Introduction to operator algebra

Homework assignment # 1

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Exercise 1

Let $\mathcal{H}$ be an infinite-dimensional Hilbert space with orthonormal basis $(e_n)_{n\in\mathbb{Z}}$.

(i)

Justify that there exists an operator U in $B(\mathcal{H})$ such that $Ue_n = e_{n-1}$ for all $n \in \mathbb{Z}$. Show that U is unitary, i.e. that $U^*U = I = UU^*$.

First a lemma.

Lemma 1. *Let $\mathcal{H}$ be a Hilbert space with an orthonormal basis $(e_n)_{n\in\mathbb{Z}}$. If two bounded operators $S, T \in B(\mathcal{H})$ satisfy $\langle Sx, e_n\rangle = \langle Tx, e_n\rangle$ for all $n \in \mathbb{Z}$, then $S = T$.*

Proof. Let $x \in \mathcal{H}$. For any $y \in \mathcal{H}$, we have $\sum_{n=-\infty}^{\infty}\langle y, e_n\rangle e_n$ and thus continuity of the inner product yields

$$\begin{aligned}\langle Sx, y\rangle &= \left\langle Sx, \sum_{n=-\infty}^{\infty}\langle y, e_n\rangle e_n\right\rangle = \sum_{n=-\infty}^{\infty}\overline{\langle y, e_n\rangle}\,\langle Sx, e_n\rangle = \sum_{n=-\infty}^{\infty}\overline{\langle y, e_n\rangle}\,\langle Tx, e_n\rangle \\ &= \sum_{n=-\infty}^{\infty}\langle Tx, \langle e_n, y\rangle e_n\rangle = \left\langle Tx, \sum_{n=-\infty}^{\infty}\langle y, e_n\rangle e_n\right\rangle = \langle Tx, y\rangle\end{aligned}$$

and thus $Sx = Tx$ by properties of the inner product. □

Note first that for any $x \in \mathcal{H}$ and $m \in \mathbb{N}_0$,

$$\sum_{n=-m}^{m}|\langle x, e_{n+1}\rangle|^2 = \sum_{n=-m+1}^{m+1}|\langle x, e_n\rangle|^2 \le \sum_{n=-\infty}^{\infty}|\langle x, e_n\rangle|^2 = \|x\|^2 < \infty \tag{1}$$

by Parseval's identity. Thus the series $\sum_{n=-\infty}^{\infty}\langle x, e_{n+1}\rangle e_n$ converges in $\mathcal{H}$ for all $x \in \mathcal{H}$, and we define $U : \mathcal{H} \to \mathcal{H}$ by

$$U(x) = \sum_{n=-\infty}^{\infty}\langle x, e_{n+1}\rangle e_n.$$

By linearity of the inner product on $\mathcal{H}$, we see that

$$\sum_{n=-m}^{m}(\lambda_1\langle x_1, e_{n+1}\rangle + \lambda_2\langle x_2, e_{n+1}\rangle)e_n = \lambda_1\sum_{n=-m}^{m}\langle x_1, e_{n+1}\rangle e_n + \lambda_2\sum_{n=-m}^{m}\langle x_2, e_{n+1}\rangle e_n$$

for all $m \in \mathbb{N}$; as both sides converge in $\mathcal{H}$ for $m \to \infty$, we obtain that U is linear. From (1), it is seen that U is bounded, so $U \in B(\mathcal{H})$. Finally, for $n \in \mathbb{Z}$,

$$Ue_n = \sum_{k=-\infty}^{\infty}\langle e_n, e_{k+1}\rangle e_k = \langle e_n, e_n\rangle e_{n-1} = e_{n-1},$$

as $\langle e_n, e_{k+1}\rangle = 0$ for $k \ne n-1$.

Let $x \in \mathcal{H}$. For $n \in \mathbb{Z}$, note that $U^*e_n = e_{n+1}$ for all $n \in \mathbb{Z}$, since

$$\begin{aligned}\langle U^*e_n, x\rangle &= \langle e_n, Ux\rangle = \left\langle e_n, \sum_{k=-\infty}^{\infty} \langle x, e_{k+1}\rangle e_k \right\rangle = \sum_{k=-\infty}^{\infty} \overline{\langle x, e_{k+1}\rangle}\langle e_n, e_k\rangle \\ &= \overline{\langle x, e_{n+1}\rangle}\langle e_n, e_n\rangle = \langle e_{n+1}, x\rangle,\end{aligned}$$

by continuity and linearity of the inner product, as well as the fact that $\langle e_k, e_{n-1}\rangle = 0$ for $k \neq n-1$. Hence $UU^*e_n = e_n = U^*Ue_n$ for all $n \in \mathbb{Z}$, so $\langle UU^*x, e_n\rangle = \langle x, UU^*e_n\rangle = \langle x, e_n\rangle$ and $\langle U^*Ux, e_n\rangle = \langle x, UU^*e_n\rangle = \langle x, e_n\rangle$ for all $n \in \mathbb{Z}$, and therefore Lemma 1 yields $UU^* = I = U^*U$. Hence U is unitary.

(ii)

Show that $\sigma(U) = \mathbb{T}$.

From ((1), Proposition 8.2), the unitary operator U – contained in the C^*-algebra $B(\mathcal{H})$ – has spectrum contained in $\mathbb{T}$. Hence it suffices to show that every $\lambda \in \mathbb{T}$ belongs to $\sigma(U)$.

Let $\lambda \in \mathbb{T}$ and define the sequence

$$x_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \lambda^i e_i, \quad n \in \mathbb{N}.$$

Then for all $n \in \mathbb{N}$,

$$\|x_n\|^2 = \frac{1}{n}\left\|\sum_{i=1}^{n} \lambda^i e_i\right\|^2 = \frac{1}{n}\sum_{i=1}^{n}\|\lambda^i e_i\|^2 = \sum_{i=1}^{n}\frac{1}{n}|\lambda|^{2i}\|e_i\|^2 = \sum_{i=1}^{n}\frac{1}{n} = 1$$

by Pythagoras' theorem, so $\|x_n\| = 1$. Also,

$$Ux_n - \lambda x_n = \frac{1}{\sqrt{n}}\left(\sum_{i=1}^{n}\lambda^i e_{i-1} - \sum_{i=1}^{n}\lambda^{i+1}e_i\right) = \frac{1}{\sqrt{n}}\left(\sum_{i=0}^{n-1}\lambda^{i+1}e_i - \sum_{i=1}^{n}\lambda^{i+1}e_i\right) = \frac{1}{\sqrt{n}}(e_0 - \lambda^{n+1}e_n),$$

so

$$\|(U-\lambda I)x_n\|^2 = \frac{1}{n}\|e_0 - \lambda^{n+1}e_n\|^2 = \frac{1}{n}(\|e_0\|^2 + |\lambda|^{2(n+1)}\|e_n\|^2) = \frac{2}{n}$$

by Pythagoras' theorem, so $\|(U-\lambda I)x_n\| \to 0$ for $n \to \infty$. Assuming that $\lambda \notin \sigma(U)$, then the operator $U - \lambda I$ is invertible in $B(\mathcal{H})$ with an inverse T with $\|T\| > 0$, so

$$\|(U-\lambda I)x_n\| = \frac{1}{\|T\|}\|T\|\|(U-\lambda I)x_n\| \geq \frac{1}{\|T\|}\|T(U-\lambda I)x_n\| = \frac{1}{\|T\|}\|x_n\| = \frac{1}{\|T\|}$$

for all $n \in \mathbb{N}$, contradicting that $\|(U-\lambda I)x_n\| \to 0$ for $n \to \infty$. Hence $\lambda \in \sigma(\mathbb{T})$, and this proves $\sigma(U) = \mathbb{T}$.

(iii)

Consider now the case where $\mathcal{H} = L^2(-\pi,\pi)$ *and* $e_n(t) = (2\pi)^{-1/2}e^{int}$, $t \in [-\pi,\pi]$. *It is well-known that* $\{e_n\}_{n\in\mathbb{Z}}$ *is an orthonormal basis for* $\mathcal{H}$. *Let* U *be defined as in (i). Show that* $U = M_g$ *for a suitable (bounded) continuous function* $g : (-\pi,\pi) \to \mathbb{C}$, *where* M_g *is the multiplication operator defined by* $M_g f = gf$ *for all* $f \in \mathcal{H}$*; and identify* g.

Define $g : (-\pi,\pi) \to \mathbb{C}$ by $g(t) = e^{-it}$. g is clearly continuous and bounded, as $|g(t)| = 1$ for all $t \in (-\pi,\pi)$. For any $f \in \mathcal{H}$, then since

$$\int_{-\pi}^{\pi}|g(t)f(t)|^2\mathrm{d}t = \int_{-\pi}^{\pi}|g(t)|^2|f(t)|^2\mathrm{d}t = \int_{-\pi}^{\pi}|f(t)|^2\mathrm{d}t = \|f\|^2 < \infty,$$

we have $M_g f = gf \in \mathcal{H}$. Since $U^*e_n = e_{n+1}$ for all $n \in \mathbb{Z}$, then

$$\langle Uf, e_n\rangle = \langle f, e_{n+1}\rangle = \int_{-\pi}^{\pi} f(t)\overline{\frac{1}{\sqrt{2\pi}}e^{i(n+1)t}}\mathrm{d}t = \int_{-\pi}^{\pi} f(t)\overline{\frac{1}{\sqrt{2\pi}}e^{int}e^{it}}\mathrm{d}t = \int_{-\pi}^{\pi} e^{-it}f(t)\overline{\frac{1}{\sqrt{2\pi}}e^{int}}\mathrm{d}t.$$

Thus $\langle Uf, e_n\rangle = \langle M_g f, e_n\rangle$ for all $n \in \mathbb{Z}$, and therefore $U = M_g$ with g as defined above.

Exercise 2

Let $\mathcal{A}$ be a unital Banach algebra, and let $x, y \in \mathcal{A}$ be such that $xy = 1_{\mathcal{A}}$ and $yx \neq 1_{\mathcal{A}}$.

(i)

Let $z \in \mathcal{A}$ be such that $\|x - z\| < \|y\|^{-1}$. Show that z is not invertible. Conclude that

$$\operatorname{dist}(x, G(\mathcal{A})) \geq \|y\|^{-1}.$$

Note that x and y are non-zero elements. Since

$$\|1_{\mathcal{A}} - zy\| = \|xy - zy\| = \|(x - z)y\| \leq \|x - z\|\|y\| < \|y\|^{-1}\|y\| = 1,$$

we obtain from ((1), Proposition 2.1), that $zy = 1_{\mathcal{A}} - (1_{\mathcal{A}} - zy)$ is invertible. If z were invertible, $y = z^{-1}(zy)$ would be invertible and have an inverse w, but then

$$yx = (yx)(yw) = y(xy)w = yw = 1_{\mathcal{A}},$$

contradicting the original assumption. Thus z is not invertible.

Hence for all invertible elements $z \in G(\mathcal{A})$, we have the inequality $\|x - z\| \geq \|y\|^{-1}$. Thus

$$\operatorname{dist}(x, G(\mathcal{A})) = \inf_{z \in G(\mathcal{A})} \|x - z\| \geq \|y\|^{-1}.$$

(ii)

Give an example of a unital Banach algebra $\mathcal{A}$ and elements $x, y \in \mathcal{A}$ such that $xy = 1_{\mathcal{A}}$ and $yx \neq 1_{\mathcal{A}}$.

Let $\mathcal{A} = B(\ell_2(\mathbb{N})) = B(\ell^2)$ be the Banach algebra of bounded linear operators $\ell^2 \to \ell^2$ with composition as its product and define $x : \ell^2 \to \ell^2$ and $y : \ell^2 \to \ell^2$ by

$$x(s) = (s_2, s_3, s_4, \ldots), \quad y(s) = (0, s_1, s_2, s_3, \ldots), \quad s = (s_n)_{n \in \mathbb{N}} \in \ell^2.$$

These maps are clearly well-defined, since $\|x(s)\|_2^2 = \sum_{n=2}^{\infty} |s_n|^2 \leq \sum_{n=1}^{\infty} |s_n|^2 = \|s\|_2^2 < \infty$ and $\|y(s)\|_2^2 = 0 + \sum_{n=2}^{\infty} |s_{n-1}|^2 = \sum_{n=1}^{\infty} |s_n|^2 = \|s\|_2^2 < \infty$ for all $s = (s_n) \in \ell^2$. It is also clear that they are both linear, as, in the case of x, we have

$$x(\lambda_1 s + \lambda_2 t) = (\lambda_1 s_2 + \lambda_2 t_2, \lambda_1 s_3 + \lambda_2 t_3, \ldots) = \lambda_1(s_2, s_3, \ldots) + \lambda_2(t_2, t_3, \ldots) = \lambda_1 x(s) + \lambda_2 x(t)$$

for all $s = (s_n)$, $t = (t_n) \in \ell^2$ and $\lambda_1, \lambda_2 \in \mathbb{C}$. Since $\|x(s)\|_2 \leq \|s\|_2$ and $\|y(s)\|_2 = \|s\|_2$ as proven above, we conclude that $x, y \in \mathcal{A}$; x and y are the so-called *shift operators* on ℓ^2. As

$$xy(s) = x(y(s)) = x((0, s_1, s_2, s_3, \ldots)) = (s_1, s_2, s_3, \ldots) = s$$

for all $s = (s_n) \in \ell^2$, we have $xy = 1_{\mathcal{A}}$. However, letting $e_1 = (1, 0, 0, \ldots) \in \ell^2$, we note that

$$yx(e_1) = y(x(e_1)) = y((0, 0, \ldots)) = (0, 0, \ldots),$$

so that yx is not the identity operator $\ell^2 \to \ell^2$ and thus $yx \neq 1_{\mathcal{A}}$.

Exercise 3

((1), Exercise 6.7) Suppose $\mathcal{A}$ is a Banach algebra and x is a non-zero element in $\mathcal{A}$. Let $\mathcal{B}$ be the closed subalgebra generated by 1 and x. Show that $\mathcal{B}$ is commutative and that the maximal ideal space (the space of characters $\Delta(\mathcal{B})$) of $\mathcal{B}$ is homeomorphic to $\sigma_{\mathcal{B}}(x)$.

The closure of any subalgebra is a subalgebra and therefore a Banach algebra with norm inherited from the overlying Banach algebra. The subalgebra $\mathcal{X}$ generated by 1 and x is the intersection of all subalgebras containing 1 and x. Since all polynomials in x with complex coefficients, i.e. elements of the form

$$\sum_{k=0}^{n} \lambda_k x^k, \quad n \in \mathbb{N}_0, \ (\lambda_k)_{k=0}^{n} \subseteq \mathbb{C},$$

where $x^0 = 1$, are contained in any such subalgebra, and the set of these polynomials actually *is* a subalgebra, we obtain that $\mathcal{X}$ is the set of all polynomials in x. All these polynomials commute; it follows from the formula

$$\left(\sum_{k=0}^{n} \lambda_k x^k\right)\left(\sum_{l=0}^{n'} \lambda'_l x^l\right) = \sum_{j=0}^{n+n'} \left(\sum_{k+l=j} \lambda_k \lambda'_l\right) x^j, \quad n, n' \in \mathbb{N}_0, \quad (\lambda_k)_{k=0}^{n}, (\lambda'_l)_{l=0}^{n'} \subseteq \mathbb{C}.$$

For all $y, z \in \mathcal{B}$, then $y = \lim_n y_n$ and $z = \lim_n z_n$ for sequences of polynomials (y_n) and (z_n) in $\mathcal{X}$; by continuity of the product in $\mathcal{A}$, $yz = \lim_n y_n z_n = \lim_n z_n y_n = zy$. Hence $\mathcal{B}$ is a unital commutative Banach algebra.

By ((1), Corollary 6.2),

$$\sigma_{\mathcal{B}}(x) = \{\Gamma(x)(\varphi) \mid \varphi \in \Delta(\mathcal{B})\} = \{\varphi(x) \mid \varphi \in \Delta(\mathcal{B})\},$$

$\Delta(\mathcal{B})$ being the space of characters on $\mathcal{B}$. This inspires us to define a map $\Phi : \Delta(\mathcal{B}) \to \sigma_{\mathcal{B}}(x)$ given by $\Phi(\varphi) = \varphi(x)$, and the above equalities of sets immediately yield surjectivity of Φ. If $\varphi_1(x) = \varphi_2(x)$ for $\varphi_1, \varphi_2 \in \Delta(\mathcal{B})$, then clearly

$$\varphi_1\left(\sum_{k=0}^{n} \lambda_k x^k\right) = \sum_{k=0}^{n} \lambda_k \varphi_1(x)^k = \sum_{k=0}^{n} \lambda_k \varphi_2(x)^k = \varphi_2\left(\sum_{k=0}^{n} \lambda_k x^k\right), \quad n \in \mathbb{N}_0, \ (\lambda_k)_{k=0}^{n} \subseteq \mathbb{C}$$

by φ_1 and φ_2 being characters (and $\varphi_i(x^0) = \varphi_i(1) = 1 = \varphi_i(x)^0$, $i = 1, 2$). Thus $\varphi_1(y) = \varphi_2(y)$ for $y \in \mathcal{X}$. As all characters on unital Banach algebras are continuous ((1), Proposition 4.1), then for $z \in \mathcal{B}$, $z = \lim_n z_n$ for some sequence (z_n) in $\mathcal{X}$ and thus

$$\varphi_1(z) = \varphi_1(\lim_n z_n) = \lim_n \varphi_1(z_n) = \lim_n \varphi_2(z_n) = \varphi_2(\lim_n z_n) = \varphi_2(z).$$

Therefore $\varphi_1 = \varphi_2$, and thus Φ is injective.

If the net $(\varphi_\lambda)_{\lambda \in \Lambda}$ of $\Delta(\mathcal{B})$ converges to φ, then since the topology on $\Delta(\mathcal{B})$ is the weak* topology, $\varphi_\lambda(z) \to \varphi(z)$ in $\mathbb{C}$ for all $z \in \mathcal{B}$; in particular,

$$\Phi(\varphi_\lambda) = \varphi_\lambda(x) \to \varphi(x) = \Phi(\varphi),$$

so Φ is continuous. Thus $\Phi : \Delta(\mathcal{B}) \to \sigma_{\mathcal{B}}(x)$ is a continuous bijection, and as $\Delta(\mathcal{B})$ is compact with the weak* topology ((1), Proposition 4.2) and $\sigma_{\mathcal{B}}(x)$ is Hausdorff (being a subspace of $\mathbb{C}$), a result from general topology yields that Φ is a homeomorphism, and we are done.

Exercise 4

Let $H^\infty(\mathbb{D})$ be the algebra of bounded holomorphic functions defined on the open unit disc $\mathbb{D}$ and let $A(\mathbb{D})$ be the set of all $f \in C(\overline{\mathbb{D}})$ which are holomorphic on $\mathbb{D}$. Notice that $A(\mathbb{D}) \subseteq H^\infty(\mathbb{D})$. Put

$$f(z) = \exp\left(\frac{z+1}{z-1}\right), \quad z \in \mathbb{D}.$$

(i)

Show that $\mathrm{Re}(\frac{z+1}{z-1}) < 0$ *for all* $z \in \mathbb{D}$.

For $z \in \mathbb{D}$, write $z = a + ib$, $a, b \in \mathbb{R}$. Then we have

$$\frac{z+1}{z-1} = \frac{(a+1)+ib}{z-1} = \frac{((a+1)+ib)((a-1)-ib)}{|z-1|^2} = \frac{(a+1)(a-1)+b^2+i(b(a-1)-b(a+1))}{|z-1|^2}.$$

Therefore

$$\mathrm{Re}\left(\frac{z+1}{z-1}\right) = \frac{(a+1)(a-1)+b^2}{|z-1|^2} = \frac{a^2+b^2-1}{|z-1|^2} = \frac{|z|^2-1}{|z-1|^2} < 0,$$

because $|z|^2 < 1$.

(ii)

Show that $f \in H^\infty(\mathbb{D})$.

f is clearly holomorphic, being a composition of the entire function exp and the rational function $z \mapsto \frac{z+1}{z-1}$, defined and holomorphic on $\mathbb{D}$. From (a), we obtain that

$$|f(z)| = \left|\exp\left(\frac{z+1}{z-1}\right)\right| = \exp\left(\operatorname{Re}\left(\frac{z+1}{z-1}\right)\right) < e^0 = 1$$

for all $z \in \mathbb{D}$, exp as a real function being strictly increasing. Hence $\|f\|_\infty \leq 1$, so f is bounded, and we conclude that $f \in H^\infty(\mathbb{D})$.

(iii)

Show that $f \notin A(\mathbb{D})$.

We will show that it is not possible to extend f to a continuous function $\overline{\mathbb{D}} \to \mathbb{C}$. Assume for contradiction that $g : \overline{\mathbb{D}} \to \mathbb{C}$ is a continuous function satisfying $g|_{\mathbb{D}} = f$, and define another continuous extension $h : \overline{\mathbb{D}} \setminus \{1\} \to \mathbb{C}$ of f by

$$h(z) = \exp\left(\frac{z+1}{z-1}\right), \quad z \in \overline{\mathbb{D}} \setminus \{1\}.$$

For any $z \in \overline{\mathbb{D}} \setminus \{1\}$, there exists a sequence (z_n) of complex numbers in $\mathbb{D}$ converging to z. By continuity of g, we must have $g(z_n) \to g(z)$ for $n \to \infty$. However, we also have $g(z_n) = h(z_n) \to h(z)$ by continuity of h, so

$$g(z) = h(z) = \exp\left(\frac{z+1}{z-1}\right), \quad z \in \overline{\mathbb{D}} \setminus \{1\}$$

by $\mathbb{C}$ being Hausdorff. For $z \in \partial\mathbb{D} \setminus \{1\}$, we then have

$$|g(z)| = \exp\left(\operatorname{Re}\left(\frac{z+1}{z-1}\right)\right) = \exp\left(\frac{|z|^2 - 1}{|z-1|^2}\right) = \exp(0) = 1,$$

as $|z|^2 = 1$. As the composite function $|g|$ of g with the absolute value is continuous at $z = 1$, then by taking a sequence (z_n) of $\partial\mathbb{D} \setminus \{1\}$ converging to 1, we obtain $|g(z_n)| \to |g(1)|$; since the sequence $(|g(z_n)|)$ is constantly equal to 1, then $|g(1)| = 1$, once again by $\mathbb{C}$ being Hausdorff. We conclude that $|g(z)| = 1$ for all $z \in \partial\mathbb{D}$.

Now, define $t_n = 1 - \frac{1}{n}$ for $n \in \mathbb{N}$, and note that

$$|g(t_n)| = \exp\left(\frac{t_n + 1}{t_n - 1}\right) = \exp\left(\frac{2 - \frac{1}{n}}{-\frac{1}{n}}\right) = \exp(1 - 2n) \to 0$$

for $n \to \infty$. As $t_n \to 1$ for $n \to \infty$ in $\overline{\mathbb{D}}$, then continuity of $|g|$ at $z = 1$ yields $|g(t_n)| \to |g(1)|$, so $|g(1)| = 0$ by $\mathbb{C}$ being Hausdorff. This contradicts the fact that $|g(z)| = 1$ for all $z \in \partial\mathbb{D}$ as proven above, and thus there exists no continuous extension of f to $\overline{\mathbb{D}}$.

(It follows from (ii) and (iii) that $H^\infty(\mathbb{D}) \neq A(\mathbb{D})$.)

Exercise 5

In this exercise we shall explain what it means that condition (iv) in Theorem 6.4 (the version given in Week Sheet 4) is a universal property.

Let Ω be a locally compact Hausdorff space, let $\beta\Omega$ be its Stone-Čech compactification, and let $\iota : \Omega \to \beta\Omega$ be the inclusion mapping (as in the version of Theorem 6.4 given in Week Sheet 4). (Often times one suppresses ι and writes $\Omega \subseteq \beta\Omega$.)

Let $(K, j : \Omega \to K)$ be another compactification of Ω, that is, (i), (ii) and (iii) in Theorem 6.4 are satisfied with K in the place of $\beta\Omega$ and j in the place of ι.

(i)

Show that there exists a unique continuous surjection $\rho : \beta\Omega \to K$ such that $\rho \circ \iota = j$ (as mappings $\Omega \to K$).

Define $\varphi : C(K) \to C_b(\Omega)$ by $\varphi(\xi)(\omega) = \xi(j(\omega))$; it is well-defined, since $\xi \circ j$ is a composition of two continuous maps and thus continuous; it is also bounded, as

$$|\xi(j(\omega))| \leq \sup_{\kappa \in K} |\xi(\kappa)| < \infty$$

for all $\omega \in \Omega$, by K being compact and ξ being continuous, so that ξ has compact image in $\mathbb{C}$. By (iv) in Theorem 6.4, $\varphi(\xi)$ has a (unique) continuous extension to a continuous function $\overline{\xi} : \beta\Omega \to \mathbb{C}$ such that $\overline{\xi} \circ \iota = \varphi(\xi)$. If $\xi_1 = \xi_2$ for $\xi_1, \xi_2 \in C(K)$, then clearly $\varphi(\xi_1) = \varphi(\xi_2)$. Both extend to continuous functions $\overline{\xi_1}, \overline{\xi_2} : \beta\Omega \to \mathbb{C}$, and they are in fact identical: for $x \in \beta\Omega$, $x = \lim_\lambda \iota(\omega_\lambda)$ for some net $(\omega_\lambda)_{\lambda \in \Lambda} \subseteq \Omega$, yielding

$$\overline{\xi_1}(x) = \lim_\lambda \overline{\xi_1}(\iota(\omega_\lambda)) = \lim_\lambda \varphi(\xi_1)(\omega_\lambda) = \lim_\lambda \varphi(\xi_2)(\omega_\lambda) = \lim_\lambda \overline{\xi_2}(\iota(\omega_\lambda)) = \overline{\xi_2}(x)$$

by continuity. Defining $\Phi(\xi) = \overline{\xi}$ for $\xi \in C(K)$, we obtain a well-defined map $\Phi : C(K) \to C(\beta\Omega)$.

We now prove that Φ is in fact a unital algebra homomorphism. Let $\lambda \in \mathbb{C}$ and $\xi_1, \xi_2 \in C(K)$. It is clear from the definition that $\varphi(\xi_1 + \xi_2) = \varphi(\xi_1) + \varphi(\xi_2)$, $\varphi(\lambda\xi_1) = \lambda\varphi(\xi_1)$ and $\varphi(\xi_1\xi_2) = \varphi(\xi_1)\varphi(\xi_2)$, using how the algebra operations in $C_b(\Omega)$ are defined. Once again we have $x = \lim_\lambda \iota(\omega_\lambda)$ for some net $(\omega_\lambda)_{\lambda \in \Lambda} \subseteq \Omega$ for a given $x \in \beta\Omega$, and so

$$\begin{aligned} \Phi(\xi_1\xi_2)(x) &= \lim_\lambda \Phi(\xi_1\xi_2)(\iota(\omega_\lambda)) = \lim_\lambda \varphi(\xi_1\xi_2)(\omega_\lambda) = \lim_\lambda \varphi(\xi_1)(\omega_\lambda)\varphi(\xi_2)(\omega_\lambda) \\ &= \lim_\lambda \Phi(\xi_1)(\iota(\omega_\lambda))\Phi(\xi_2)(\iota(\omega_\lambda)) = \lim_\lambda (\Phi(\xi_1)\Phi(\xi_2))(\iota(\omega_\lambda)) = (\Phi(\xi_1)\Phi(\xi_2))(x), \end{aligned}$$

using that the product of the two continuous functions $\Phi(\xi_1)$ and $\Phi(\xi_2)$ is continuous. Similarly one can prove that $\Phi(\xi_1 + \xi_2) = \Phi(\xi_1) + \Phi(\xi_2)$ and $\Phi(\lambda\xi_1) = \lambda\Phi(\xi_1)$. Finally,

$$\Phi\left(1_{C(K)}\right)(x) = \lim_\lambda \Phi\left(1_{C(K)}\right)(\iota(\omega_\lambda)) = \lim_\lambda \varphi\left(1_{C(K)}\right)(\omega_\lambda)) = \lim_\lambda 1 = 1,$$

so $\Phi\left(1_{C(K)}\right) = 1_{C(\beta\Omega)}$, and we conclude that Φ is a unital homomorphism.

We now prove that Φ is injective (one could say quite straightforwardly). Assume that $\Phi(\xi_1) = \Phi(\xi_2)$ for $\xi_1, \xi_2 \in C(K)$; this implies $\Phi(\xi_1)(x) = \Phi(\xi_2)(x)$ for all $x \in \beta\Omega$ which in turn implies

$$\xi_1(j(\omega)) = \varphi(\xi_1)(\omega) = \Phi(\xi_1)(\iota(\omega)) = \Phi(\xi_2)(\iota(\omega)) = \varphi(\xi_2)(\omega) = \xi_2(j(\omega))$$

for all $\omega \in \Omega$. As $j(\Omega)$ is dense in K, we obtain by continuity of ξ_1 and ξ_2 that $\xi_1(\kappa) = \xi_2(\kappa)$ for all $\kappa \in K$. Thus $\xi_1 = \xi_2$.

From Exercise 2 and Exercise 1 from Week Sheet 3, we now obtain a continuous surjection $\rho : \beta\Omega \to K$ between compact Hausdorff spaces such that $\Phi(\xi) = \xi \circ \rho$ for all $\xi \in C(K)$.

By Urysohn's lemma, then since K is compact Hausdorff and thus normal, $C(K)$ separates points in K: for two points $\kappa_1 \neq \kappa_2$, there exists a continuous function $\xi : K \to [0,1] \subseteq \mathbb{C}$ such that $\xi(\kappa_1) \neq \xi(\kappa_2)$. Since $\xi(\rho(\iota(\omega))) = \Phi(\xi)(\iota(\omega)) = \varphi(\xi)(\omega) = \xi(j(\omega))$ for *all* $\xi \in C(K)$ and $\omega \in \Omega$, we can therefore conclude that $\rho(\iota(\omega)) = j(\omega)$ for all $\omega \in \Omega$. ρ is unique with this property – if $\rho' : \beta\Omega \to K$ is continuous and surjective satisfying $\rho' \circ \iota = j : \Omega \to K$, then $\rho'(\iota(\omega)) = \rho(\iota(\omega))$ for all $\omega \in \Omega$, and continuity of ρ and ρ' as well as the fact that $\iota(\Omega)$ is dense in $\beta\Omega$ yields $\rho' = \rho$.

(This tells us that $\beta\Omega$ is in some sense "larger than or equal to" any compactification of Ω. Indeed, as what we have just shown, any imbedding of Ω in a compact Hausdorff space with dense image factors uniquely through $\beta\Omega$.)

(ii)

Suppose that $(K, j : \Omega \to K)$ also satisfies condition (iv) in Theorem 6.4. Show that ρ is a homeomorphism.

In (a), we did not use any facts about $\beta\Omega$ other than its properties from Theorem 6.4. Therefore, since $\beta\Omega$ is a compactification of Ω and $(K, j : \Omega \to K)$ satisfies condition (iv), then the proof of (a) yields a unique continuous surjection $\gamma : K \to \beta\Omega$ that satisfies $\gamma \circ j = \iota$ (just by exchanging i with j and K with $\beta\Omega$). Since $\iota(\omega) = \gamma(j(\omega)) = \gamma(\rho(\iota(\omega)))$ for all $\omega \in \Omega$, then because $\iota(\Omega)$ is dense in $\beta\Omega$ and $\gamma \circ \rho$ is a composition of continuous function and thus continuous itself, we obtain $x = \gamma(\rho(x))$ for all $x \in \beta\Omega$. Similarly one proves that $\kappa = \rho(\gamma(\kappa))$ for all $\kappa \in K$. Therefore ρ is a bijection $\beta\Omega \to K$ with the inverse γ. Since ρ is a continuous bijection, $\beta\Omega$ is compact and K is Hausdorff, a result from general topology says that ρ is a homeomorphism.

Thus any compactification $(K, j : \Omega \to K)$ of Ω that satisfies all the conditions of Theorem 6.4 must be homeomorphic to $\beta\Omega$, and condition (iv) in Theorem uniquely classifies $\beta\Omega$ as a compactification. Hence condition (iv) is the universal property of the Stone-Čech compactification.

References

[1] K. Zhu, *An Introduction to Operator Algebras*, Studies in Advanced Mathematics, CRC Press, 1993.