Homological Algebra

Assignment 2

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(1)

Let k be a field, let $\Lambda = k[x, y, z]$ and let $\Gamma = \Lambda/(xz^2, yz^2, xyz)$.

(a)

Construct a free resolution of Γ over Λ and use this to calculate $\operatorname{Tor}_i^\Lambda(\Gamma, k)$ for all i.

We will immediately state a free resolution of Γ and then prove that it actually is a free resolution. Set $P_0 = \Lambda$, $P_1 = \Lambda^3$, $P_2 = \Lambda^2$ and $P_i = 0$ for $i > 2$, where Λ^n are free Λ-modules in the usual way. Consider the sequence

$$P_\bullet : \cdots \longrightarrow 0 \xrightarrow{\partial_3 = 0} P_2 \xrightarrow{\partial_2 = \begin{pmatrix} y & 0 \\ 0 & x \\ -z & -z \end{pmatrix}} P_1 \xrightarrow{\partial_1 = \begin{pmatrix} xz^2 & yz^2 & xyz \end{pmatrix}} P_0 \xrightarrow{\partial_0 = 0} 0, \qquad P_0 \xrightarrow{\pi} \Gamma$$

where π is the quotient homomorphism; define $\partial_i = 0 : P_i \to P_{i-1}$ for all $i > 2$. $P_\bullet$ is a chain complex, since for any $p, q \in \Lambda$, we have

$$\partial_1\partial_2 \begin{pmatrix} p \\ q \end{pmatrix} = \partial_1 \begin{pmatrix} py \\ qx \\ -z(p+q) \end{pmatrix} = pxyz^2 + qxyz^2 - z(p+q)xyz = 0,$$

and clearly $\partial_2\partial_3 = 0$ and $\partial_0\partial_1 = 0$. We will now check that the sequence

$$P_\bullet \to \Gamma : \cdots \longrightarrow 0 \xrightarrow{\partial_3} P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\pi} \Gamma \longrightarrow 0$$

is exact. Clearly, for $i > 2$, $\ker \partial_i = 0 = \operatorname{im}\partial_{i+1}$. Next, if $\binom{p}{q} \in \Lambda^2$ satisfies $\partial_2 \binom{p}{q} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, then

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} y & 0 \\ 0 & x \\ -z & -z \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} py \\ qx \\ -z(p+q) \end{pmatrix},$$

from which we can conclude (from examining the first two coordinates) that $p = q = 0$ because Λ is an integral domain. Alas ∂_2 is injective. We have already shown that $\operatorname{im}\partial_2 \subseteq \ker \partial_1$, so assume that $\begin{pmatrix} p \\ q \\ r \end{pmatrix} \in \ker \partial_1$; then

$$0 = \begin{pmatrix} xz^2 & yz^2 & xyz \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix} = pxz^2 + qyz^2 + rxyz = z(pxz + qyz + rxy),$$

so we conclude $pxz + qyz + rxy = 0$, once again since Λ is an integral domain. Since $pxz = -qyz - rxy$, y must divide the left side, so $p = sy$ for some $s \in \Lambda$; thus $sxz = -qz - rx$, so $qz = -sxz - rx$, so

$q = tx$ for some $t \in \Lambda$ and finally $tz = -sz - r$, so $r = -sz - tz$, using in abundance that Λ is an integral domain. Now, since

$$\partial_2 \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} y & 0 \\ 0 & x \\ -z & -z \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} sy \\ tx \\ -sz - tz \end{pmatrix} = \begin{pmatrix} p \\ q \\ r \end{pmatrix},$$

we conclude that $\ker \partial_1 = \mathrm{im}\partial_2$. Finally, π is surjective; if $\pi(p) = 0$, then $p = axz^2 + byz^2 + cxyz$ for some $a, b, c \in \Lambda$, and clearly, $\partial_1(a, b, c) = 0$. Likewise, for $p, q, r \in \Lambda$,

$$\pi\partial_1 \begin{pmatrix} p \\ q \\ r \end{pmatrix} = \pi(pxz^2 + qyz^2 + rxyz) = 0,$$

since $pxz^2 + qyz^2 + rxyz \in (xz^2, yz^2, rxyz)$. We have therefore proved exactness of $P_\bullet \to \Gamma$. Finally, the homology of $P_\bullet$ at degree 0 is

$$H_0(P_\bullet) = \ker \partial_0 / \mathrm{im}\partial_1 = \Lambda/(xz^2, yz^2, xyz) = \Gamma,$$

so we conclude that $P_\bullet$ is a free resolution of Γ over Λ.

Since $P_\bullet$ is a free (projective) resolution of Γ, we can calculate $\mathrm{Tor}_i^\Lambda(\Gamma, k)$ by recalling that

$$\mathrm{Tor}_i^\Lambda(\Gamma, k) = H_i(P_\bullet \otimes_\Lambda k), \quad i \in \mathbb{Z},$$

where k is made into a left Λ-module by defining $\lambda \cdot \omega = [\lambda\omega]_{(x,y,z)}$ for $\lambda \in \Lambda$ and $\omega \in k$. Therefore consider the diagram

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & 0 \otimes_\Lambda k & \xrightarrow{\partial_3 \otimes \mathrm{id}} & \Lambda^2 \otimes_\Lambda k & \xrightarrow{\partial_2 \otimes \mathrm{id}} & \Lambda^3 \otimes k & \xrightarrow{\partial_1 \otimes \mathrm{id}} & \Lambda \otimes_\Lambda k & \xrightarrow{\partial_0 \otimes \mathrm{id}} & 0 \otimes_\Lambda k & \longrightarrow & \cdots \\
 & & \downarrow \varphi_3 & & \downarrow \varphi_2 & & \downarrow \varphi_1 & & \downarrow \varphi_0 & & \downarrow \varphi_{-1} & & \\
\cdots & \longrightarrow & 0 & \xrightarrow{\partial_3'} & k^2 & \xrightarrow{\partial_2'} & k^3 & \xrightarrow{\partial_1'} & k & \xrightarrow{\partial_0'} & 0 & \longrightarrow & \cdots
\end{array}$$

with $P_\bullet \otimes_\Lambda k$ in the top row with φ_i and ∂_i' still undefined (not to worry, we will define them now). In particular, $0 \otimes_\Lambda k \cong 0$, so defining $\varphi_i = 0$ for $i \geq 3$ and $i \leq -1$, they become isomorphisms. Additionally, $\Lambda^n \otimes_\Lambda k \simeq (\Lambda \otimes_\Lambda k)^n$ by the isomorphism $(\lambda_1, \ldots, \lambda_n) \otimes \omega \mapsto (\lambda_1 \otimes \omega, \ldots, \lambda_n \otimes \omega)$ with the inverse $(\lambda_1 \otimes \omega_1, \ldots, \lambda_n \otimes \omega_n) \mapsto \sum_{i=1}^n (0, \ldots, 0, \lambda_i, 0, \ldots, 0) \otimes \omega_i$, and $(\Lambda \otimes_\Lambda k)^n \cong k^n$ by the isomorphism $(\lambda_1 \otimes \omega_1, \ldots, \lambda_n \otimes \omega_n) \mapsto (\lambda_1\omega_1, \ldots, \lambda_n\omega_n)$ (see (1), p. 109) with the inverse $(\omega_1, \ldots, \omega_n) \mapsto (1 \otimes \omega_1, \ldots, 1 \otimes \omega_n)$. Thus, defining φ_i, $i = 0, 1, 2$ to be the composite of these, i.e.

$$\begin{aligned}
\varphi_2((\lambda_1, \lambda_2) \otimes \omega) &= (\lambda_1\omega, \lambda_2\omega), \\
\varphi_1((\lambda_1, \lambda_2, \lambda_3) \otimes \omega) &= (\lambda_1\omega, \lambda_2\omega, \lambda_3\omega), \\
\varphi_0(\lambda_1 \otimes \omega) &= \lambda_1\omega,
\end{aligned}$$

for $\lambda_1, \lambda_2, \lambda_3 \in \Lambda$ and $\omega \in k$, they become isomorphisms with inverses

$$\begin{aligned}
\varphi_2^{-1}(\omega_1, \omega_2) &= (1, 0) \otimes \omega_1 + (0, 1) \otimes \omega_2, \\
\varphi_1^{-1}(\omega_1, \omega_2, \omega_3) &= (1, 0, 0) \otimes \omega_1 + (0, 1, 0) \otimes \omega_2 + (0, 0, 1) \otimes \omega_3, \\
\varphi_0^{-1}(\omega_1) &= 1 \otimes \omega_1,
\end{aligned}$$

for $\omega_1, \omega_2, \omega_3 \in k$. Therefore, define $\partial_i' = \varphi_{i-1}(\partial_i \otimes \mathrm{id})\varphi_i^{-1}$, so that the squares commute. It is now directly seen that $\partial_3' = 0$ and $\partial_0' = 0$. Additionally,

$$\begin{aligned}
\partial_2' \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} &= \varphi_1(\partial_2 \otimes \mathrm{id}) \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \omega_1 + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \omega_2 \right) \\
&= \varphi_1 \left(\begin{pmatrix} y \\ 0 \\ -z \end{pmatrix} \otimes \omega_1 + \begin{pmatrix} 0 \\ x \\ -z \end{pmatrix} \otimes \omega_2 \right) \\
&= \begin{pmatrix} y\omega_1 \\ x\omega_2 \\ -z(\omega_1 + \omega_2) \end{pmatrix} = 0,
\end{aligned}$$

and

$$\begin{aligned}
\partial_1' \begin{pmatrix}\omega_1\\ \omega_2\\ \omega_3\end{pmatrix} &= \varphi_0(\partial_1\otimes \mathrm{id})\left(\begin{pmatrix}1\\0\\0\end{pmatrix}\otimes\omega_1+\begin{pmatrix}0\\1\\0\end{pmatrix}\otimes\omega_2+\begin{pmatrix}0\\0\\1\end{pmatrix}\otimes\omega_3\right)\\
&= \varphi_0\left(xz^2\otimes\omega_1+yz^2\otimes\omega_2+xyz\otimes\omega_3\right)\\
&= xz^2\omega_1+yz^2\omega_2+xyz\omega_3=0,
\end{aligned}$$

so $\partial_1'=0$ and $\partial_2'=0$ as well. It is clear that the lower row $Q_\bullet$ is a chain complex, and since the isomorphism and chain map $\varphi_\bullet$ is in particular a quasi-isomorphism, we obtain

$$\mathrm{Tor}_i^\Lambda(\Gamma,k)=H_i(P_\bullet\otimes_\Lambda k)\cong H_i(Q_\bullet)=\frac{\ker\partial_i'}{\mathrm{im}\partial_{i+1}'}\cong\ker\partial_i'=\begin{cases}0 & i\geq 3 \text{ or } i\leq -1\\ k^2 & i=2\\ k^3 & i=1\\ k & i=0.\end{cases}$$

(b)

Calculate the homology of the Koszul complex $H_i(K_\bullet^\Gamma(x,y,z))$ for all i.

Let $\{e_1,e_2,e_3\}$ be a set of three elements; then the Koszul complex $K_\bullet^\Gamma(x,y,z)$ is defined by

$$\begin{aligned}
K_0^\Gamma(x,y,z) &= \Gamma\\
K_1^\Gamma(x,y,z) &= \Gamma e_1\oplus\Gamma e_2\oplus\Gamma e_3\\
K_2^\Gamma(x,y,z) &= \Gamma(e_2\wedge e_3)\oplus\Gamma(e_1\wedge e_3)\oplus\Gamma(e_1\wedge e_2),\\
K_3^\Gamma(x,y,z) &= \Gamma(e_1\wedge e_2\wedge e_3),
\end{aligned}$$

and $K_i^\Gamma(x,y,z)=0$ for $i>3$ or $i<0$. The differentials are given by

$$\begin{aligned}
&\partial_1(e_1)=x,\quad \partial_1(e_2)=y,\quad \partial_1(e_3)=z,\\
&\partial_2(e_2\wedge e_3)=ye_3-ze_2,\quad \partial_2(e_1\wedge e_3)=xe_3-ze_1,\quad \partial_2(e_1\wedge e_2)=xe_2-ye_1,\\
&\partial_3(e_1\wedge e_2\wedge e_3)=x(e_2\wedge e_3)-y(e_1\wedge e_3)+z(e_1\wedge e_2),
\end{aligned}$$

and $\partial_i=0$ for $i>3$ or $i<0$, so that we obtain a chain complex

$$\cdots\longrightarrow 0\xrightarrow{\partial_4=0}\Gamma\xrightarrow{\partial_3=\begin{pmatrix}x\\-y\\z\end{pmatrix}}\Gamma^3\xrightarrow{\partial_2=\begin{pmatrix}0&-z&-y\\-z&0&x\\y&x&0\end{pmatrix}}\Gamma^3\xrightarrow{\partial_1=(x\ y\ z)}\Gamma\xrightarrow{\partial_0=0}0\longrightarrow\cdots.$$

It's clear that $H_i(K_\bullet^\Gamma(x,y,z))=0$ for $i>3$ and $i<0$.

Homology at degree 0. Consider $H_0(K_\bullet^\Gamma)=\Gamma/(x,y,z)$, and define a Λ-module homomorphism $\Gamma/(x,y,z)\to k$ by $[[p]]\mapsto p_0$ for $p\in\Lambda$ where p_0 denotes the constant term of p. It is well-defined, since if $p=q$ in Γ, then $p-q$ has constant term 0, so p and q have the same constant term. It is injective since if the constant term of $p\in\Lambda$ is 0, then $p=ax+by+cz$ for some $a,b,c\in\Lambda$, so $p\in(x,y,z)$ in Γ and we obtain $[[p]]=0$. It is also surjective, since for any $\omega\in k$, then $[[\omega]]$ clearly maps to ω. Alas we obtain $H_0(K_\bullet^\Gamma)\cong k$.

We will jump to (c) here, as it presents a more effective (and easy) way of calculating the desired homology.

(c)

Compare your answers in (a) and (b) and explain the observations you make.

It can be seen from (a) and (b) that somehow we managed to obtain $\mathrm{Tor}_0^\Lambda(\Gamma,k)\cong H_0(K_\bullet^\Gamma(x,y,z))$. The way to realize why this is so is to note that there's a way to calculate $\mathrm{Tor}_i^\Lambda(\Gamma,k)$ different from the method used in (a). By having a projective resolution $Q_\bullet$ over k, we obtain

$$\mathrm{Tor}_i^\Lambda(\Gamma,k)=H_i(\Gamma\otimes_\Lambda Q_\bullet)$$

by the balancing of the Tor functor. Since the Koszul complex $K^\Lambda_\bullet(x, y, z)$ (note the Λ!) is in fact a projective resolution of k (constructed in the same way as $K^\Gamma_\bullet$ in (b)), we note that

$$\mathrm{Tor}^\Lambda_i(\Gamma, k) = H_i(\Gamma \otimes_\Lambda K^\Lambda_\bullet(x, y, z)).$$

The observation then follows from using the isomorphism (and thus quasi-isomorphism) $\Gamma \otimes_\Lambda \Lambda^n \to \Gamma^n$ given by $\gamma \otimes (\lambda_1, \ldots, \lambda_n) \mapsto (\gamma\lambda_1, \ldots, \gamma\lambda_n)$ constructed in the same manner as in (a), clearly making the diagram

$$\begin{array}{ccccccccc}
\Gamma \otimes_\Lambda \Lambda & \xrightarrow{\mathrm{id}\otimes\begin{pmatrix} x \\ -y \\ z \end{pmatrix}} & \Gamma \otimes_\Lambda \Lambda^3 & \xrightarrow{\mathrm{id}\otimes\begin{pmatrix} 0 & -z & -y \\ -z & 0 & x \\ y & x & 0 \end{pmatrix}} & \Gamma \otimes_\Lambda \Lambda^3 & \xrightarrow{\mathrm{id}\otimes(\,x\ y\ z\,)} & \Gamma \otimes_\Lambda \Lambda & \longrightarrow & 0 \\
\cong\downarrow & & \cong\downarrow & & \cong\downarrow & & \cong\downarrow & & \cong\downarrow \\
\Gamma & \xrightarrow{\begin{pmatrix} x \\ -y \\ z \end{pmatrix}} & \Gamma^3 & \xrightarrow{\begin{pmatrix} 0 & -z & -y \\ -z & 0 & x \\ y & x & 0 \end{pmatrix}} & \Gamma^3 & \xrightarrow{(\,x\ y\ z\,)} & \Gamma & \longrightarrow & 0
\end{array}$$

commute. Alas

$$\mathrm{Tor}^\Lambda_i(\Gamma, k) = H_i(\Gamma \otimes_\Lambda K^\Lambda_\bullet(x, y, z)) \cong H_i(K^\Gamma_\bullet(x, y, z))$$

for all $i \in \mathbb{Z}$.

(2)

Let $\varphi : C_\bullet \to D_\bullet$ be a morphism of chain complexes and let $E(\varphi)$ denote the mapping cone of φ (see Exercise IV.1.2 in (1)).

(a)

Establish a short exact sequence of chain complexes

$$0 \to D_\bullet \to E(\varphi) \to \Sigma C_\bullet \to 0$$

and identify the connecting homomorphism associated to this sequence.

Note that the mapping cone $E(\varphi)$ is the chain complex defined by $E(\varphi)_n = C_{n-1} \oplus D_n$ with differential $\partial^E_n(c, d) = (-\partial^C_{n-1}c, \varphi_{n-1}(c) + \partial^D_n d)$ for $n \in \mathbb{Z}$ and that $\Sigma C_\bullet$ is the chain complex $(\Sigma C_\bullet)_n = C_{n-1}$ with differential $\partial^{\Sigma C}_n c = -\partial^C_{n-1} c$ for $n \in \mathbb{Z}$.

It's obvious how to establish the desired short exact sequence. For $n \in \mathbb{Z}$, define the homomorphisms $\iota_n : D_n \to C_{n-1} \oplus D_n$ by $\iota_n(d) = (0, d)$ and $\pi_n : C_{n-1} \oplus D_n \to C_{n-1}$ by $\pi_n(c, d) = c$. π_n is surjective (for any $c \in C_{n-1}$, $\pi_n(c, 0) = c$), ι_n is injective ($\iota_n(d) = (0, 0)$ implies $d = 0$) and $\ker \pi_n = \{(c, d) \in E(\varphi)_n \,|\, c = 0\} = \mathrm{im}\iota_n$, so the sequence

$$0 \longrightarrow D_\bullet \xrightarrow{\iota_\bullet} E(\varphi) \xrightarrow{\pi_\bullet} \Sigma C_\bullet \longrightarrow 0$$

of chain complexes is exact. Additionally, we have for $n \in \mathbb{Z}$ that

$$\pi_{n-1}\partial^E_n(c, d) = -\partial^C_{n-1}c = -\partial^C_{n-1}\pi_n(c, d) = \partial^{\Sigma C}_n \pi_n(c, d)$$

and $\partial^E_n \iota_n(d) = (0, \partial^D_n d) = \iota_{n-1}\partial^D_n d$ for any $c \in C_{n-1}$ and $d \in D_n$, so that $\iota_\bullet$ and $\pi_\bullet$ are morphisms of chain complexes and induce well-defined homomorphisms under homology.

Now, any short exact sequence of chain complexes induces a long exact sequence in homology ((1), Theorem IV.2.1) with the well-defined connecting homomorphism $\omega : H(\Sigma C_\bullet) \to H(D_\bullet)$:

$$\cdots \longrightarrow H_{n+1}(\Sigma C_\bullet) \xrightarrow{\omega_{n+1}} H_n(D_\bullet) \xrightarrow{\iota^*_n} H_n(E(\varphi)) \xrightarrow{\pi^*_n} H_n(\Sigma C_\bullet) \xrightarrow{\omega_n} H_{n-1}(D_\bullet) \longrightarrow \cdots$$

In order to determine the connecting homomorphism, we need only use how ω is defined and apply the procedure to this sequence. Alas for $n \in \mathbb{Z}$, let $[c] \in H_n(\Sigma C_\bullet)$. Then $c \in \ker \partial^{\Sigma C}_n$ or $c \in \ker \partial^C_{n-1}$ and since $\pi_n(c, 0) = c$, take the image of $(c, 0)$ under ∂^E_n, i.e. $\partial^E_n(c, 0) = (-\partial^C_{n-1}c, \varphi_{n-1}(c)) = (0, \varphi_{n-1}(c))$, since c is a cycle of C_{n-1}. Finally, since $\iota_{n-1}(\varphi_{n-1}(c)) = (0, \varphi_{n-1}(c))$, we realize by how ω is defined that $\omega_n([c]) = [\varphi_{n-1}(c)]$. We need not worry about independence of choices of elements and well-definedness, since that has already been taken care of by the theorem. Alas the connecting homomorphism ω is the homomorphism induced by the morphism φ under homology taken one degree lower, i.e. $\omega_n = H_{n-1}(\varphi)$ for $n \in \mathbb{Z}$.

(b)

Prove that φ is a quasi-isomorphism if and only if $E(\varphi)$ is exact.

In the following, we will use the sequences constructed in (a). Assume that φ is a quasi-isomorphism; then the induced homomorphisms $H_n(\varphi) : H_n(C_\bullet) \to H_n(D_\bullet)$ are isomorphisms for all $n \in \mathbb{Z}$, but we identified the connecting homomorphism ω from the long exact sequence in homology in (a) with $H(\varphi)$, so ω_n is an isomorphism for all $n \in \mathbb{Z}$. Let $n \in \mathbb{Z}$. Since $\ker \iota_n^* = \mathrm{im}\omega_{n+1} = H_n(D_\bullet)$ since ω_{n+1} is surjective, we obtain $\iota_n^* = 0$. Additionally, $\mathrm{im}\pi_n^* = \ker \omega_n = 0$ since ω_n is injective, so $\pi_n^* = 0$. Therefore, $H_n(E(\varphi)) = \ker \pi_n^* = \mathrm{im}\iota_n^* = 0$. Therefore, $E(\varphi)$ is exact since $H_n(E(\varphi)) = 0$ is equivalent to $\ker \partial_n^E = \mathrm{im}\partial_{n+1}^E$ for all $n \in \mathbb{Z}$.

Assuming that $E(\varphi)$ is exact or $H_n(E(\varphi)) = 0$ for all $n \in \mathbb{Z}$, then it is seen for all $n \in \mathbb{Z}$ from the above long exact sequence that ω_{n+1} is surjective for all $n \in \mathbb{Z}$ ($\mathrm{im}\omega_{n+1} = \ker \iota_n^* = H_n(D_\bullet)$ since $\iota_n^* = 0$) and that ω_n is injective ($\ker \omega_n = \mathrm{im}\pi_{n+1}^* = 0$ since $H_n(E(\varphi)) = 0$), so $\omega_n = H_n(\varphi)$ is an isomorphism for all $n \in \mathbb{Z}$. Therefore φ is a quasi-isomorphism.

(c)

Prove that φ is a homotopy equivalence if and only if $E(\varphi)$ is split exact.

Assume that $\varphi : C_\bullet \to D_\bullet$ is a homotopy equivalence. Thus there exist a chain map $\psi : D_\bullet \to C_\bullet$ and homotopies $\sigma' = \{\sigma_n' : C_n \to C_{n+1}\}_{n\in\mathbb{Z}}$ and $\sigma'' = \{\sigma_n'' : D_n \to D_{n+1}\}_{n\in\mathbb{Z}}$ such that $\psi_n\varphi_n - 1_{C_n} = \partial_{n+1}^C\sigma_n' + \sigma_{n-1}'\partial_n^C$ and $\varphi_n\psi_n - 1_{D_n} = \partial_{n+1}^D\sigma_n'' + \sigma_{n-1}''\partial_n^D$.

For all $n \in \mathbb{Z}$, define $\sigma_n : E(\varphi)_n \to E(\varphi)_{n+1}$ by[1]

$$\sigma_n(c,d) = (\sigma_{n-1}'(c)+\psi_n(d)+\psi_n\sigma_{n-1}''\varphi_{n-1}(c)-\psi_n\varphi_n\sigma_{n-1}'(c),\ -\sigma_n''(d)+\sigma_n''\varphi_n\sigma_{n-1}'(c)-\sigma_n''\sigma_{n-1}''\varphi_{n-1}(c)).$$

For $(c,d) \in (E(\varphi))_n$, we have

$$\begin{aligned}\partial_{n+1}^E\sigma_n(c,d) &= (-\partial_n^C\sigma_{n-1}'(c) - \partial_n^C\psi_n(d) - \partial_n^C\psi_n\sigma_{n-1}''\varphi_{n-1}(c) + \partial_n^C\psi_n\varphi_n\sigma_{n-1}'(c),\\ &\quad \varphi_n\sigma_{n-1}'(c) + \varphi_n\psi_n(d) + \varphi_n\psi_n\sigma_{n-1}''\varphi_{n-1}(c)\\ &\quad -\varphi_n\psi_n\varphi_n\sigma_{n-1}'(c) - \partial_{n+1}^D\sigma_n''(d) + \partial_{n+1}^D\sigma_n''\varphi_n\sigma_{n-1}'(c) - \partial_{n+1}^D\sigma_n''\sigma_{n-1}''\varphi_{n-1}(c))\end{aligned}$$

and

$$\begin{aligned}\sigma_{n-1}\partial_n^E(c,d) &= (-\sigma_{n-2}'\partial_{n-1}^C(c) + \psi_{n-1}\varphi_{n-1}(c) + \psi_{n-1}\partial_n^D(d) - \psi_{n-1}\sigma_{n-2}''\varphi_{n-2}\partial_{n-1}^C(c)\\ &\quad +\psi_{n-1}\varphi_{n-1}\sigma_{n-2}'\partial_{n-1}^C(c),\ -\sigma_{n-1}''\varphi_{n-1}(c) - \sigma_{n-1}''\partial_n^D(d)\\ &\quad -\sigma_{n-1}''\varphi_{n-1}\sigma_{n-2}'\partial_{n-1}^C(c) + \sigma_{n-1}''\sigma_{n-2}''\varphi_{n-2}\partial_{n-1}^C(c))\end{aligned}$$

Summing the coordinates, we realize that the first coordinate becomes

$$\begin{aligned} & -\partial_n^C\sigma_{n-1}'(c) - \partial_n^C\psi_n(d) - \partial_n^C\psi_n\sigma_{n-1}''\varphi_{n-1}(c) + \partial_n^C\psi_n\varphi_n\sigma_{n-1}'(c) - \sigma_{n-2}'\partial_{n-1}^C(c)\\ &\quad +\psi_{n-1}\varphi_{n-1}(c) + \psi_{n-1}\partial_n^D(d) - \psi_{n-1}\sigma_{n-2}''\varphi_{n-2}\partial_{n-1}^C(c) + \psi_{n-1}\varphi_{n-1}\sigma_{n-2}'\partial_{n-1}^C(c)\\ = {} & c - \psi_{n-1}\partial_n^D(d) - \psi_{n-1}\partial_n^D\sigma_{n-1}''\varphi_{n-1}(c) + \partial_n^C\psi_n\varphi_n\sigma_{n-1}'(c)\\ &\quad +\psi_{n-1}\partial_n^D(d) - \psi_{n-1}\sigma_{n-2}''\partial_{n-1}^D\varphi_{n-1}(c) + \psi_{n-1}\varphi_{n-1}\sigma_{n-2}'\partial_{n-1}^C(c)\\ = {} & c - \psi_{n-1}(\partial_n^D\sigma_{n-1}'' + \sigma_{n-2}''\partial_{n-1}^D)\varphi_{n-1}(c) + \partial_n^C\psi_n\varphi_n\sigma_{n-1}'(c) + \psi_{n-1}\varphi_{n-1}\sigma_{n-2}'\partial_{n-1}^C(c)\\ = {} & c - \psi_{n-1}\varphi_{n-1}\psi_{n-1}\varphi_{n-1}(c) + \psi_{n-1}\varphi_{n-1}(c) + \psi_{n-1}\varphi_{n-1}\partial_n^C\sigma_{n-1}'(c) + \psi_{n-1}\varphi_{n-1}\sigma_{n-2}'\partial_{n-1}^C(c)\\ = {} & c - \psi_{n-1}\varphi_{n-1}\psi_{n-1}\varphi_{n-1}(c) + \psi_{n-1}\varphi_{n-1}(c + \partial_n^C\sigma_{n-1}'(c) + \sigma_{n-2}'\partial_{n-1}^C(c))\\ = {} & c - \psi_{n-1}\varphi_{n-1}\psi_{n-1}\varphi_{n-1}(c) + \psi_{n-1}\varphi_{n-1}\psi_{n-1}\varphi_{n-1}(c)\\ = {} & c,\end{aligned}$$

since $\partial_n^C\psi_n = \psi_{n-1}\partial_n^D$, ψ is a chain map, and $\psi\varphi \cong 1_{C_\bullet}$ by the homotopy σ'. The second coordinate

[1] This idea stems from the proof of ((2), Theorem 1.7.7.)

becomes

$$\begin{aligned}
&\varphi_n\sigma'_{n-1}(c)+\varphi_n\psi_n(d)+\varphi_n\psi_n\sigma''_{n-1}\varphi_{n-1}(c)-\varphi_n\psi_n\varphi_n\sigma'_{n-1}(c)-\partial^D_{n+1}\sigma''_n(d)\\
&\quad+\partial^D_{n+1}\sigma''_n\varphi_n\sigma'_{n-1}(c)-\partial^D_{n+1}\sigma''_n\sigma''_{n-1}\varphi_{n-1}(c)-\sigma''_{n-1}\varphi_{n-1}(c)-\sigma''_{n-1}\partial^D_n(d)\\
&\quad-\sigma''_{n-1}\varphi_{n-1}\sigma'_{n-2}\partial^C_{n-1}(c)+\sigma''_{n-1}\sigma''_{n-2}\varphi_{n-2}\partial^C_{n-1}(c)\\
=\;&d+(1_{D_n}-\varphi_n\psi_n+\partial^D_{n+1}\sigma''_n)\varphi_n\sigma'_{n-1}(c)+(\varphi_n\psi_n-\partial^D_{n+1}\sigma''_n-1_{D_n})\sigma''_{n-1}\varphi_{n-1}(c)\\
&\quad-\sigma''_{n-1}\varphi_{n-1}\sigma'_{n-2}\partial^C_{n-1}(c)+\sigma''_{n-1}\sigma''_{n-2}\varphi_{n-2}\partial^C_{n-1}(c)\\
=\;&d-\sigma''_{n-1}\partial^D_n\varphi_n\sigma'_{n-1}(c)+\sigma''_{n-1}\partial^D_n\sigma''_{n-1}\varphi_{n-1}(c)-\sigma''_{n-1}\varphi_{n-1}\sigma'_{n-2}\partial^C_{n-1}(c)\\
&\quad+\sigma''_{n-1}\sigma''_{n-2}\varphi_{n-2}\partial^C_{n-1}(c)\\
=\;&d-\sigma''_{n-1}\varphi_{n-1}\partial^C_n\sigma'_{n-1}(c)+\sigma''_{n-1}\partial^D_n\sigma''_{n-1}\varphi_{n-1}(c)-\sigma''_{n-1}\varphi_{n-1}\sigma'_{n-2}\partial^C_{n-1}(c)\\
&\quad+\sigma''_{n-1}\sigma''_{n-2}\partial^D_{n-1}\varphi_{n-1}(c)\\
=\;&d-\sigma''_{n-1}\varphi_{n-1}(\partial^C_n\sigma'_{n-1}+\sigma'_{n-2}\partial^C_{n-1})(c)+\sigma''_{n-1}(\partial^D_n\sigma''_{n-1}+\sigma''_{n-2}\partial^D_{n-1})\varphi_{n-1}(c)\\
=\;&d-\sigma''_{n-1}\varphi_{n-1}\psi_{n-1}\varphi_{n-1}(c)+\sigma''_{n-1}\varphi_{n-1}(c)+\sigma''_{n-1}\varphi_{n-1}\psi_{n-1}\varphi_{n-1}(c)-\sigma''_{n-1}\varphi_{n-1}(c)\\
=\;&d,
\end{aligned}$$

by φ being a chain map and the properties of the homotopies σ' and σ''. Thus, $\partial^E\sigma+\sigma\partial^E=1_\bullet$ on $E(\varphi)$, so $1_\bullet\cong 0_\bullet$ and $E(\varphi)$ is split exact.

Assume now that $E(\varphi)$ is split exact. Then there is a homotopy $\sigma=\{\sigma_n : E(\varphi)_n\to E(\varphi)_{n+1}\}_{n\in\mathbb{Z}}$ such that $\sigma\partial^E+\partial^E\sigma=1_\bullet$ on $E(\varphi)$. For any $n\in\mathbb{Z}$, we obtain by ((1), Proposition I.3.3) that $\sigma_n(c,d)=(\sigma^1_n(c,d),\sigma^2_n(c,d))$ for homomorphisms $\sigma^1_n : E(\varphi)_n\to C_n$ and $\sigma^2_n : E(\varphi)_n\to D_{n+1}$ (these are the projections onto the first and second coordinates of σ_n). For these, it now holds that

$$\begin{aligned}
(c,d)&=\sigma_{n-1}\partial^E_n(c,d)+\partial^E_{n+1}\sigma_n(c,d)\\
&=\sigma_{n-1}(-\partial^C_{n-1}(c),\varphi_{n-1}(c)+\partial^D_n(d))+\partial^E_{n+1}(\sigma^1_n(c,d),\sigma^2_n(c,d))\\
&=(\sigma^1_{n-1}(-\partial^C_{n-1}(c),\varphi_{n-1}(c)+\partial^D_n(d)),\sigma^2_{n-1}(-\partial^C_{n-1}(c),\varphi_{n-1}(c)+\partial^D_n(d)))\\
&\qquad+(-\partial^C_n\sigma^1_n(c,d),\varphi_n(\sigma^2_n(c,d))+\partial^D_{n+1}(\sigma^2_n(c,d))),
\end{aligned}$$

so

$$\begin{aligned}
c&\overset{(\dagger)}{=}\sigma^1_{n-1}(-\partial^C_{n-1}(c),\varphi_{n-1}(c)+\partial^D_n(d))-\partial^C_n\sigma^1_n(c,d)\quad\text{and}\\
d&=\sigma^2_{n-1}(-\partial^C_{n-1}(c),\varphi_{n-1}(c)+\partial^D_n(d))+\varphi_n(\sigma^2_n(c,d))+\partial^D_{n+1}(\sigma^2_n(c,d)).
\end{aligned}$$

These two equations hold for *any* $(c,d)\in C_{n-1}\oplus D_n$, $n\in\mathbb{Z}$. In particular, we can put $d=0$ in the first and $c=0$ in the second equation and use the properties of the direct sum (as well as raising n one degree in the first equation), so that

$$\begin{aligned}
c&=-\sigma^1_n(\partial^C_n(c),0)+\sigma^1_n(0,\varphi_n(c))-\partial^C_{n+1}\sigma^1_{n+1}(c,0)\quad\text{and}\\
d&=\sigma^2_{n-1}(0,\partial^D_n(d))+\varphi_n(\sigma^2_n(0,d))+\partial^D_{n+1}(\sigma^2_n(0,d))
\end{aligned}$$

for any $n\in\mathbb{Z}$, $c\in C_n$ and $d\in D_n$. These equations in fact reveal that φ is a homotopy equivalence: define $\psi : D_\bullet\to C_\bullet$, $\sigma'=\{\sigma'_n : C_n\to C_{n+1}\}_{n\in\mathbb{Z}}$ and $\sigma''=\{\sigma''_n : D_n\to D_{n+1}\}_{n\in\mathbb{Z}}$ by

$$\psi_n(d)=\sigma^1_n(0,d),\quad \sigma'_n(c)=-\sigma^1_{n+1}(c,0),\quad \sigma''_n(d)=\sigma^2_n(0,d)$$

for $n\in\mathbb{Z}$, $c\in C_n$ and $d\in D_{n-1}$. They are clearly homomorphisms, and from the above equations, it follows that $1_{C_n}-\psi_n\varphi_n=\sigma'_{n-1}\partial^C_n+\partial^C_{n+1}\sigma'_n$ and $1_{D_n}-\varphi_n\psi_n=\sigma''_{n-1}\partial^D_n+\partial^D_{n+1}\sigma''_n$ for all $n\in\mathbb{Z}$. Finally, ψ is a chain map: the equality marked (†) from earlier yields

$$0=\sigma^1_{n-1}(0,\partial^D_n(d))-\partial^C_n\sigma^1_n(0,d)=\psi_{n-1}\partial^D_n(d)-\partial^C_n\psi_n(d)$$

for all $d\in D_n$.

(d)

Prove that a morphism between bounded below chain complexes of projective modules is a quasi-isomorphism if and only if it is a homotopy equivalence.

Let $\varphi : P_\bullet \to Q_\bullet$ be a morphism between bounded below chain complex of projective modules. Thus there exist $m, n \in \mathbb{Z}$ such that $P_i = 0$ for $i < m$ and $Q_j = 0$ for $j < n$. In the manner of (a), we can construct the mapping cone $E(\varphi)$. For $i < m+1$, we have $P_{i-1} = 0$, so $E(\varphi)_i = P_{i-1} \oplus Q_i = 0$ for $i < M := \min\{m+1, n\}$, meaning that $E(\varphi)$ is a bounded below chain complex as well. Additionally, $E(\varphi)_i$ is projective for all $i \in \mathbb{Z}$ since it is a direct sum of projective modules ((1), Proposition I.4.5).

If φ is a homotopy equivalence, then $E(\varphi)$ is split exact, i.e. the identity morphism $1_\bullet$ on $E(\varphi)$ is homotopy equivalent to the zero map $0_\bullet$ on $E(\varphi)$. By ((1), Proposition IV.3.1), then for $n \in \mathbb{Z}$, we have $1_{H_n(E(\varphi))} = H_n(1_\bullet) = H_n(0_\bullet) = 0$ or $x = 0$ for all $x \in H_n(E(\varphi))$, meaning that $H_n(E(\varphi)) = 0$. Thus $E(\varphi)$ is exact (note that this holds for any split exact chain complex).

Assume now that $E(\varphi)$ is exact. Then the zero map from $E(\varphi)$ to the zero complex $0_\bullet$ is a surjective quasi-isomorphism, since $H_n(E(\varphi)) = 0$ for all $n \in \mathbb{Z}$. Now, in the diagram

$$\begin{array}{ccc} & & E(\varphi) \\ & \overset{\lambda}{\nearrow} & \big\downarrow 0 \\ E(\varphi) & \xrightarrow{0} & 0 \end{array}$$

there exists a morphism λ such that the diagram commutes, since $E(\varphi)$ is a bounded below chain complex of projective modules. Furthermore, λ is unique up to homotopy. But clearly, λ can be *any* chain map $E(\varphi) \to E(\varphi)$, so all morphisms $E(\varphi) \to E(\varphi)$ are homotopic. In particular, $1_\bullet \cong 0_\bullet$, so $E(\varphi)$ is split exact. Alas it follows from our assumptions that $E(\varphi)$ is exact if and only if it is split exact. (Note that this holds for *any* bounded below and projective chain complex.)

We can now prove the original statement, which follows from the following biimplications:

$$\varphi \text{ quasi-isomorphism} \overset{(b)}{\Leftrightarrow} E(\varphi) \text{ exact} \Leftrightarrow E(\varphi) \text{ split exact} \overset{(c)}{\Leftrightarrow} \varphi \text{ homotopy equivalence.}$$

(3)

Consider a commutative diagram of chain complexes with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C'_\bullet & \xrightarrow{\mu} & C_\bullet & \xrightarrow{\varepsilon} & C''_\bullet & \longrightarrow & 0 \\ & & \big\downarrow \alpha & & \big\downarrow \beta & & \big\downarrow \gamma & & \\ 0 & \longrightarrow & D'_\bullet & \xrightarrow{\mu'} & D_\bullet & \xrightarrow{\varepsilon'} & D''_\bullet & \longrightarrow & 0. \end{array}$$

(a)

Show that if two out of α, β, γ are quasi-isomorphisms, then so is the third.

The rows of short exact sequences of chain complexes induce long exact sequences in homology with connecting homomorphisms $\omega_n : H_n(C''_\bullet) \to H_{n-1}(C'_\bullet)$ and $\omega'_n : H_n(D''_\bullet) \to H_{n-1}(D'_\bullet)$ for all $n \in \mathbb{Z}$ by ((1), Theorem IV.2.1). Likewise, the morphisms α, β and γ induce homomorphisms in homology, yielding the diagram

$$\begin{array}{ccccccccccccc} \cdots & \longrightarrow & H_{n+1}(C''_\bullet) & \xrightarrow{\omega_{n+1}} & H_n(C'_\bullet) & \xrightarrow{\mu_n^*} & H_n(C_\bullet) & \xrightarrow{\varepsilon_n^*} & H_n(C''_\bullet) & \xrightarrow{\omega_n} & H_{n-1}(C'_\bullet) & \longrightarrow & \cdots \\ & & \big\downarrow \gamma_{n+1}^* & & \big\downarrow \alpha_n^* & & \big\downarrow \beta_n^* & & \big\downarrow \gamma_n^* & & \big\downarrow \alpha_{n-1}^* & & \\ \cdots & \longrightarrow & H_{n+1}(D''_\bullet) & \xrightarrow{\omega'_{n+1}} & H_n(D'_\bullet) & \xrightarrow{\mu_n'^*} & H_n(D_\bullet) & \xrightarrow{\varepsilon_n'^*} & H_n(D''_\bullet) & \xrightarrow{\omega'_n} & H_{n-1}(D'_\bullet) & \longrightarrow & \cdots . \end{array}$$

For $n \in \mathbb{Z}$, the two middle squares commute, since $\mu_n'^* \alpha_n^*([x]) = [\mu'_n \alpha_n(x)] = [\beta_n \mu_n(x)] = \beta_n^* \mu_n^*([x])$ for all $x \in \ker \partial_n^{C'}$ and $\varepsilon_n'^* \beta_n^*([x]) = [\varepsilon'_n \beta_n(x)] = [\gamma_n \varepsilon_n(x)] = \gamma_n^* \varepsilon_n^*([x])$ for all $x \in \ker \partial_n^C$, $\partial^{C'}$ and ∂^C denoting the differentials on the chain complexes $C'_\bullet$ and $C_\bullet$, respectively.

In fact, the outer squares commute as well, by recalling how the connecting homomorphisms are defined. Indeed, let $c'' \in \ker \partial_n^{C''}$. The image of $[c'']$ under ω_n arises by taking $c \in C_n$ such that $\varepsilon_n(c) = c''$ and then taking $c' \in C'_{n-1}$ such that $\mu_{n-1}(c') = \partial_n^C(c)$, whereupon $\omega_n([c'']) = [c']$ and

$\alpha^*_{n-1}\omega_n([c'']) = [\alpha_{n-1}(c')]$. In the same way, the image of $\gamma^*_n([c'']) = [\gamma_n(c'')]$ under ω'_n arises by taking $d \in D_n$ such that $\varepsilon'_n(d) = \gamma_n(c'')$ and then taking $d' \in D'_{n-1}$ such that $\mu'_{n-1}(d') = \partial^D_n(d)$, so $\omega'_{n-1}\gamma^*_n([c'']) = [d']$. What is important here is that the image of $\gamma_n(c'')$ under ω'_{n-1} is *independent* of the choices of d and (then) d' as long as they satisfy the above equalities. Hence we can take $d = \beta_n(c)$, since $\varepsilon'_n\beta_n(c) = \gamma_n\varepsilon_n(c) = \gamma_n(c'')$, and furthermore we can replace d' by $\alpha_{n-1}(c')$ since

$$\partial^D_n\beta_n(c) = \beta_{n-1}\partial^C_n(c) = \beta_{n-1}\mu_{n-1}(c') = \mu'_{n-1}\alpha_{n-1}(c').$$

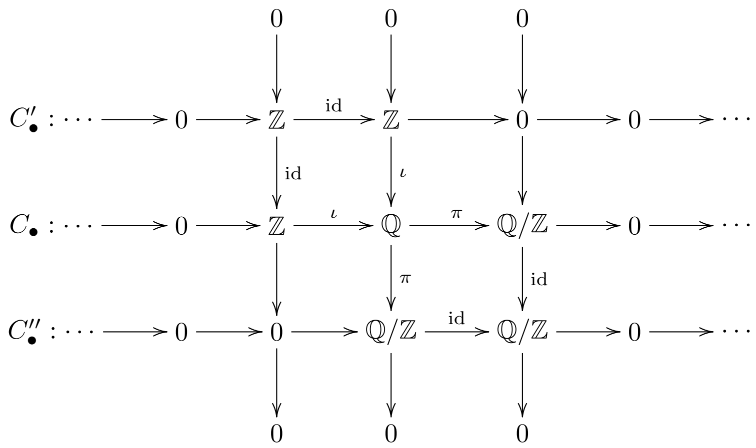
Alas $\omega'_{n-1}\gamma^*_n([c'']) = [\alpha_{n-1}(c')] = \alpha^*_{n-1}\omega_n([c''])$, so the outer squares commute.

Now assume that γ and α are quasi-isomorphisms. Then the induced homomorphisms γ^*_n and α^*_n on homology are isomorphisms for all $n \in \mathbb{Z}$, and since the diagram above is commutative with exact rows, it follows from the Five Lemma ((1), Exercise I.1.2) that the induced homomorphism β^*_n is an isomorphism for all $n \in \mathbb{Z}$, i.e. β is a quasi-isomorphism itself. The two other cases follow in the same manner by just shifting the above diagram one place to the left and to the right.

(b)

Is the same true if you replace 'quasi-isomorphism' by 'homotopy equivalence'? If yes, provide a proof, if no provide a counterexample.

It is *not* true the above statement holds if we replace 'quasi-isomorphism' by 'homotopy equivalence'. Consider the chain complexes connected by homomorphisms



where ι and π denote the inclusion and quotient homomorphisms respectively, and let $D'_\bullet$, $D_\bullet$ and $D''_\bullet$ be the zero complex. The above diagram is clearly commutative with exact rows and columns. Since $C'_\bullet$, $C_\bullet$ and $C''_\bullet$ are exact complexes, all their homology groups are zero, whereupon the zero maps $C'_\bullet \to D'_\bullet$, $C_\bullet \to D_\bullet$ and $C''_\bullet \to D''_\bullet$ are quasi-isomorphisms.

Now, the chain complexes $C'_\bullet$ and $C''_\bullet$ are in fact split exact. Indeed, the identity maps have inverses equal to themselves, and so the desired homotopies are families of the identity and the zero map, put in the right places. However, the chain complex $C_\bullet$ is not split exact. Indeed, assume that there exist homomorphisms $\sigma : \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}$ and $\sigma' : 0 \to \mathbb{Q}/\mathbb{Z}$ such that $\pi\sigma + \sigma' 0_{\mathbb{Q}/\mathbb{Z}} = 1_{\mathbb{Q}/\mathbb{Z}}$; then $\sigma' = 0$ and $\sigma = 0$ since all elements in $\mathbb{Q}/\mathbb{Z}$ have finite order and all elements in $\mathbb{Q}$ except for 0 have infinite order. Thus $1_{\mathbb{Q}/\mathbb{Z}} = 0_{\mathbb{Q}/\mathbb{Z}}$, clearly a falsehood, so there exists no splitting of $C_\bullet$. What good does this do, one may ask, but the answer comes in the following lemma:

Lemma 1. *A chain complex $C_\bullet$ is split exact if and only if the zero map $\varphi : C_\bullet \to 0$ is a homotopy equivalence.*

Proof. If $C_\bullet$ is split exact, then $1_{C_\bullet} \cong 0_{C_\bullet} = \psi\varphi$ and $1_{0_\bullet} = 0 = \varphi\psi$, ψ denoting the zero morphism $0 \to C_\bullet$ so that φ is a homotopy equivalence. If φ is a homotopy equivalence, there exists a morphisms $\psi : 0 \to C_\bullet$ (i.e. ψ is the zero map) and a homotopy $\sigma' = \{\sigma'_n : C_n \to C_{n+1}\}_{n\in\mathbb{Z}}$ such that $\sigma'_{n-1}\partial^C_n + \partial^C_{n+1}\sigma'_n = 1_{C_n} - \psi_n\varphi_n = 1_{C_n}$, so $C_\bullet$ is split exact. ∎

Alas in the commutative diagram of chain complexes with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C'_\bullet & \longrightarrow & C_\bullet & \xrightarrow{\varepsilon} & C''_\bullet & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \\ 0 & \longrightarrow & D'_\bullet & \longrightarrow & D_\bullet & \longrightarrow & D''_\bullet & \longrightarrow & 0. \end{array}$$

the zero maps $C'_\bullet \to D'_\bullet$ and $C''_\bullet \to D''_\bullet$ are homotopy equivalences, whereas the zero maps $C_\bullet \to D_\bullet$ is not. Thus the statement does not hold for homotopy equivalences.

(4)

(a)

Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be an adjunction (F left adjoint to G). Prove that if F preserves monomorphisms then G preserves injective objects.

By definition of left adjointness, there is a natural transformation $\eta = \eta_{X,Y} : \mathcal{D}(FX, Y) \to \mathcal{C}(X, GY)$ between the functors $\mathcal{D}(F(-_1), -_2), \mathcal{C}(-_1, G(-_2)) : \mathcal{C}^{opp} \times \mathcal{D} \to \textbf{Sets}$ that is an isomorphism for any objects X in $\mathcal{C}$ and Y in $\mathcal{D}$. Thus for any morphism $(f_1, f_2) \in \mathcal{C}^{opp} \times \mathcal{D}((X, Y), (X', Y')) = \mathcal{C}(X', X) \times \mathcal{D}(Y, Y')$, the diagram

$$\begin{array}{ccc} \mathcal{D}(FX, Y) & \xrightarrow[\cong]{\eta_{X,Y}} & \mathcal{C}(X, GY) \\ \downarrow{\scriptstyle \mathcal{D}(f_1,f_2)} & & \downarrow{\scriptstyle \mathcal{C}(f_1,f_2)} \\ \mathcal{D}(FX', Y') & \xrightarrow[\cong]{\eta_{X',Y'}} & \mathcal{C}(X', GY'). \end{array}$$

commutes, where $\mathcal{D}(f_1, f_2)(\omega) = f_2 \circ \omega \circ Ff_1$ for $\omega \in \mathcal{D}(FX, Y)$ and $\mathcal{C}(f_1, f_2)(\omega) = Gf_2 \circ \omega \circ f_1$ for $\omega \in \mathcal{C}(X, GY)$.

Let J be an injective object in $\mathcal{D}$ and consider the diagram

$$\begin{array}{ccc} & & GJ \\ & & \uparrow{\scriptstyle \varphi} \\ A & \xleftarrow[\mu]{} & B, \end{array}$$

where μ is a monomorphism. We will find a morphism $\rho : A \to GJ$ such that $\rho \circ \mu = \varphi$, so that GJ is injective and G preserves injective objects. By using the natural equivalence $\eta_{B,J}$, we obtain a diagram

$$\begin{array}{ccc} & & J \\ & & \uparrow{\scriptstyle \eta^{-1}_{B,J}(\varphi)} \\ FA & \xleftarrow[F\mu]{} & FB, \end{array}$$

where $F\mu$ is a monomorphism by the assumption that F preserves monomorphisms. Because J is injective, there exists $\rho' \in \mathcal{D}(FA, J)$ such that the diagram

$$\begin{array}{ccc} & \overset{\rho'}{\nearrow} & J \\ & & \uparrow{\scriptstyle \eta^{-1}_{B,J}(\varphi)} \\ FA & \xleftarrow[F\mu]{} & FB \end{array}$$

commutes. This yields a morphism $\rho = \eta_{A,J}(\rho') \in \mathcal{C}(A, GJ)$. Now since $G\mathrm{id}_J = \mathrm{id}_{GJ}$, we have

$$\rho \circ \mu = \eta_{A,J}(\rho') \circ \mu = (\mathcal{C}(\mu, \mathrm{id}_J) \circ \eta_{A,J})(\rho') = (\eta_{B,J} \circ \mathcal{D}(\mu, \mathrm{id}_J))(\rho') = \eta_{B,J}(\rho' \circ F\mu) = \varphi$$

by the properties of the natural equivalence η, so GJ is injective.

(b)

Prove that if A is a flat right Λ-module and D is a divisible abelian group then $\mathrm{Hom}_{\mathbb{Z}}(A, D)$ *is an injective left Λ-module.*

Since A is flat, the functor $A \otimes_\Lambda - : \mathfrak{M}^\ell_\Lambda \to \mathbf{Ab}$ preserves monomorphisms by definition, since it sends short exact sequences of left Λ-modules to exact sequences of abelian groups. Additionally, $A \otimes_\Lambda -$ is left adjoint to the functor $\mathrm{Hom}_{\mathbb{Z}}(A, -) : \mathbf{Ab} \to \mathfrak{M}^\ell_\Lambda$ by ((1), Proposition IV.7.2). By (a), $\mathrm{Hom}_{\mathbb{Z}}(A, -)$ preserves injective objects. Since D is a divisible $\mathbb{Z}$-module and $\mathbb{Z}$ is a PID, then by ((1), Theorem I.7.1), D is an injective abelian group, so $\mathrm{Hom}_{\mathbb{Z}}(A, D)$ thus becomes an injective left Λ-module.

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