A_1 \cap A_2) = \mathcal{V}(S_1 S_2)$. If it were true that $\mathcal{V}(S_1 S_2) = \mathbb{K}^k$, we would have $S_1 S_2 = \{0\}$ by the above lemma. Therefore either $S_1 = \{0\}$ or $S_2 = \{0\}$, implying that $\mathcal{V}(S_i) = \mathbb{K}^k$ for some $i = 1, 2$, which contradicts the assumption that both A_1 and A_2 are non-empty. Hence $A_1 \cap A_2$ is non-empty. ✺

We have the following result, essential for purposes soon to be realized:

Proposition 6.2.7. *The special linear group* SL$(n, \mathbb{R})$ *is irreducible in* $\mathbb{R}^{n^2}$.

This is quite a fact, and it can proved as follows. For all subsets $X \subseteq \mathbb{K}^k$, we define the *ideal of* X by

$$\mathfrak{I}(X) = \{f \in \mathbb{K}[x_1, \dots, x_k] \mid f(\alpha) = 0 \text{ for all } \alpha \in X\}.$$

It is easy to check that $X_1 \subseteq X_2 \subseteq \mathbb{K}^k$ implies $\mathfrak{I}(X_2) \subseteq \mathfrak{I}(X_1)$. Moreover,

$$S \subseteq \mathfrak{I}(\mathcal{V}(S)), \quad X \subseteq \mathcal{V}(\mathfrak{I}(X)), \quad \mathcal{V}(S) = \mathcal{V}(\mathfrak{I}(\mathcal{V}(S))) \quad \text{and} \quad \mathfrak{I}(X) = \mathfrak{I}(\mathcal{V}(\mathfrak{I}(X)))$$

for all subsets $S \subseteq \mathbb{K}[x_1, \dots, x_k]$ and $X \subseteq \mathbb{K}^k$. We then have the following easy, but essential result to the topic of algebraic geometry:

Proposition 6.2.8. *A non-empty Zariski-closed subset* $X \subseteq \mathbb{K}^k$ *is irreducible if and only if* $\mathfrak{I}(X)$ *is a prime ideal of* $\mathbb{K}[x_1, \dots, x_k]$.

Proof. If $f_1, f_2 \notin \mathfrak{I}(X)$, then $X_1 = X \setminus \mathcal{V}(f_1)$ and $X_2 = X \setminus \mathcal{V}(f_2)$ are non-empty and Zariski-open in X. If X is irreducible, then $X \setminus \mathcal{V}(f_1 f_2) \neq \emptyset$, so that $f_1 f_2 \notin \mathfrak{I}(X)$. Hence $\mathfrak{I}(X)$ is prime. Conversely, if X is reducible and $X_1, X_2 \subseteq X$ are non-empty proper Zariski-closed subsets with $X = X_1 \cup X_2$, write $X_i = \mathcal{V}(S_i)$ for subsets $S_i \subseteq \mathbb{K}[x_1, \dots, x_k]$, $i = 1, 2$. Then for $i = 1, 2$ there must exist $f_i \in S_i$ such that f_i vanishes on all of X_i, but not on all of X. Therefore $f_1 \notin \mathfrak{I}(X)$ and $f_2 \notin \mathfrak{I}(X)$, but clearly $f_1 f_2 \in \mathfrak{I}(X)$, so $\mathfrak{I}(X)$ is not prime. ☐

We now claim that the polynomial $f = \det -1$ on $R = \mathbb{R}[x_{11}, \dots, x_{nn}]$ is irreducible; indeed, if we write f as a product $f = gh$ for $g, h \in R$, then x_{11} must occur in one of the factors, say g. Then for all $j = 2, \dots, n$, x_{j1} and x_{1j} also occurs in g, since no monomial summand of f contains a product of the form $x_{1j} x_{1k}$ or $x_{j1} x_{k1}$ for $j \neq k$. This finally implies that all x_{jk} occur in g for all $j, k = 1, \dots, n$, so h has to be a constant.

Let $k = n^2$ in the following. To prove that SL$(n, \mathbb{R})$ is irreducible, the above proposition requires us to show that $\mathfrak{I}(\mathrm{SL}(n, \mathbb{R})) = \mathfrak{I}(\mathcal{V}(f))$ is prime. Note that f is a smooth function on $\mathbb{R}^k$ and that for all $i, j = 1, \dots, n$, we have

$$\frac{\partial f}{\partial x_{ij}}(x_{11}, \dots, x_{nn}) = \sum_{\sigma \in F_{ij}} \left[\mathrm{sign}(\sigma) \prod_{m \neq i} x_{m\sigma(m)} \right],$$

where F_{ij} is the set of permutations of $A = \{1, \dots, n\}$ such that $\sigma(i) = j$. If I_n denotes the n-by-n identity matrix, then clearly

$$f(I_n) = 0 \quad \text{and} \quad \frac{\partial f}{\partial x_{ii}}(I_n) = 1$$

for all $i = 1, \dots, n$. By [13, Theorem 4.5.1], this is enough to ensure that $\mathfrak{I}(\mathcal{V}(f))$ in fact equals the principal ideal generated by f; the cited theorem is a consequence of the so-called *real Nullstellensatz*. Since f is irreducible, the principal ideal (f) is necessarily prime, so by Proposition 6.2.8 ensures that SL$(n, \mathbb{R})$ is an irreducible, Zariski-closed subset of $M_n(\mathbb{R})$.

Note further that we also have that SL$(n, \mathbb{C})$ is irreducible in $\mathbb{C}^{n^2}$ (it is a direct consequence of Hilbert's Nullstellensatz). On that account, for $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$ the special linear group SL$(n, \mathbb{K})$ satisfies the three properties of Proposition 6.2.4, and these properties carry over to PSL$(n, \mathbb{K})$ by definition.

We now briefly discuss the notion of lattices in locally compact groups. Recall the following definition:

Definition 6.2.9. Let G be a locally compact group. A discrete subgroup Γ of G is called a *lattice* if G/Γ has a finite G-invariant measure.

We will need the classical result that $\mathrm{SL}(n,\mathbb{Z})$ is a lattice in the connected simple Lie group $\mathrm{SL}(n,\mathbb{R})$ (cf. [48, Theorem 7.1]). Related to the Zariski topology is the following well-known (but extremely non-trivial) theorem:

Theorem 6.2.10 (The Borel density theorem, 1960)**.** *Let G be a connected semi-simple linear real Lie group and let $\Gamma \subseteq G$ be a lattice. Then Γ is Zariski-dense in G.*

Proof. See [48, Corollary 4.47] or [75, Theorem 3.2.5]. □

Hence we have the following result essential for the next section:

Corollary 6.2.11. *The discrete subgroup $\mathrm{SL}(n,\mathbb{Z})$ is Zariski-dense in $\mathrm{SL}(n,\mathbb{R})$. Consequently, the projective special linear group $\mathrm{PSL}(n,\mathbb{Z})$ is Zariski-dense in $\mathrm{PSL}(n,\mathbb{R})$.*

6.3 The result of Bekka, Cowling and de la Harpe

Keeping the previous two sections in the back of our heads, we now focus on the case $\mathbb{K} = \mathbb{R}$. We want to prove the following theorem:

Theorem 6.3.1. *Let Γ be a subgroup of $\mathrm{PSL}(n,\mathbb{R})$ containing $\mathrm{PSL}(n,\mathbb{Z})$, equipped with the discrete topology. Then $C^*_r(\Gamma)$ is simple and has unique trace.*

The proof is in three parts, the first of which concerns another combinatorial property for groups.

Definition 6.3.2. Let Γ be a group. We say that Γ has *property* ($\mathrm{P_{com}}$) if for any finite subset F of $\Gamma \setminus \{1\}$ there exist $s_0 \in \Gamma$ and subsets $C, D_1, \dots, D_n$ of Γ such that

(i) $\Gamma \setminus C \subseteq \bigcup_{i=1}^n D_i$,
(ii) $sC \cap C = \emptyset$ for all $s \in F$, and
(iii) $s_0^{-j} D_i \cap D_i = \emptyset$ for all integers $j \geq 1$ and $i = 1, \dots, n$.

Remark 6.3.3. Note that property ($\mathrm{P_{com}}$) resembles the Powers property quite a bit – in some sense, it is an "infinite" version of the Powers property, albeit with some variations. Pierre de la Harpe proved in [31] that for $n \in \{2,3\}$ (see the note p. 253), all subgroups of $\mathrm{PSL}(n,\mathbb{R})$ or $\mathrm{PSL}(n,\mathbb{C})$ containing a lattice Γ are in fact Powers groups, by considering their actions on the so-called *flag manifolds.* ✺

Recall that if $A \in \mathrm{SL}(n,\mathbb{R})$ and $\alpha \in \mathbb{R}^\times$, then $\lambda \in \mathbb{R}$ is an eigenvalue of A if and only if $\alpha\lambda$ is an eigenvalue of αA. If so, the eigenspace of αA associated to $\alpha\lambda$ is the eigenspace of A associated to λ. Hence if we were to consider A as an element of G, the eigenvalues of A might not be well-defined, but the eigenspaces are. Therefore we can (and will) say that $x \in \mathbb{R}^n \setminus \{0\}$ is an eigenvector of $s \in G$ if and only if there exists a representative of s in $\mathrm{SL}(n,\mathbb{R})$ that has x as an eigenvector. If the eigenspace containing x is one-dimensional, we say that the image of $x \in \mathbb{R}^n$ under the quotient map $\mathbb{R}^n \setminus \{0\} \to \Omega$ is an *eigenline* of s. Concerning the problem of non-well-defined eigenvalues, we do know that eigenvalues of representatives of elements in G are well-defined up to sign.

Remark 6.3.4. There exists an element in $\mathrm{SL}(n,\mathbb{Z})$ with n distinct positive eigenvalues. Indeed, if n is even, write $n = 2m$ and define a polynomial

$$P_n(x) = \prod_{i=1}^m (x^2 - (i+2)x + 1).$$

If n is odd, write $n = 2m+1$ and define $P_n(x) = (x-1)P_{2m}(x)$. Recall that the companion matrix $C(P)$ of the polynomial

$$P(x) = x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$$

is given by

$$C(P) = \begin{pmatrix} 0 & 0 & \cdots & 0 & -c_0 \\ 1 & 0 & \cdots & 0 & -c_1 \\ 0 & 1 & \cdots & 0 & -c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -c_{n-1} \end{pmatrix}$$

and has the property that the characteristic polynomial of $C(P)$ is P. Hence $C(P_n)$ is an integer matrix of determinant 1, with n distinct positive eigenvalues. ✺

Proposition 6.3.5. *Let Γ be a subgroup of $G = \mathrm{PSL}(n,\mathbb{R})$ containing $\mathrm{PSL}(n,\mathbb{Z})$. Then Γ has property* $(\mathrm{P_{com}})$.

Proof. We consider the canonical action of Γ on the homogeneous compact Hausdorff space $\Omega = \mathbb{P}^{n-1}(\mathbb{R})$ for G, recalling that Ω is topologized by means of the quotient topology. First and foremost, let $j\colon \mathbb{R}^n \setminus \{0\} \to \Omega$ and $\pi\colon \mathrm{SL}(n,\mathbb{R}) \to G$ denote the canonical quotient maps. Let $F \subseteq \Gamma \setminus \{1\}$ be a finite subset. By the above remark, there exists an $a' \in \Gamma$ with a representative in $\mathrm{SL}(n,\mathbb{Z})$ having n distinct positive eigenvalues $\lambda_1 > \ldots > \lambda_n > 0$, with corresponding eigenlines $y_1', \ldots, y_n'$. We first show that there exists a conjugate a of a' with n eigenlines $y_1, \ldots, y_n$ such that

$$s\{y_1,\ldots,y_n\} \cap \{y_1,\ldots,y_n\} = \emptyset$$

for all $s \in F$. For all $i,j = 1,\ldots,n$ and $s \in F$, define

$$G_{s,i,j} = \{t \in G \,|\, tst^{-1}y_i' \neq y_j'\}.$$

Claim 1. *Each $G_{s,i,j}$ is a non-empty Zariski-open subset of G.*

Proof: Let $A \in \mathrm{SL}(n,\mathbb{R})$ such that $s = \pi(A)$ and consider the subset

$$\Lambda = \{B \in \mathrm{SL}(n,\mathbb{R}) \,|\, \pi(BAB^{-1})y_i' = y_j'\} = \{B \in \mathrm{SL}(n,\mathbb{R}) \,|\, BAB^{-1}x_i \in \mathbb{R}^\times x_j\} \subseteq \mathrm{SL}(n,\mathbb{R}),$$

where $x_i, x_j \in \mathbb{R}^n \setminus \{0\}$ satisfy $j(x_i) = y_i'$ and $j(x_j) = y_j'$. Once we show that Λ is Zariski-closed, we have then shown that $G_{s,i,j}$ is Zariski-open.

We consider the *second exterior power* $W = \wedge^2(\mathbb{R}^n)$ of $\mathbb{R}^n$. As a vector space, W can be viewed as the tensor product space $\mathbb{R}^n \odot \mathbb{R}^n$ modulo repetition, i.e., under the quotient map $\mathbb{R}^n \odot \mathbb{R}^n \to W$ the image $x \wedge y$ of an elementary tensor $x \otimes y$ is zero if and only if x and y are linearly dependent. The vector $x \wedge y$ is called a *wedge product*. It is well-known that W has dimension $\binom{n}{2}$ over $\mathbb{R}$. Now, for all $B \in \mathrm{SL}(n,\mathbb{R})$, note that the entries of B^{-1} can be expressed solely by determinants of entries of B, as it is the adjugate matrix of B. Hence the coordinates of $BAB^{-1}x_i$ consist of polynomials in the entries of B. We then define a map $f\colon \mathrm{SL}(n,\mathbb{R}) \to W$ by

$$f(B) = BAB^{-1}x_i \wedge x_j.$$

Since $BAB^{-1}x_i \neq 0$ for all $B \in \mathrm{SL}(n,\mathbb{R})$, we have $f(B) = 0$ if and only if $B \in \Lambda$. Under the isomorphism

$$W \cong \mathbb{R}^{\binom{n}{2}},$$

then the wedge product $x \wedge y \in W$ has coordinates $x_p y_q - x_q y_p$ for $1 \le p < q \le n$ in the standard basis, and these are homogeneous polynomials in the coordinates of x and y. Hence it follows that Λ is the zero set of $\binom{n}{2}$ real n^2-variable polynomials, so that Λ is a Zariski-closed subset of $\mathrm{SL}(n,\mathbb{R})$.

Suppose now that $G_{s,i,j}$ is empty. Then $sy_i' = y_j'$, so that $sty_i' = ty_j' = tsy_i'$ for all $t \in G$. If $t \in G$ stabilizes y_i', then $ty_j' = tsy_i' = sty_i' = sy_i' = y_j'$, so that t also stabilizes y_j' and if $t \in G$ stabilizes y_j', then $ty_i' = s^{-1}(sty_i') = s^{-1}(tsy_i') = s^{-1}y_j' = y_i'$. Hence y_i' and y_j' have the same stabilizer, so they must be equal by Corollary 6.1.5. Therefore $s(ty_i') = ty_i'$ for all $t \in G$, so s fixes every point of Ω. By faithfulness, we obtain $s = 1$, a contradiction. ♠

The finite intersection G' of all the subsets $G_{s,i,j}$ is a non-empty Zariski-open subset of G. By Zariski density of Γ in G (cf. Corollary 6.2.11), Γ contains an element $t_0 \in G'$. Defining $a = t_0^{-1}a't_0 \in \Gamma$, then a has eigenlines $y_i = t_0^{-1}y_i'$ for $i = 1,\ldots,n$ and for all $s \in F$ and $i,j = 1,\ldots,n$ we have

$$sy_i = st_0^{-1}y_i' \neq t_0^{-1}y_j' = y_j,$$

as wanted. Moreover, we can choose $A \in \mathrm{SL}(n,\mathbb{R})$ such that A has eigenvalues $\lambda_1 > \ldots > \lambda_n > 0$ and $\pi(A) = a$.

Choosing non-zero vectors x_i in the eigenlines y_i for all $i = 1,\ldots,n$ yields a basis $x_1,\ldots,x_n$ of $\mathbb{R}^n$ with respect to which we can express each y_i by the homogeneous coordinates

$$y_i = [0 : \cdots : 0 : 1 : 0 : \cdots : 0],$$

the 1 being in the i'th place for all $i = 1, \dots, n$. For all $i = 1, \dots, n$ and $\varepsilon > 0$, we now define

$$V_{i,\varepsilon} = \left\{ [\alpha_1 : \cdots : \alpha_n] \;\middle|\; \alpha_i \neq 0 \text{ and } \left|\frac{\alpha_j}{\alpha_i}\right| < \varepsilon \text{ whenever } j \neq i \right\} \subseteq \Omega$$

and $V_\varepsilon = \bigcup_{i=1}^n V_{i,\varepsilon}$. It is clear that each $V_{i,\varepsilon}$ is an open subset of Ω: if $B\colon \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ is the homeomorphism given by

$$B(\alpha_1, \dots, \alpha_n) = \sum_{i=1}^n \alpha_i x_i,$$

then $B^{-1}(\pi^{-1}(V_{i,\varepsilon}))$ is open in $\mathbb{R}^n \setminus \{0\}$. Note also that $V_{i,\varepsilon} \subseteq V_{i,\varepsilon'}$ and $V_\varepsilon \subseteq V_{\varepsilon'}$ for all $0 < \varepsilon < \varepsilon'$.

We now establish three claims, paving the way for a smashing conclusion.

Claim 2. *For all $\varepsilon > 0$, we have $\Omega = \bigcup_{k=0}^\infty a^{-k} V_\varepsilon$.*

Proof: Let $x \in \mathbb{R}^n \setminus \{0\}$ and write $x = \sum_{i=1}^n \alpha_i x_i$. For all integers $k \geq 0$, we now have

$$A^k x = \sum_{i=1}^n \alpha_i \lambda_i^k x_i.$$

Let i_0 be the smallest $i = 1, \dots, n$ such that $\alpha_i \neq 0$. Then for all $j < i_0$ we have

$$\left| \frac{\alpha_j}{\alpha_{i_0}} \frac{\lambda_j^k}{\lambda_{i_0}^k} \right| = 0,$$

while we have for all $j > i_0$ that

$$\left| \frac{\alpha_j}{\alpha_{i_0}} \frac{\lambda_j^k}{\lambda_{i_0}^k} \right| \to 0$$

for $k \to \infty$. Thus there exists $k \geq 0$ such that $a^k j(x) \in V_{i_0,\varepsilon} \subseteq V_\varepsilon$. ♠

Claim 3. *There exists $\varepsilon > 0$ such that $sV_{i,\varepsilon} \cap V_{j,\varepsilon} = \emptyset$ for all $s \in F$ and $i, j = 1, \dots, n$.*

Proof: Given $i \in \{1, \dots, n\}$, define a continuous map $f\colon \mathbb{R}^{n-1} \to \mathbb{R}^n \setminus \{0\}$ by

$$f(x_1, \dots, x_{n-1}) = (x_1, \dots, x_{i-1}, 1, x_i, \dots, x_{n-1}).$$

If $U \subseteq \Omega$ is an open neighbourhood of y_i, then there exists $\varepsilon > 0$ such that

$$(-\varepsilon, \varepsilon)^{n-1} \subseteq f^{-1}(B^{-1}(\pi^{-1}(U))),$$

and it follows that $V_{i,\varepsilon} \subseteq U$.

For any $s \in F$ and $i, j \in \{1, \dots, n\}$, then since $sy_i \neq y_j$, there exist disjoint open neighbourhoods $U_{s,i,j}$ of sy_i and $V_{s,i,j}$ of y_j. By virtue of what we saw above, there exists $\varepsilon_{s,i,j} > 0$ such that $V_{i,\varepsilon_{s,i,j}} \subseteq s^{-1}U_{s,i,j}$ and $V_{j,\varepsilon_{s,i,j}} \subseteq V_{s,i,j}$. We now get our $\varepsilon > 0$ by taking the smallest of all the $\varepsilon_{s,i,j}$. ♠

Take $\varepsilon > 0$ such that Claim 3 is satisfied, and define $V = V_\varepsilon$ and $V_i = V_{i,\varepsilon}$ for all $i = 1, \dots, n$. Claim 3 then implies that $sV \cap V = \emptyset$ for all $s \in F$.

Claim 4. *For all $i = 1, \dots, n$ and integers $j \geq 1$, if $y \in V_i$ and $a^j y \notin V_i$, then $a^{j+1} y \notin V_i$.*

Proof: Letting $y = [\alpha_1 : \cdots : \alpha_n]$, where $\alpha_1, \dots, \alpha_n \in \mathbb{R}$, assume that $\alpha_i \neq 0$ and that $|\alpha_k| < \varepsilon |\alpha_i|$ for all $k \neq i$. Then $a^j y = [\lambda_1^j \alpha_1 : \cdots : \lambda_n^j \alpha_n]$. If $a^j y \notin V_i$, then there exists $\ell \neq i$ such that

$$\left| \frac{\lambda_\ell^j \alpha_\ell}{\lambda_i^j \alpha_i} \right| \geq \varepsilon.$$

This implies $|\lambda_i^j| < |\lambda_\ell^j|$, so that $\lambda_i < \lambda_\ell$ and $\ell < i$. Therefore

$$\left| \frac{\lambda_\ell^{j+1} \alpha_\ell}{\lambda_i^{j+1} \alpha_i} \right| = \left| \frac{\lambda_\ell}{\lambda_i} \right| \left| \frac{\lambda_\ell^j \alpha_\ell}{\lambda_i^j \alpha_i} \right| > \varepsilon,$$

so that $a^{j+1}y \notin V_i$. ♠

For all $i = 1, \dots, n$ and integers $j \geq 1$, define

$$\Omega_{i,j} = a^{-j}V_i \setminus \bigcup_{k=0}^{j-1} a^{-k}V_i.$$

Note first that $\Omega_{i,j} \cap \Omega_{i,k} = \emptyset$ whenever $j \neq k$. It is also clear that $\Omega_{i,j+1} \subseteq a^{-1}\Omega_{i,j}$, and the reverse inclusion follows from Claim 4, so that we have equality. It follows immediately that $a^{-k}\Omega_{i,j} = \Omega_{i,j+k}$ for all integers $k \geq 1$. It is easy to check that

$$\bigcup_{k=1}^{j} \Omega_{i,k} = \left(\bigcup_{k=1}^{j} a^{-k}V_i\right) \setminus V_i.$$

By Claim 2 we have

$$\Omega \setminus V \subseteq \bigcup_{k=1}^{\infty}\bigcup_{i=1}^{n} a^{-k}V_i,$$

so compactness of $\Omega \setminus V$ yields an integer $N \geq 1$ such that

$$\Omega \setminus V \subseteq \bigcup_{i=1}^{n}\left(\bigcup_{k=1}^{N} a^{-k}V_i\right) \setminus V \subseteq \bigcup_{i=1}^{n}\bigcup_{k=1}^{N} \Omega_{i,k}.$$

Now define $s_0 = a^N$ and $\Omega_i = \bigcup_{k=1}^{N} \Omega_{i,k}$ for all $i = 1, \dots, n$. Letting $x_0 \in \Omega$ be some point and defining a map $f\colon \Gamma \to \Omega$ by $f(s) = sx_0$, we then define $C = f^{-1}(V)$ and $D_i = f^{-1}(\Omega_i)$ for all $i = 1, \dots, n$. Then condition (i) of Definition 6.3.2 is satisfied, and condition (ii) follows from Claim 3. Finally, as

$$a^{-jN}\Omega_{i,k} \cap \Omega_{i,\ell} = \Omega_{i,k+jN} \cap \Omega_{i,\ell} = \emptyset$$

for all $i = 1, \dots, n$, integers $j \geq 1$ and $k, \ell = 1, \dots, N$, it follows that Γ has property ($\mathrm{P_{com}}$). □

To relate property ($\mathrm{P_{com}}$) to the terrene of reduced group C^*-algebras, we introduce another property for groups.

Definition 6.3.6. Let Γ be a discrete group. We say that Γ has *property* ($\mathrm{P_{ana}}$) if the following condition holds: For all finite subsets $F \subseteq \Gamma \setminus \{1\}$, there exists $s_0 \in \Gamma$ and a constant $C > 0$ such that

$$\left\| \sum_{n=1}^{\infty} a_n \lambda_\Gamma(s_0^{-n} t s_0^n) \right\| \leq C\|a\|_2$$

for all $a = (a_n)_{n\geq 1} \in C_c(\mathbb{N})$ and $t \in F$, where λ_Γ is the left-regular representation of Γ.

Proposition 6.3.7. *If a discrete group Γ has property ($\mathrm{P_{ana}}$), the reduced group C^*-algebra $C_r^*(\Gamma)$ is simple with unique trace.*

Proof. If σ is a trace on $C_r^*(\Gamma)$, then for all $t \in \Gamma \setminus \{1\}$ there exist $s_0 \in \Gamma$ and $C > 0$ such that

$$\|\sigma(\lambda_\Gamma(t))\| = \left\| \sigma\left(\frac{1}{N}\sum_{n=1}^{N} \lambda_\Gamma(s_0^{-n} t s_0^n)\right)\right\| \leq C\sqrt{\frac{N}{N^2}} = \frac{C}{\sqrt{N}}$$

for all integers $N \geq 1$. Hence $\sigma(\lambda_\Gamma(t)) = 0$ for all $t \in \Gamma \setminus \{1\}$, so by linearity and continuity we conclude that σ must be the canonical faithful trace on $C_r^*(\Gamma)$.

Note now that δ_1 is a cyclic vector of the unitary representation λ_Γ. Assume that $\rho\colon \Gamma \to \mathcal{U}(\mathcal{H})$ is a unitary representation of Γ satisfying $\rho \prec \lambda_\Gamma$. To prove that $C_r^*(\Gamma)$ is simple, Corollary 1.5.10 requires us to show that $\lambda_\Gamma \prec \rho$. By Theorem 1.5.9 it suffices to show that the function $s \mapsto \langle \lambda_\Gamma(s)\delta_1, \delta_1\rangle$ is the uniform limit of finite sums of positive definite functions associated with ρ, over all finite subsets of Γ. Now, if $F \subseteq \Gamma$ is a finite subset, then by Γ having property ($\mathrm{P_{ana}}$), there exist $s_0 \in \Gamma$ and $C > 0$ such that

$$\left\| \frac{1}{N}\sum_{n=1}^{N} \lambda_\Gamma(s_0^{-n} t s_0^n) \right\| \leq \frac{C}{\sqrt{N}}$$

for all $t \in F \setminus \{1\}$ and $N \geq 1$. Since $\rho \prec \lambda_\Gamma$, it follows that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \rho(s_0^{-n} t s_0^n) = 0$$

for all $t \in F \setminus \{1\}$. Let $\xi \in \mathcal{H}$ be a unit vector and define $\eta_n = \rho(s_0^n)\xi$ for all $n \geq 1$. Then for all $t \in F \cup \{1\}$ we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \langle \rho(t)\eta_n, \eta_n\rangle = \langle \lambda_\Gamma(t)\delta_1, \delta_1\rangle.$$

Because this convergence is pointwise, it is uniform over the finite subset $F \subseteq \Gamma$. This proves that $\lambda_\Gamma \prec \rho$, so that $C_r^*(\Gamma)$ is simple. □

The next and final result puts all of the pieces together, yielding Theorem 6.3.1 straight away:

Proposition 6.3.8. *If* Γ *is a group satisfying property* ($\mathrm{P}_{\mathrm{com}}$)*, then* Γ *has property* ($\mathrm{P}_{\mathrm{ana}}$)*.*

Proof. Let $F \subseteq \Gamma \setminus \{1\}$ be a finite subset, so that there exist $s_0 \in \Gamma$ and subsets $C, D_1, \ldots, D_n \subseteq \Gamma$ such that the conditions of Definition 6.3.2 are satisfied. For all $f \in \ell^2(\Gamma)$ and subsets $S \subseteq \Gamma$ we let $\chi_S \in B(\ell^2(\Gamma))$ denote the projection given by

$$(\chi_S f)(s) = \begin{cases} f(s) & \text{if } s \in S \\ 0 & \text{else.} \end{cases}$$

Then it is easy to check that

$$\chi_S \chi_T = \chi_{S\cap T}, \quad \lambda_\Gamma(s)\chi_S = \chi_{sS}\lambda_\Gamma(s), \quad s \in \Gamma,\ S, T \subseteq \Gamma.$$

For all $t \in F$ and $f, g \in \ell^2(\Gamma)$, we now have

$$\begin{aligned}
|\langle \lambda_\Gamma(t) f, g\rangle| &\leq |\langle \lambda_\Gamma(t)\chi_C f, g\rangle| + |\langle \lambda_\Gamma(t)\chi_{\Gamma\setminus C} f, g\rangle| \\
&= |\langle \chi_{tC}\lambda_\Gamma(t) f, \chi_C g + \chi_{\Gamma\setminus C} g\rangle| + |\langle \lambda_\Gamma(t)\chi_{\Gamma\setminus C} f, g\rangle| \\
&= |\langle \chi_{tC}\lambda_\Gamma(t) f, \chi_{\Gamma\setminus C} g\rangle| + |\langle \lambda_\Gamma(t)\chi_{\Gamma\setminus C} f, g\rangle| \\
&\leq \|f\| \left\|\chi_{\Gamma\setminus C} g\right\| + \left\|\chi_{\Gamma\setminus C} f\right\| \|g\| \\
&\leq \sum_{i=1}^{n} \left(\|f\| \|\chi_{D_i} g\| + \|\chi_{D_i} f\| \|g\|\right),
\end{aligned}$$

using conditions (i) and (ii) of Definition 6.3.2. Therefore for all $j \geq 1$, we have

$$\begin{aligned}
|\langle \lambda_\Gamma(s_0^{-j} t s_0^j) f, g\rangle| &\leq \sum_{i=1}^{n} \|\lambda_\Gamma(s_0^j) f\| \left\|\chi_{D_i}\lambda_\Gamma(s_0^j) g\right\| + \left\|\chi_{D_i}\lambda_\Gamma(s_0^j) f\right\| \|\lambda_\Gamma(s_0^j) g\| \\
&= \sum_{i=1}^{n} \left(\|f\| \left\|\chi_{s_0^{-j} D_i} g\right\| + \left\|\chi_{s_0^{-j} D_i} f\right\| \|g\|\right).
\end{aligned}$$

By condition (iii) of Definition 6.3.2, the sets $s_0^{-j} D_i$ and $s_0^{-k} D_i$ are disjoint for $i = 1, \ldots, n$ and distinct $j, k \geq 1$, so that $\sum_{j=1}^{\infty} \chi_{s_0^{-j} D_i} \leq 1$. If we now let $t \in F$, $f, g \in \ell^2(\Gamma)$ and $a = (a_j)_{j\geq 1} \in C_c(\mathbb{N})$, we then have

$$\begin{aligned}
\left|\left\langle \sum_{j=1}^{\infty} a_j \lambda_\Gamma(s_0^{-j} t s_0^j) f, g\right\rangle\right| &\leq \sum_{j=1}^{\infty} |a_j| \left(\sum_{i=1}^{n} \|f\| \left\|\chi_{s_0^{-j} D_i} g\right\| + \left\|\chi_{s_0^{-j} D_i} f\right\| \|g\|\right) \\
&\leq \sum_{i=1}^{n} \left(\|f\| \sum_{j=1}^{\infty} |a_j| \left\|\chi_{s_0^{-j} D_i} g\right\| + \|g\| \sum_{j=1}^{\infty} |a_j| \left\|\chi_{s_0^{-j} D_i} f\right\|\right) \\
&\leq \sum_{i=1}^{n} \|a\|_2 \left(\|f\| \left(\sum_{j=1}^{\infty} \left\|\chi_{s_0^{-j} D_i} g\right\|^2\right)^{1/2} + \|g\| \left(\sum_{j=1}^{\infty} \left\|\chi_{s_0^{-j} D_i} f\right\|\right)^{1/2}\right) \\
&\leq \sum_{i=1}^{n} 2\|a\|_2 \|f\| \|g\| \\
&= 2n\|a\|_2 \|f\| \|g\|.
\end{aligned}$$

By setting $C = 2n$, we see that Γ has property ($\mathrm{P}_{\mathrm{ana}}$). □

Shortly after publishing the above result, Bekka, Cowling and de la Harpe generalized it in [9] to certain subgroups of connected, real, semisimple Lie groups without compact factors.

6.4 The Powers property of some subgroups of $\mathrm{PSL}(2,\mathbb{C})$

To end the thesis in a manner that combines the techniques of this chapter with concepts of the other chapters, let us show the following result:

Theorem 6.4.1. *All subgroups of* $\mathrm{PSL}(2,\mathbb{C})$ *containing* $\mathrm{PSL}(2,\mathbb{Z})$ *are Powers groups.*

As we have seen in Section 6.1, the canonical action of $G = \mathrm{PSL}(2,\mathbb{C})$ on $\Omega = \mathbb{P}^1(\mathbb{C})$ is faithful and transitive.

Definition 6.4.2. We say that an element $a \in G$ is *polar regular* if a has a representative $A \in \mathrm{SL}(2,\mathbb{C})$ with eigenvalues of distinct moduli. If so, then if $\lambda_1, \lambda_2 \in \mathbb{C}$ are the eigenvalues of A satisfying $0 < |\lambda_1| < |\lambda_2|$, with corresponding eigenvectors x_1, x_2, define

$$s_a = j(x_1), \quad r_a = j(x_2).$$

We say that s_a is the *source* and that r_a is the *range* of a.

Assume that $a \in G$ is polar regular with its representative $A \in \mathrm{SL}(2,\mathbb{C})$ having eigenvalues λ_1, λ_2 and eigenvectors x_1, x_2 as above, and let U (resp. V) be a neighbourhood of s_a (resp. r_a) in Ω. Any fixed point of a in Ω is an eigenline for A, so that s_a and r_a are the only fixed points of a. As x_1, x_2 constitute a basis for $\mathbb{C}^2$, we can write $s_a = [1:0]$ and $r_a = [0:1]$ in homogeneous coordinates. Recall also the open sets

$$V_{i,\varepsilon} = \left\{ [\alpha_1 : \alpha_2] \;\middle|\; \alpha_i \neq 0 \text{ and } \left|\frac{\alpha_j}{\alpha_i}\right| < \varepsilon \text{ for } j \neq i \right\} \subseteq \Omega$$

from the proof of Proposition 6.3.5, where $i = 1, 2$ and $\varepsilon > 0$. As in the proof of Claim 3 of that result, there exists $\varepsilon > 0$ such that $V_{1,\varepsilon} \subseteq U$ and $V_{2,\varepsilon} \subseteq V$. For $y \in \Omega \setminus V_{1,\varepsilon}$, write $y = [\alpha_1 : \alpha_2]$ and note that $a^n y = [\lambda_1^n \alpha_1 : \lambda_2^n \alpha_2]$. It must hold that $\alpha_2 \neq 0$. Because $y \notin V_{1,\varepsilon}$ we either have $\alpha_1 = 0$, in which case $a^n y \in V_{2,\varepsilon}$, or $\alpha_1 \neq 0$ and $|{\alpha_2}/{\alpha_1}| \geq \varepsilon$. Taking $N \geq 1$ such that $|{\lambda_1}/{\lambda_2}|^N < \varepsilon^2$ yields

$$\left|\frac{\lambda_1^n \alpha_1}{\lambda_2^n \alpha_2}\right| \leq \frac{1}{\varepsilon}\left|\frac{\lambda_1^N}{\lambda_2^N}\right| < \varepsilon$$

for all $n \geq N$. Consequently $a^n(\Omega \setminus U) \subseteq a^n(\Omega \setminus V_{1,\varepsilon}) \subseteq V_{2,\varepsilon} \subseteq V$, so that:

Proposition 6.4.3. *All polar regular elements of* $\mathrm{PSL}(2,\mathbb{C})$ *are hyperbolic homeomorphisms of* $\mathbb{P}^1(\mathbb{C})$.

If we now let Γ be a subgroup of G containing $\mathrm{PSL}(2,\mathbb{Z})$, then Γ clearly contains two transverse hyperbolic homeomorphisms of Ω: for instance, just consider the matrices

$$\begin{pmatrix} 0 & 1 \\ -1 & 3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ -1 & 4 \end{pmatrix}.$$

Therefore the action of Γ on Ω is strongly hyperbolic in the sense of Definition 3.1.12. Moreover, it is well-known that each non-identity element of G has at most 2 fixed points in Ω, so Theorem 6.4.1 now follows from Corollary 3.1.15.

APPENDIX A

FULL, FINITE AND PROPERLY INFINITE PROJECTIONS

In the following exposition, we will clarify some of the notions and notation used in Chapter 2. The reason that these are relegated to an appendix is simply that they would otherwise disrupt the flow of the stepwise proof of Theorem 2.3. Throughout the next three section, $\mathcal{A}$ will always denote a C^*-algebra and m, n, k will be positive integers.

We define $M_{m,n}(\mathcal{A})$ to be the set of rectangular $m \times n$ matrices with entries in $\mathcal{A}$. The adjoint of $a \in M_{m,n}(\mathcal{A})$ is then the matrix in $M_{n,m}(\mathcal{A})$ obtained by transposing the matrix (i.e., the rows of a become the columns and vice versa) and adjoining all entries, and if $a \in M_{m,n}(\mathcal{A})$ and $b \in M_{n,k}(\mathcal{A})$, then $ab \in M_{m,k}(\mathcal{A})$ denotes the usual matrix product. Note that $M_{n,n}(\mathcal{A})$ is just the matrix algebra $M_n(\mathcal{A})$. The zero matrix of $M_{m,n}(\mathcal{A})$ is denoted by $0_{m,n}$ and we will always write 0_n instead of $0_{n,n}$.

A.1 The semigroup $\mathcal{P}_\infty(\mathcal{A})$

If $a \in M_m(\mathcal{A})$ and $b \in M_n(\mathcal{A})$, we define the block matrix

$$a \oplus b = \begin{pmatrix} a & 0_{m,n} \\ 0_{n,m} & b \end{pmatrix} \in M_{m+n}(\mathcal{A}).$$

It is clear that $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ for all (quadratic) matrices a, b, c with entries in $\mathcal{A}$.

Lemma A.1.1. *If $a \in M_n(\mathcal{A})$ is positive and $x \in M_{k,n}(\mathcal{A})$, then $xax^* \in M_k(\mathcal{A})$ is positive.*

Proof. The result is clear if $k = n$. If $k > n$, let $\varphi_1 \colon M_n(\mathcal{A}) \to M_k(\mathcal{A})$ be the *-homomorphism given by $\varphi_1(y) = y \oplus 0_{k-n}$ and let $x_1 \in M_k(\mathcal{A})$ be given by

$$x_1 = \begin{pmatrix} x & 0_{k,k-n} \end{pmatrix} \in M_k(\mathcal{A}).$$

Then a simple calculation shows that $x_1\varphi(a)x_1^* = xax^*$, and since $\varphi(a) \geq 0$, it follows that xax^* must be positive. If $k < n$, let $\varphi_2 \colon M_k(\mathcal{A}) \to M_n(\mathcal{A})$ be the *-homomorphism given by $\varphi_2(y) = y \oplus 0_{n-k}$ and define

$$x_2 = \begin{pmatrix} x \\ 0_{n-k,n} \end{pmatrix} \in M_n(\mathcal{A}).$$

One can now show that $x_2ax_2^* = \varphi(xax^*)$. Since $a \geq 0$, it follows that $\varphi(xax^*) \geq 0$, and because φ is injective, we have $xax^* \geq 0$. □

We now define

$$\mathcal{P}_n(\mathcal{A}) = \mathcal{P}(M_n(\mathcal{A})), \quad \mathcal{P}_\infty(\mathcal{A}) = \bigcup_{n=1}^{\infty} \mathcal{P}_n(\mathcal{A}),$$

where the sets $\mathcal{P}_n(\mathcal{A})$ are seen as pairwise disjoint inside $\mathcal{P}_\infty(\mathcal{A})$. We define a relation $\sim_0$ on $\mathcal{P}_\infty(\mathcal{A})$ by writing $p \sim_0 q$ for matrices $p \in M_n(\mathcal{A})$ and $q \in M_m(\mathcal{A})$ if there exists a matrix $v \in M_{n,m}(\mathcal{A})$ such that $p = vv^*$ and $q = v^*v$. Note that if $n = m$, then $p \sim_0 q$ simply means that p and q are Murray-von Neumann equivalent. It is easy to see that $\sim_0$ is an equivalence relation, and it satisfies the following properties:

Proposition A.1.2. *Let $\mathcal{A}$ be a C^*-algebra.*

(i) *For all $p \in \mathcal{P}_\infty(\mathcal{A})$ and $n \geq 1$, $p \sim_0 p \oplus 0_n$ where 0_n denotes the zero matrix in $M_n(\mathcal{A})$.*

(ii) *If* $p_1 \sim_0 q_1$ *and* $p_2 \sim_0 q_2$ *for* $p_1, p_2, q_1, q_2 \in \mathcal{P}_\infty(\mathcal{A})$, *then* $p_1 \oplus p_2 \sim_0 q_1 \oplus q_2$.

(iii) *For all* $p, q \in \mathcal{P}_\infty(\mathcal{A})$, $p \oplus q \sim_0 q \oplus p$.

(iv) *If* $p, q \in \mathcal{P}_n(\mathcal{A})$ *with* $pq = 0$, *then* $p + q$ *is a projection and* $p + q \sim_0 p \oplus q$.

Proof. (i) If $p \in \mathcal{P}_m(\mathcal{A})$, then by defining

$$v = \begin{pmatrix} p & 0_{m,n} \end{pmatrix} \in M_{m,m+n}(\mathcal{A}),$$

then $vv^* = p$ and $v^*v = p \oplus 0_n$.

(ii) If $p_i \in \mathcal{P}_{m_i}(\mathcal{A})$ and $q_i \in \mathcal{P}_{n_i}(\mathcal{A})$ for $i = 1, 2$ and $v \in M_{m_1,n_1}(\mathcal{A})$ and $w \in M_{m_2,n_2}(\mathcal{A})$ such that $p_1 = vv^*$, $q_1 = v^*v$, $p_2 = ww^*$ and $q_2 = w^*w$, then by defining

$$u = \begin{pmatrix} v & 0_{m_1,n_2} \\ 0_{m_2,n_1} & w \end{pmatrix} \in M_{m_1+m_2,n_1+n_2}(\mathcal{A}),$$

we have $p_1 \oplus p_2 = uu^*$ and $q_1 \oplus q_2 = u^*u$.

(iii) If $p \in \mathcal{P}_m(\mathcal{A})$ and $q \in \mathcal{P}_n(\mathcal{A})$, then by defining

$$u = \begin{pmatrix} 0_{m,n} & p \\ q & 0_{n,m} \end{pmatrix} \in M_{m+n}(\mathcal{A})$$

we find that $p \oplus q = uu^*$ and $q \oplus p = u^*u$.

(iv) Clearly $p + q$ is a projection and

$$u = \begin{pmatrix} p & q \end{pmatrix} \in M_{n,2n}(\mathcal{A})$$

satisfies $p + q = uu^*$ and $p \oplus q = u^*u$. □

We now define another relation $\precsim$ on $\mathcal{P}_\infty(\mathcal{A})$ by writing $p \precsim q$ for $p \in \mathcal{P}_n(\mathcal{A})$ and $q \in \mathcal{P}_m(\mathcal{A})$ if there exists a projection $q_0 \in \mathcal{P}_m(\mathcal{A})$ such that $p \sim_0 q_0 \leq q$.

Proposition A.1.3. *The relation* $\precsim$ *is transitive and has the following properties:*

(i) *For all* $p, q \in \mathcal{P}_\infty(\mathcal{A})$, $p \precsim q$ *if and only if* $q \sim_0 p \oplus p_0$ *for some projection* $p_0 \in \mathcal{P}_\infty(\mathcal{A})$.

(ii) *If* $p_1 \precsim q_1$ *and* $p_2 \precsim q_2$ *for* $p_1, p_2, q_1, q_2 \in \mathcal{P}_\infty(\mathcal{A})$, *then* $p_1 \oplus p_2 \precsim q_1 \oplus q_2$.

Proof. We first prove (i). If $p \precsim q$, then there exists $q_0 \in \mathcal{P}_\infty(\mathcal{A})$ such that $p \sim_0 q_0 \leq q$. Define $p_0 = q - q_0$ and note that $q_0 \perp p_0$, so that Proposition A.1.2 (iv) and (ii) tells us that

$$q = q_0 + p_0 \sim_0 q_0 \oplus p_0 \sim_0 p \oplus p_0.$$

For the converse, assume that $p \in \mathcal{P}_m(\mathcal{A})$, $q \in \mathcal{P}_n(\mathcal{A})$ and $p_0 \in \mathcal{P}_k(\mathcal{A})$ and that there exists $v \in M_{n,m+k}(\mathcal{A})$ such that $vv^* = q$ and $v^*v = p \oplus p_0$. Defining $w = v(p \oplus 0_k)v^* \in M_n(\mathcal{A})$, then w is a projection and

$$p \sim_0 p \oplus 0_k = (v^*w)(w^*v) \sim_0 (w^*v)(v^*w) = wqw = w = v(p \oplus 0_k)v^* \leq v(p \oplus p_0)v^* = q.$$

Therefore $p \sim_0 w \leq q$, so the proof of (i) is complete. Transitivity of $\precsim$ immediately follows from (i), and (ii) follows from (i) and Proposition A.1.2 (iii). □

Lemma A.1.4. *If* p *and* q *are projections in* $\mathcal{A}$, *then:*

(i) p *is properly infinite if and only if* $p \oplus p \precsim p$.

(ii) *If* $p \precsim q \precsim p$ *and* p *is properly infinite, then* q *is properly infinite.*

Proof. (i) Assume that p is properly infinite and let $e, f \in \mathcal{A}$ be mutually orthogonal projections such that $e \sim f \sim p$, $e \leq p$ and $f \leq p$. Then $p \oplus p \sim_0 e \oplus f \sim_0 e + f \leq p$ by Proposition A.1.2 (ii) and (iv). Conversely, if $p \oplus p \precsim p$, then there exists a projection $r \in \mathcal{A}$ such that $p \oplus p \sim_0 r \leq p$. Therefore there exist $v_1, v_2 \in \mathcal{A}$ such that

$$\begin{pmatrix} v_1 v_1^* & v_1 v_2^* \\ v_2 v_1^* & v_2 v_2^* \end{pmatrix} = \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}, \quad v_1^* v_1 + v_2^* v_2 = r.$$

Defining $e = v_1^* v_1$ and $f = v_2^* v_2$, then because $v_1 v_1^*$ and $v_2 v_2^*$ are projections (they are equal to p), it follows that e and f are projections equivalent to p satisfying $e + f = r \leq p$. Moreover, $ef = v_1^*(v_1 v_2^*)v_2 = 0$, so p is properly infinite.

(ii) If p is properly infinite, then Proposition A.1.3 and (i) yield

$$q \oplus q \precsim p \oplus p \precsim p \precsim q,$$

so q is properly infinite. □

For any element $x \in \mathcal{A}$, we let $\Delta_i(x)$ be the diagonal matrix in $M_n(\mathcal{A})$ with x in the i'th diagonal entry and zeros everywhere else. It is clear that the map $\mathcal{A} \to M_n(\mathcal{A})$ given by $x \mapsto \Delta_i(x)$ is an injective *-homomorphism. Moreover, we let

$$x^{\oplus n} = \sum_{i=1}^{n} \Delta_i(x) = \underbrace{x \oplus x \oplus \cdots \oplus x}_{n}.$$

Lemma A.1.5. *Let $p \in \mathcal{A}$ be non-zero. If p is a finite projection, then $\Delta_i(p)$ is finite in $M_n(\mathcal{A})$ for all $i = 1, \ldots, n$. If $\Delta_i(p)$ is a finite projection for some $i = 1, \ldots, n$, then p is a finite projection.*

Proof. Assume that $q = (q_{jk})_{j,k=1}^n \in M_n(\mathcal{A})$ is a projection satisfying $q \sim \Delta_i(p)$ and $q \leq \Delta_i(p)$. Then there exists

$$v = \begin{pmatrix} v_{11} & \cdots & v_{n1} \\ \vdots & \ddots & \vdots \\ v_{n1} & \cdots & v_{nn} \end{pmatrix} \in M_n(\mathcal{A})$$

such that $vv^* = \Delta_i(p)$ and $v^*v = q$. Since $\Delta_i(p)v = v$, we must have

$$v_{ik} = pv_{ik}, \quad v_{jk} = 0, \quad j, k = 1, \ldots, n, \ j \neq i.$$

Since $vv^* = p$, we have

$$p = \sum_{k=1}^{n} v_{ij} v_{ij}^*.$$

We now claim that $v_{ik} = 0$ for all $k \neq i$. Note first that because $q = v^*v$ by assumption, then $q_{jk} = v_{ij}^* v_{ik}$ for all j, k. Since $q \leq \Delta_i(p)$, we have

$$\Delta_k(v_{ik}^* v_{ik}) = \Delta_k(q_{kk}) = \Delta_k(1_{\mathcal{A}}) q \Delta_k(1_{\mathcal{A}}) \leq \Delta_k(1_{\mathcal{A}}) \Delta_i(p) \Delta_k(1_{\mathcal{A}})$$

for all k. If $k \neq i$, then the latter matrix is the zero matrix, so $v_{ik}^* v_{ik} = 0$ and $v_{ik} = 0$. The case $k = i$ implies $v_{ii}^* v_{ii} \leq p$. Moreover, we now know that $p = v_{ii} v_{ii}^*$, so since p is finite, we have $q_{ii} = v_{ii}^* v_{ii} = p$. Hence $q = \Delta_i(q_{ii}) = \Delta_i(p)$.

Conversely, if $\Delta_i(p)$ is a finite projection for some i, then p is first and foremost a projection. If $q \in \mathcal{A}$ is a projection with $q \sim p$ and $q \leq p$, then there exists a partial isometry $v \in \mathcal{A}$ with $v^*v = q$ and $vv^* = p$. Then $\Delta_i(v)^* \Delta_i(v) = \Delta_i(q)$ and $\Delta_i(v) \Delta_i(v)^* = \Delta_i(p)$. Moreover, $\Delta_i(q) \leq \Delta_i(p)$, so $\Delta_i(p) = \Delta_i(q)$ as $\Delta_i(p)$ is finite. Hence $p = q$. □

Lemma A.1.6. *If $p, q \in \mathcal{P}(\mathcal{A})$ and q is central, then $r \precsim p^{\oplus n}$ implies $rq \precsim (pq)^{\oplus n}$ for all $r \in \mathcal{P}(\mathcal{A})$.*

Proof. If $r \precsim p^{\oplus n}$, then there exists $v \in M_{1,n}(\mathcal{A})$ such that $vv^* = r$ and $v^*v \leq p^{\oplus n}$. Defining $w = vq^{\oplus n} \in M_{1,n}(\mathcal{A})$, then $ww^* = rq$ and $w^*w = q^{\oplus n} v^* v q^{\oplus n} \leq q^{\oplus n} p^{\oplus n} q^{\oplus n} = (pq)^{\oplus n}$ by virtue of q being central. □

The final result of this section serves as a reminder that finite projections behave much better in von Neumann algebras:

Lemma A.1.7. *Let $\mathscr{M}$ be a von Neumann algebra. Assume that $p \in \mathscr{M}$ and $q \in \mathscr{M}$ satisfy $p \precsim q^{\oplus n}$ for some n. If q is finite, then p is finite.*

Proof. By Proposition A.1.3 (i) there exists $p_0 \in \mathcal{P}_k(\mathscr{M})$ for some k such that $p \oplus p_0 \sim_0 q^{\oplus n}$. Hence the matrices $p \oplus p_0 \oplus 0_n$ and $q^{\oplus n} \oplus 0_{k+1}$ are Murray-von Neumann equivalent. Lemma A.1.5 yields that $\Delta_i(q) \in \mathcal{P}_{n+k+1}(\mathscr{M})$ is finite for all $i = 1, \dots, n$, so since finite sums of orthogonal finite projections in a von Neumann algebra are finite (cf. [40, Theorem 6.3.8]), it follows that $q^{\oplus n} \oplus 0_{k+1} = \sum_{i=1}^{n} \Delta_i(q)$ is finite as well. Therefore $p \oplus p_0 \oplus 0_n$ is finite, and because $p \oplus 0_{n+k} \leq p \oplus p_0 \oplus 0_n$, Lemma A.1.5 now tells us that p is finite. □

A.2 Full projections

Henceforth, $\mathcal{A}$ always denotes a *unital* C^*-algebra. The deus ex machina in our proof of Theorem 2.3 comes in the form of a certain full projection, and as we shall see, such projections have some very nice properties (some of which are quite beautiful when formulated with the relations of the previous section). Full elements in a unital C^*-algebra are defined on page 35.

Recall the following basic fact.

Lemma A.2.1. *If $\mathcal{A}$ is a unital Banach algebra and $\mathfrak{I} \subseteq \mathcal{A}$ is a proper, two-sided ideal, then $\overline{\mathfrak{I}}$ is a proper, closed, two-sided ideal.*

Proof. It is easy to see that $\overline{\mathfrak{I}}$ is a two-sided ideal. Supposing that $\overline{\mathfrak{I}} = \mathcal{A}$, then there exists $x \in \mathfrak{I}$ such that $\|x - 1_{\mathcal{A}}\| < 1$. Hence x is invertible, so $1_{\mathcal{A}} = xx^{-1} \in \mathfrak{I}$. □

Lemma A.2.2. *If $a \in \mathcal{A}$ is full, then there exist $n \geq 1$ and $x_1, \dots, x_n, y_1, \dots, y_n \in \mathcal{A}$ such that*

$$1_{\mathcal{A}} = \sum_{i=1}^{n} x_i a y_i.$$

Proof. Let

$$\mathfrak{I} = \left\{ \sum_{i=1}^{n} x_i a y_i \;\middle|\; n \geq 1,\ x_1, \dots, x_n, y_1, \dots, y_n \in \mathcal{A} \right\}.$$

Then $\mathfrak{I}$ is a two-sided ideal in $\mathcal{A}$, so $\overline{\mathfrak{I}}$ is a closed two-sided ideal in $\mathcal{A}$. Since $a \in \overline{\mathfrak{I}}$, it follows from fullness of a that $\overline{\mathfrak{I}} = \mathcal{A}$. Hence $\mathfrak{I} = \mathcal{A}$, from which the claim follows. □

Lemma A.2.3. *If $a \in \mathcal{A}$ is full and positive and $q \in \mathcal{A}$ is a projection, then there exist a positive integer $n \geq 1$ and $x_1, \dots, x_n \in \mathcal{A}$ such that*

$$q = \sum_{i=1}^{n} x_i a x_i^*.$$

Proof. Since a is full, it is clear that $2a$ is also full. By the previous lemma, there exist elements $y_1, z_1, \dots, y_m, z_m \in \mathcal{A}$ such that $1_{\mathcal{A}} = 2\sum_{i=1}^{m} y_i a z_i$. Hence $1_{\mathcal{A}} = 2\sum_{i=1}^{m} z_i^* a y_i^*$, by a being self-adjoint. We then see that

$$1_{\mathcal{A}} = \sum_{i=1}^{m} (y_i a z_i + z_i^* a y_i^*) \leq \sum_{i=1}^{m} (y_i a y_i^* + z_i^* a z_i),$$

the inequality following from the fact that $(y - z^*)a(y^* - z) \geq 0$ for all $y, z \in \mathcal{A}$. Defining $n = 2m$ as well as $w_i = y_i$ and $w_{i+n} = z_i^*$ for $i = 1, \dots, m$, we have

$$1_{\mathcal{A}} \leq \sum_{i=1}^{n} w_i a w_i^*.$$

Since $b = \sum_{i=1}^{n} w_i a w_i^*$ is invertible and positive, there exists a self-adjoint $z \in \mathcal{A}$ such that $1_{\mathcal{A}} = zbz$, namely $z = b^{-1/2}$. Defining $x_i = qzw_i$ for $i = 1, \dots, n$, we have the desired composition. □

With the above properties established, we now turn to showing what fullness of a projection means in matrix terms.

Proposition A.2.4. *If p and q are projections in $\mathcal{A}$ and p is full, then there exists $n \geq 1$ such that $q \precsim p^{\oplus n}$.*

Proof. Let $n \geq 1$ and $x_1, \dots, x_n \in \mathcal{A}$ such that $q = \sum_{i=1}^n x_i p x_i^*$. Define $P = p^{\oplus n}$ and let $v \in M_{1,n}(\mathcal{A})$ be the rectangular matrix given by

$$v = \begin{pmatrix} x_1 p & x_2 p & \cdots & x_n p \end{pmatrix}.$$

Then $vv^* = q$. Note that $(v^*v)^2 = v^*qv \leq v^*v$. If we define $f(z) = z - z^2$ for $z \in \sigma(v^*v)$, then we have $f(\sigma(v^*v)) = \sigma(f(v^*v)) \subseteq [0, \infty)$ by the continuous functional calculus, so $\sigma(v^*v) \subseteq [0,1]$. If 1_n denotes the identity element of $M_n(\mathcal{A})$, we then see that $\sigma(1_n - v^*v) \subseteq [0,1]$, so $v^*v \leq 1_n$. Hence $v^*v = Pv^*vP \leq P$, so $q \precsim P$. □

Proposition A.2.5. *A unital C^*-algebra $\mathcal{A}$ is properly infinite if and only if it contains a properly, infinite full projection.*

Proof. If $\mathcal{A}$ is properly infinite, then $1_{\mathcal{A}}$ is properly infinite and full. Conversely, if p is properly infinite and full, then by the previous result there exists $n \geq 1$ such that $1_{\mathcal{A}} \precsim p^{\oplus n}$. Since p is properly infinite, we have $p^{\oplus n} \precsim p$ by Lemma A.1.4 (i), so that $1_{\mathcal{A}} \precsim p$. As $p \precsim 1_{\mathcal{A}}$ trivially, it follows that $1_{\mathcal{A}}$ is properly infinite by Lemma A.1.4 (ii). □

Proposition A.2.6. *Let $\mathcal{A}$ be a unital C^*-algebra. If $p \in \mathcal{P}_\infty(\mathcal{A})$ is properly infinite and full, then $q \precsim p$ for all $q \in \mathcal{P}_\infty(\mathcal{A})$.*

Proof. Let $n \geq 1$ such that $p \in \mathcal{P}_n(\mathcal{A})$. As in the previous proof, we then have $1_n \precsim p$ by p being properly infinite and full. For any $q \in \mathcal{P}_k(\mathcal{A})$, then by taking $m \geq 1$ such that $nm > k$ we have

$$q \precsim 1_k \sim_0 1_k \oplus 0_{nm-k} \precsim 1_{nm} = (1_n)^{\oplus m} \precsim p^{\oplus m} \precsim p$$

by Lemmas A.1.2 (i) and A.1.4 (i), completing the proof. □

We finally pass to von Neumann algebras, proving the results necessary to give a thorough proof of Lemma 2.14.

Corollary A.2.7. *If $\mathscr{M}$ is a von Neumann algebra and $p, q \in \mathscr{M}$ are properly infinite, full projections, then they are equivalent.*

Proof. This follows from the preceding proposition and the Schröder-Bernstein theorem [74, Lemma 25.1]. □

Proposition A.2.8. *Let $\mathscr{M}$ be a properly infinite von Neumann algebra. Then any full projection $p \in \mathscr{M}$ is properly infinite.*

Proof. Let $p \in \mathscr{M}$ be a full projection. By Proposition A.2.4 there exists $n \geq 1$ such that $1_{\mathscr{M}} \precsim p^{\oplus n}$. Assuming that p is not properly infinite, then there exists a central projection $q \in \mathscr{M}$ such that pq is finite and non-zero by [11, III.1.3.5]. As $1_{\mathscr{M}} \precsim p^{\oplus n}$ now implies $q \precsim (pq)^{\oplus n}$ and $(pq)^{\oplus n}$ is finite by Lemma A.1.5, q is consequently finite by Lemma A.1.7, contradicting that $\mathscr{M}$ is properly infinite by the type classification [11, III.1.4.1]. Therefore p is indeed properly infinite. □

Appendix B

An isomorphism theorem for reduced twisted crossed products

The proofs of many results in Chapter 4 rely on a quite deep isomorphism theorem, allowing us to realize the reduced twisted crossed product of a group Γ with a normal subgroup Λ as an iterated crossed product with respect to Λ and the quotient group Γ/Λ. We give a proof of it here, but consider yourselves warned: it is very heavy *and* very ugly. We have nonetheless elected to include it because we construct the reduced twisted crossed product a bit differently from how it is done in [5].

Theorem 4.1.15. *Let $(\mathcal{A}, \Gamma, \alpha, u)$ be a twisted dynamical system, let Λ be a normal subgroup of Γ, let $Q = \Gamma/\Lambda$ and let $j\colon \Gamma \to Q$ denote the canonical epimorphism. Moreover, let (α', u') denote the restriction of (α, u) to Λ. Then for any $s \in \Gamma$, there exists $\gamma_s \in \operatorname{Aut}(\mathcal{A} \rtimes^{u'}_{\alpha',r} \Lambda)$ such that*

$$\gamma_s(\pi_{\alpha'}(a)) = \pi_{\alpha'}(\alpha_s(a)), \quad \gamma_s(\lambda_{u'}(t)) = \pi_{\alpha'}(\tilde{u}(s,t))\lambda_{u'}(sts^{-1}), \quad a \in \mathcal{A},\ t \in \Lambda.$$

Moreover, if $k\colon \Gamma/\Lambda \to \Gamma$ is a cross-section for j with $k(1) = 1$, define maps

- *$\beta\colon Q \to \operatorname{Aut}(\mathcal{A} \rtimes^{u'}_{\alpha',r} \Lambda)$ by $\beta = \gamma \circ k$,*
- *$m\colon Q \times Q \to \Lambda$ by $m(x,y) = k(x)k(y)k(xy)^{-1}$ for $x, y \in Q$, and*
- *$v\colon Q \times Q \to \mathcal{U}(\mathcal{A} \rtimes^{u'}_{\alpha',r} \Lambda)$ by*

$$v(x,y) = \pi_{\alpha'}(u(k(x),k(y))u(m(x,y),k(xy))^*)\lambda_{u'}(m(x,y)), \quad x, y \in Q.$$

Then (β, v) is a twisted action of Q on $\mathcal{A} \rtimes^{u'}_{\alpha',r} \Lambda$ such that

$$\mathcal{A} \rtimes^{u}_{\alpha,r} \Gamma \cong (\mathcal{A} \rtimes^{u'}_{\alpha',r} \Lambda) \rtimes^{v}_{\beta,r} Q.$$

Proof. We can assume that $\mathcal{A} \subseteq B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. By Remark 4.1.14 we can also assume the existence of a map $a\colon \Gamma \to \mathcal{U}(\mathcal{H})$ such that $a(s)xa(s)^* = \alpha_s(x)$ for all $s \in \Gamma$ and $x \in \mathcal{A}$. For $s \in \Gamma$, we now define $u_s \in B(\mathcal{H} \otimes \ell^2(\Lambda))$ by

$$u_s(\xi \otimes \delta_t) = u(st^{-1}s^{-1}, s)a(s^{-1})^*u(s^{-1}, st^{-1})\xi \otimes \delta_{sts^{-1}}$$

for $\xi \in \mathcal{H}$ and $t \in \Lambda$. Then u_s is a unitary operator on $\mathcal{H} \otimes \ell^2(\Lambda)$ with

$$u_s^*(\xi \otimes \delta_t) = u(s^{-1}, t^{-1}s)^*a(s^{-1})u(t^{-1}, s)^*\xi \otimes \delta_{s^{-1}ts}, \quad \xi \in \mathcal{H},\ t \in \Lambda.$$

For all $t \in \Lambda$,

$$\begin{aligned} u_s\pi_{\alpha'}(a)u_s^*(\xi \otimes \delta_t) &= u(t^{-1}, s)a(s^{-1})^* \left[u(s^{-1}, t^{-1}s)\alpha_{s^{-1}t^{-1}s}(a)u(s^{-1}, t^{-1}s)^*\right] a(s^{-1})u(t^{-1}, s)^*\xi \otimes \delta_t \\ &= u(t^{-1}, s) \left[a(s^{-1})^*\alpha_{s^{-1}}(\alpha_{t^{-1}s}(a))a(s^{-1})\right] u(t^{-1}, s)^*\xi \otimes \delta_t \\ &= u(t^{-1}, s)\alpha_{t^{-1}s}(a)u(t^{-1}, s)^*\xi \otimes \delta_t \\ &= \alpha_{t^{-1}}(\alpha_s(a))\xi \otimes \delta_t \\ &= \pi_{\alpha'}(\alpha_s(a))(\xi \otimes \delta_t). \end{aligned}$$

Further, for all $t, w \in \Lambda$, then by defining $p = stws^{-1}$ we see that

$$\begin{aligned}
u_s \lambda_{u'}(t)(\xi \otimes \delta_w) &= u(p^{-1}, s) a(s^{-1})^* \left[u(s^{-1}, p^{-1}s) u(s^{-1}p^{-1}s, t)\right] \xi \otimes \delta_p \\
&= u(p^{-1}, s) \left[a(s^{-1})^* \alpha_{s^{-1}}(u(p^{-1}s, t))\right] u(s^{-1}, p^{-1}st) \xi \otimes \delta_p \\
&= \left[u(p^{-1}, s) u(p^{-1}s, t)\right] a(s^{-1})^* u(s^{-1}, p^{-1}st) \xi \otimes \delta_p \\
&= \alpha_{p^{-1}}(u(s,t)) u(p^{-1}, st) \left[u(p^{-1}sts^{-1}, s)^* u(p^{-1}, sts^{-1})^*\right] \\
&\qquad u(p^{-1}, sts^{-1}) u(sw^{-1}s^{-1}, s) a(s^{-1})^* u(s^{-1}, sw^{-1}) \xi \otimes \delta_p \\
&= \alpha_{p^{-1}}(u(s,t)) u(p^{-1}, st) \left[u(p^{-1}, st)^* \alpha_{p^{-1}}(u(sts^{-1}, s))^*\right] \\
&\qquad u(p^{-1}, sts^{-1}) u(sw^{-1}s^{-1}, s) a(s^{-1})^* u(s^{-1}, sw^{-1}) \xi \otimes \delta_p \\
&= \left[\alpha_{p^{-1}}(u(s,t) u(sts^{-1}, s)^*)\right] u(p^{-1}, sts^{-1}) u(sw^{-1}s^{-1}, s) a(s^{-1})^* u(s^{-1}, sw^{-1}) \xi \otimes \delta_p \\
&= \pi_{\alpha'}(\tilde{u}(s,t)) \left(u(p^{-1}, sts^{-1}) u(sw^{-1}s^{-1}, s) a(s^{-1})^* u(s^{-1}, sw^{-1}) \xi \otimes \delta_p\right) \\
&= \pi_{\alpha'}(\tilde{u}(s,t)) \lambda_{u'}(sts^{-1}) \left(u(sw^{-1}s^{-1}, s) a(s^{-1})^* u(s^{-1}, sw^{-1}) \xi \otimes \delta_{sws^{-1}}\right) \\
&= \pi_{\alpha'}(\tilde{u}(s,t)) \lambda_{u'}(sts^{-1}) u_s(\xi \otimes \delta_w).
\end{aligned}$$

Therefore, if we restrict the automorphism $\mathrm{Ad}(u_s)$ of $B(\mathcal{H} \otimes \ell^2(\Lambda))$ to $\mathcal{A} \rtimes^{u'}_{\alpha', r} \Lambda$, then we obtain an automorphism

$$\gamma_s \in \mathrm{Aut}(\mathcal{A} \rtimes^{u'}_{\alpha', r} \Lambda)$$

with the desired properties. Some tedious computations in the same vein as above show that (β, v) defines a twisted action of Q on $\mathcal{A} \rtimes^{u'}_{\alpha', r} \Lambda$; we refer the reader to [51, pp. 306-307] for these. We can then define a unitary operator

$$F \colon \mathcal{H} \otimes \ell^2(Q) \otimes \ell^2(\Lambda) \to \mathcal{H} \otimes \ell^2(\Gamma)$$

by

$$F(\xi \otimes \delta_s \otimes \delta_x) = u(s^{-1}, k(x^{-1}))^* \xi \otimes \delta_{k(x^{-1})^{-1}s}, \quad \xi \in \mathcal{H},\ x \in Q,\ s \in \Lambda.$$

Now note that for all $a \in \mathcal{A}$, $t \in \Lambda$ and $x \in Q$, we have

$$\begin{aligned}
F\pi_\beta(\pi_{\alpha'}(a))(\xi \otimes \delta_t \otimes \delta_x) &= F(\gamma_{k(x^{-1})}(\pi_{\alpha'}(a))(\xi \otimes \delta_t) \otimes \delta_x) \\
&= F(\pi_{\alpha'}(\alpha_{k(x^{-1})}(a))(\xi \otimes \delta_t) \otimes \delta_x) \\
&= F(\alpha_{t^{-1}}(\alpha_{k(x^{-1})}(a)) \xi \otimes \delta_t \otimes \delta_x) \\
&= u(t^{-1}, k(x^{-1}))^* \alpha_{t^{-1}}(\alpha_{k(x^{-1})}(a)) \xi \otimes \delta_{k(x^{-1})^{-1}t} \\
&= \alpha_{t^{-1}k(x^{-1})}(a) u(t^{-1}, k(x^{-1}))^* \xi \otimes \delta_{k(x^{-1})^{-1}t} \\
&= \pi_\alpha(a) u(t^{-1}, k(x^{-1}))^* \xi \otimes \delta_{k(x^{-1})^{-1}t} \\
&= \pi_\alpha(a) F(\xi \otimes \delta_t \otimes \delta_x),
\end{aligned}$$

so that

$$F\pi_\beta(\pi_{\alpha'}(a))F^* = \pi_\alpha(a), \quad a \in \mathcal{A}. \tag{B.1}$$

Moreover, for all $s, t \in \Lambda$ and $x \in Q$, note that

$$\begin{aligned}
F\pi_\beta(\lambda_{u'}(s))(\xi \otimes \delta_t \otimes \delta_x) &= F(\gamma_{k(x^{-1})}(\lambda_{u'}(s))(\xi \otimes \delta_t) \otimes \delta_x) \\
&= F(\pi_{\alpha'}(\tilde{u}(k(x^{-1}), s)) \lambda_u(k(x^{-1})s^{-1}))(\xi \otimes \delta_t) \otimes \delta_x) \\
&= F(\lambda_u(k(x^{-1})) \lambda_u(s) \lambda_u(k(x^{-1}))^* (\xi \otimes \delta_t) \otimes \delta_x) \\
&= u(t^{-1}k(x^{-1})s^{-1}, s) u(t^{-1}, k(x^{-1}))^* \xi \otimes \delta_{sk(x^{-1})^{-1}t} \\
&= \lambda_u(s)(u(t^{-1}, k(x^{-1}))^* \xi \otimes \delta_{k(x^{-1})^{-1}t}) \\
&= \lambda_u(s) F(\xi \otimes \delta_t \otimes \delta_x),
\end{aligned}$$

so that

$$F\pi_\beta(\lambda_{u'}(s))F^* = \lambda_u(s), \quad s \in \Lambda. \tag{B.2}$$

Finally, let $s,t \in \Lambda$ and $x,y \in Q$. If we set $s = m(x^{-1}y^{-1},y)$ and $g = k(x^{-1}y^{-1})$, it follows that $sk(x^{-1}) = gk(y)$ and therefore

$$\begin{aligned}
F\lambda_v(y)(\xi \otimes \delta_t \otimes \delta_x) &= F(v(x^{-1}y^{-1},y)(\xi \otimes \delta_t) \otimes \delta_{yx}) \\
&= F(\pi_{\alpha'}(u(g,k(y))u(s,k(x^{-1}))^*)\lambda_{u'}(s)(\xi \otimes \delta_t) \otimes \delta_{yx}) \\
&= F(\alpha_{t^{-1}s^{-1}}(u(g,k(y))u(s,k(x^{-1}))^*)u(t^{-1}s^{-1},s)\xi \otimes \delta_{st} \otimes \delta_{yx}) \\
&= u(t^{-1}s^{-1},g)^*\alpha_{t^{-1}s^{-1}}(u(g,k(y))u(s,k(x^{-1}))^*)u(t^{-1}s^{-1},s)\xi \otimes \delta_{g^{-1}st} \\
&= u(t^{-1}s^{-1}g,k(y))u(t^{-1}s^{-1},gk(y))^*u(t^{-1}s^{-1},sk(x^{-1}))u(t^{-1},k(x^{-1}))^*\xi \otimes \delta_{g^{-1}st} \\
&= u(t^{-1}s^{-1}g,k(y))u(t^{-1},k(x^{-1}))^*\xi \otimes \delta_{g^{-1}st} \\
&= u(t^{-1}k(x^{-1})k(y)^{-1},k(y))u(t^{-1},k(x^{-1}))^*\xi \otimes \delta_{k(y)k(x^{-1})^{-1}t} \\
&= \lambda_u(k(y))(u(t^{-1},k(x^{-1}))^*\xi \otimes \delta_{k(x^{-1})^{-1}t}) \\
&= \lambda_u(k(y))F(\xi \otimes \delta_t \otimes \delta_x),
\end{aligned}$$

since $u(c,d)^*\alpha_c(u(d,f)) = u(cd,f)u(c,df)^*$ and $\alpha_c(u(d,f))^*u(c,d) = u(c,df)u(cd,f)^*$. Hence

$$F\lambda_v(y)F^* = \lambda_u(k(y)), \quad y \in Q. \tag{B.3}$$

It now follows from (B.1), (B.2) and (B.3) that F implements an isomorphism of the C^*-algebras $(\mathcal{A} \rtimes^{u'}_{\alpha',r} \Lambda) \rtimes^{v}_{\beta,r} Q$ and $\mathcal{A} \rtimes^{u}_{\alpha,r} \Gamma$. □

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