Homological Algebra

Exercises

Rasmus Sylvester Bryder

November 4, 2011

2.6

We will find the following groups of homomorphisms of $\mathbb{Z}$-modules:

$\mathrm{Hom}(\mathbb{Z}, \mathbb{Z}_n) \simeq \mathbb{Z}_n$: Define a map $\alpha : \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}_n) \to \mathbb{Z}_n$ by $\alpha(\varphi) = \varphi(1)$. The map is a homomorphism by the definition of composition on homomorphism groups, and it's clearly surjective (for $k \in \mathbb{Z}$, define a homomorphism $\mathbb{Z} \to \mathbb{Z}_n$ by $n \to \overline{nk}$). Injectivity follows from the fact that if $\varphi(1) = \overline{0}$, then $\varphi(x) = x\varphi(1) = x\overline{0} = \overline{0}$, so $\varphi = 0$.

$\mathrm{Hom}(\mathbb{Z}_m, \mathbb{Z}) \simeq \{0\}$: A homomorphism $\varphi : \mathbb{Z}_m \to \mathbb{Z}$ satisfies $0 = \varphi(\overline{0}) = \varphi(m\overline{x}) = m\varphi(\overline{x})$ for all $x \in \mathbb{Z}$, and since $\varphi(\overline{x}) \in \mathbb{Z}$, then $\varphi(\overline{x}) = 0$ for all $x \in \mathbb{Z}$.

$\mathrm{Hom}(\mathbb{Z}_m, \mathbb{Z}_n) \simeq \mathbb{Z}_{(m,n)}$: We have an exact sequence $\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_m \to 0$ where the first map is multiplication by m and the second is the canonical mapping q. From Theorem 2.2, then if we consider $\mathbb{Z}_n$ as a $\mathbb{Z}$-module, this forms an exact sequence $0 \to \mathrm{Hom}(\mathbb{Z}_m, \mathbb{Z}_n) \to \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}_n) \to \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}_n)$, where the second map is q^*, induced by the projection q. q^* is injective, and so is α from the first thing we proved, so $\alpha \circ q^* : \mathrm{Hom}(\mathbb{Z}_m, \mathbb{Z}_n) \to \mathbb{Z}_n$ is a monomorphism, so that $\mathrm{Hom}(\mathbb{Z}_m, \mathbb{Z}_n)$ is isomorphic to a subgroup of $\mathbb{Z}_n$, and therefore cyclic. We just need to find the order, and for this, we will count the number of homomorphisms in $\mathrm{Hom}(\mathbb{Z}_m, \mathbb{Z}_n)$.

Every $\varphi \in \mathrm{Hom}(\mathbb{Z}_m, \mathbb{Z}_n)$ can be determined, whether or not well-defined, by the image of $\overline{1} \in \mathbb{Z}_m$. Let $\overline{a} \in \mathbb{Z}_n$ be $\varphi(\overline{1})$ where $0 \leq a < n$. Assuming that φ is well-defined, then $0 = \varphi(m\overline{1}) = \overline{ma}$, so $n \mid ma$. On the other hand, if $n \mid ma$, then for $\overline{x} = \overline{y}$ in $\mathbb{Z}_m$, we have $m \mid x - y$, so $n \mid xa - xy$, so $\varphi(\overline{x}) = \varphi(x\overline{1}) = \overline{ax} = \overline{ay} = \varphi(y\overline{1}) = \varphi(\overline{y})$. Alas φ is well-defined exactly when $n \mid ma$.

If $n \mid ma$, then $ma = rn$ for some $r \in \mathbb{Z}$. Letting $\tilde{m} = m/(m,n)$ and $\tilde{n} = n/(m,n)$, then $(\tilde{m}, \tilde{n}) = 1$ since all the common divisors of m and n were factorized away. We have $\tilde{m}a = r\tilde{n}$, so $\tilde{n} \mid a$. Then $a \in \{0, \tilde{n}, ((m,n) - 1)\tilde{n}\}$ by our restriction of a; conversely, if a is one of these, then $ma = mk\tilde{n} = k\tilde{m}n$ for some $k \in \{0, \ldots, (m,n) - 1\}$. Thus φ is a well-defined homomorphism exactly when $a \in \{0, \tilde{n}, ((m,n) - 1)\tilde{n}\}$, so we have (m,n) choices; since $\mathrm{Hom}(\mathbb{Z}_m, \mathbb{Z}_n)$ was cyclic, we obtain $\mathrm{Hom}(\mathbb{Z}_m, \mathbb{Z}_n) \simeq \mathbb{Z}_{(m,n)}$.

$\mathrm{Hom}(\mathbb{Q}, \mathbb{Z}) = \{0\}$: Assume first that $\varphi \in \mathrm{Hom}(\mathbb{Q}, \mathbb{Z})$ is injective. Then $\varphi \neq 0$, since otherwise φ wouldn't be injective. Take $k \in \mathbb{Q}$ such that $\varphi(k) \neq 0$. By scaling, we obtain that φ is surjective. Therefore φ is an isomorphism which is a contradiction, since any map $\mathbb{Z} \to \mathbb{Q}$ is not surjective. Therefore φ has non-trivial kernel. Therefore we can take $a/b \in \ker \varphi$ such that $a \neq 0$, $b \neq 0$. Now $\varphi(a) = b\varphi(a/b) = 0$, and $a\varphi(1) = \varphi(a) = 0$, so $\varphi(1) = 0$. For any $p/q \in \mathbb{Q}$, we then have $q\varphi(1/q) = \varphi(1) = 0$, so $\varphi(1/q) = 0$, and therefore $\varphi(p/q) = p\varphi(1/q) = 0$, so $\varphi = 0$.

$\mathrm{Hom}(\mathbb{Q}, \mathbb{Q}) \simeq \mathbb{Q}$: For $q \neq 0$ and $\varphi \in \mathrm{Hom}(\mathbb{Q}, \mathbb{Q})$, we have $\varphi(1) = q\varphi(1/q)$. This equation has a unique rational solution if we put $\varphi(1)$ equal to some rational number, and thus $\varphi(1/q) = \varphi(1)/q$. Therefore $\varphi(p/q) = (p\varphi(1))/q$, so φ is uniquely determined by what its image of 1 is. Thus, if we define a homomorphism $\mathrm{Hom}(\mathbb{Q}, \mathbb{Q}) \to \mathbb{Q}$ by $\varphi \mapsto \varphi(1)$, it is injective. It is also surjective since we can define a homomorphism by $\varphi(x) = kx$ for any $k \in \mathbb{Q}$ and all $x \in \mathbb{Q}$, and so we obtain what we desired.