HomAlg Blok 2, 2011-2012. Assignment 3

Due Tuesday January 24

(1) Let Γ be a ring, let $x \in \Gamma$ be a central non-zerodivisor and let $\Lambda = \Gamma/x\Gamma$.
   (a) Suppose that A and B are Γ-modules on which multiplication by x is zero respectively injective. Prove that there is an isomorphism
   $$\mathrm{Ext}^n_\Lambda(A, B/xB) \cong \mathrm{Ext}^{n+1}_\Gamma(A, B).$$
   (Hint: Fix a projective resolution of A as a Λ-module and use this to construct a projective resolution of A as a Γ-module, or do an inductive argument using dimension shifting.)
   (b) Let A be a Λ-module such that $\mathrm{pd}_\Lambda A = n < \infty$. Prove that there exists a free Λ-module F such that $\mathrm{Ext}^n_\Lambda(A, F) \neq 0$.
   (c) Prove that $\mathrm{gldim}\,\Gamma \geq \mathrm{gldim}\,\Lambda + 1$ if Λ has finite global dimension.

(2) Let Λ be a ring and let $\Lambda[x]$ denote the polynomial ring over Λ. If A is a Λ-module then let $A[x] = \Lambda[x] \otimes_\Lambda A$. Clearly, $A[x]$ is a $\Lambda[x]$-module.
   (a) Prove that $\mathrm{pd}_{\Lambda[x]} A[x] = \mathrm{pd}_\Lambda A$ for any Λ-module A.
   (b) Let A be a $\Lambda[x]$-module. Prove that there is a short exact sequence of $\Lambda[x]$-modules
   $$0 \to A[x] \to A[x] \xrightarrow{\epsilon} A \to 0,$$
   where $\epsilon(x^i \otimes a) = x^i a$.
   (c) Prove that $\mathrm{gldim}\,\Lambda[x] \leq \mathrm{gldim}\,\Lambda + 1$.

   Observe that (1) and (2) imply that $\mathrm{gldim}\,\Lambda[x] = \mathrm{gldim}\,\Lambda + 1$ for any ring Λ. As a corollary, we obtain Hilbert's syzygy theorem: $\mathrm{gldim}\,k[x_1, \ldots, x_n] = n$ if k is a field.

(3) For a group G let $B_\bullet G$ denote the chain complex $(E_\bullet G)_G$ where $E_\bullet G$ is some choice of a projective resolution of the trivial G-module $\mathbb{Z}$.
   (a) Let G and H be groups. Prove that there is a homotopy equivalence of chain complexes of abelian groups $B_\bullet(G \times H) \simeq B_\bullet G \otimes_\mathbb{Z} B_\bullet H$.
   (b) Calculate the integral homology of the group $C_2 \times C_2$ in all dimensions.

(4) Do Exercise IV.9.4 and IV.9.6 from [HS].