Introduction to operator algebra

Homework assignment # 2

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Remark 1. *Let $\mathcal{A}$ be a C^*-algebra. Note that the positive square root of a positive element x for any $x \in \mathcal{A}$ is defined by taking the function $f : \sigma(x) \subseteq \mathbb{R}^+ \to \mathbb{C}$ given by $f(t) = t^{1/2}$ and using the continuous functional calculus for x; the element $x^{1/2}$ is then positive and is a unique positive element $y \in \mathcal{A}$ such that $y^2 = x$. If x is invertible as well, then it follows from defining the continuous function $g : \sigma(x) \subseteq \mathbb{R}^+ \setminus \{0\} \to \mathbb{C}$ by $g(t) = t^{-1/2}$ that we obtain an element $x^{-1/2} \in \mathcal{A}$ with the properties*

$$x^{1/2}x^{-1/2} = f(x)g(x) = (fg)(x) = 1(x) = 1_{\mathcal{A}},$$

and similarly, $x^{-1/2}x^{1/2} = 1_{\mathcal{A}}$; therefore $x^{1/2}$ and $x^{-1/2}$ are invertible with $(x^{1/2})^{-1} = x^{-1/2}$, and $x^{-1/2}$ is positive as well. This also yields $x^{-1/2}x^{-1/2} = x^{-1}$.

Remark 2. *Let $\mathcal{A}$ be a C^*-algebra and $n \in \mathbb{N}$. If $x \in \mathcal{A}$ is positive, then we can clearly define a function $f : \sigma(x) \subseteq \mathbb{R}_+ \to \mathbb{C}$ by $f(t) = t^{n/2}$, yielding an element $x^{n/2} \in \mathcal{A}$. Since $(f \cdot f)(t) = t^n$, it also holds that $f(x)^2 = x^n$. As $f(x)$ and x^n are both positive (as seen by spectral mapping) and the positive square root of a positive element is unique ((1), Theorem 10.5), it holds that $x^{n/2} = (x^n)^{1/2}$.*

Exercise 1

Let $\mathcal{A}$ be a unital C^*-algebra. Recall that the "absolute value" of an element $x \in \mathcal{A}$ is the positive operator given by $|x| = (x^*x)^{1/2}$.

(i)

Show that $x|x| = |x^*|x$ and $|x|x^* = x^*|x^*|$.

We will be using results from Exercise 2 in the following. Let $x \in \mathcal{A}$. Note that from Exercise 2(ii), we obtain $x^*|x^*|x \geq 0$ since $|x^*| \geq 0$. Therefore, as $(x^*x)^3 = (xx^*x)^*(xx^*x)$, $(x^*x)^3$ is positive and

$$(x^*|x^*|x)^2 = x^*(|x^*|xx^*|x^*|)x = x^*(xx^*)^2x = (x^*x)^3,$$

then because the positive square root is unique ((1), Theorem 10.5), $x^*|x^*|x = ((x^*x)^3)^{1/2} = (x^*x)^{3/2}$ by Remark 2. Since $((x^*x)^4)^{1/2} = (x^*x)^2$ by Remark 2 as well, then because

$$\begin{aligned}
(x|x| - |x^*|x)^*(x|x| - |x^*|x) &= |x|x^*x|x| - x^*|x^*|x|x| - |x|x^*|x^*|x + x^*|x^*||x^*|x \\
&= 2(x^*x)^2 - ((x^*x)^3)^{1/2}(x^*x)^{1/2} - (x^*x)^{1/2}((x^*x)^3)^{1/2} \\
&= 2(x^*x)^2 - (x^*x)^{1/2}(x^*x)^{3/2} - (x^*x)^{1/2}(x^*x)^{3/2} \\
&= 2(x^*x)^2 - (x^*x)^2 - (x^*x)^2 \\
&= 0
\end{aligned}$$

we obtain

$$\|(x|x| - |x^*|x)\|^2 = \|(x|x| - |x^*|x)^*(x|x| - |x^*|x)\| = \|0\| = 0$$

from the C^* axiom, so $x|x| - |x^*|x = 0$. This proves the first equation, and the second equation is obtained from the first by replacing x with x^*.

(ii)

Suppose that x is invertible. Show that $|x|$ is invertible and $u := x|x|^{-1}$ is a unitary element in $\mathcal{A}$. Show also that $u|x|u^* = |x^*|$. Conclude that $x = u|x| = |x^*|u$. This is called the polar decomposition *of x.*

As x is invertible, x^* is also invertible, yielding finally that x^*x is invertible. As x^*x is positive, Remark 1 then yields that $|x|$ is invertible. Since $|x|^2 = x^*x$, $(|x|^{-1})^2 = (|x|^2)^{-1} = x^{-1}(x^*)^{-1}$ and the positive square roots are self-adjoint, it follows that

$$u^*u = (|x|^*)^{-1}x^*x|x|^{-1} = |x|^{-1}x^*x|x|^{-1} = |x|^{-1}|x|^2|x|^{-1} = 1_{\mathcal{A}}$$

and

$$uu^* = x|x|^{-1}(|x|^{-1})^*x^* = x(|x|^{-1})^2x^* = xx^{-1}(x^*)^{-1}x^* = 1_{\mathcal{A}}.$$

Hence u is unitary. Furthermore,

$$u|x|u^* = (x|x|^{-1})|x|u^{-1} = xu^{-1} = x|x|x^{-1} = |x^*|$$

by using the first equality from (i). It follows from the definition of u that $x = u|x|$, and from the fact that $u|x|u^* = |x^*|$ we deduce that $x = u|x| = |x^*|(u^*)^{-1} = |x^*|u$.

Exercise 2

Let $\mathcal{A}$ be a unital C^*-algebra, let $\mathcal{A}_{\text{sa}}$ denote the set of self-adjoint elements in $\mathcal{A}$, and let $\mathcal{A}^+$ denote the set of positive elements in $\mathcal{A}$. Define a relation $\leq$ on $\mathcal{A}_{\text{sa}}$ by $x \leq y$ if $y - x \in \mathcal{A}^+$ for $x, y \in \mathcal{A}_{\text{sa}}$.

(i)

Show that $\leq$ is a partial order relation on $\mathcal{A}_{\text{sa}}$.

Reflexiveness. Since $0 = 0^*0$, it follows that $x - x = 0 \in \mathcal{A}^+$ and thus $x \leq x$ for any $x \in \mathcal{A}_{\text{sa}}$.

Anti-symmetry. Assuming that $x \leq y$ and $y \leq x$ for $x, y \in \mathcal{A}_{\text{sa}}$, then $y - x \in \mathcal{A}^+$ and $x - y \in \mathcal{A}^+$. Since $y-x$ and $x-y$ are both normal, then by ((1), Theorem 11.5), $\sigma(y-x) \subseteq \mathbb{R}^+$ and $\sigma(x-y) \subseteq \mathbb{R}^+$. Defining $f : \sigma(x - y) \to \mathbb{C}$ by $f(t) = -t$, f is continuous, and by spectral mapping for $x - y$ ((1), Theorem 10.3), we obtain

$$\sigma(y - x) = \sigma(-(x - y)) = \sigma(f(x - y)) = f(\sigma(x - y)) \subseteq \mathbb{R}^-,$$

so $\sigma(y - x) \subseteq \mathbb{R}^+ \cap \mathbb{R}^- = \{0\}$. Since $y - x$ is normal, then $\|y - x\| = r(y - x) = 0$ by ((1), Theorem 8.1), so $x = y$.

Transitivity. Assuming that $x \leq y$ and $y \leq z$ for $x, y, z \in \mathcal{A}_{\text{sa}}$, then $y - x$, $z - y$ and $z - x$ are self-adjoint, as e.g. $(z-x)^* = z^* - x^* = z - x$. By ((1), Theorem 11.5), $\sigma(y-x) \subseteq \mathbb{R}^+$ and $\sigma(z-y) \subseteq \mathbb{R}^+$. Since $y - x, z - y \in \mathcal{A}_{\text{sa}}$, then by ((1), Theorem 11.4), $\sigma(z - x) = \sigma((z - y) + (y - x)) \subseteq \mathbb{R}^+$, and therefore by ((1), Theorem 11.5), $z - x \in \mathcal{A}^+$, so $x \leq z$.

(ii)

Let $x, y \in \mathcal{A}_{\text{sa}}$ be such that $x \leq y$, and let $z \in \mathcal{A}$. Show that $zxz^* \leq zyz^*$.

Since for instance $(zxz^*)^* = (z^*)^*x^*z^* = zxz^*$ by $x \in \mathcal{A}_{\text{sa}}$, zxz^* and zyz^* are self-adjoint. As $y - x \in \mathcal{A}^+$, there exists $w \in \mathcal{A}$ such that $y - x = w^*w$, but then

$$zyz^* - zxz^* = z(y - x)z^* = z(w^*w)z^* = (wz^*)^*(wz^*),$$

so $zyz^* - zxz^* \in \mathcal{A}^+$ and thus $zxz^* \leq zyz^*$.

(iii)

Let $x, y \in \mathcal{A}_{\text{sa}}$ be such that $x \not\leq y$. Show that there exists $z \in \mathcal{A}$ such that $zyz^* \leq zxz^*$ and $zyz^* \neq zxz^*$.

Since $x - y \in \mathcal{A}_{\text{sa}}$, we can define a continuous function $f : \sigma(x - y) \subseteq \mathbb{R} \to \mathbb{C}$ by

$$f(t) = \begin{cases} 0 & \text{for } t \leq 0 \\ t & \text{for } t > 0. \end{cases}$$

Additionally, define a function $g : \sigma(x - y) \to \mathbb{C}$ by $g(t) = f(t)tf(t)$ for $t \in \sigma(x - y) \subseteq \mathbb{R}$, so that

$$g(t) = \begin{cases} 0 & \text{for } t \leq 0 \\ t^3 & \text{for } t > 0. \end{cases}$$

The continous functional calculus ((1), Theorem 10.3) for $x - y$ now allows us to define $f(x - y)$ and $g(x-y)$ and additionally, spectral mapping yields $\sigma(f(x-y)) = f(\sigma(x-y)) \subseteq \mathbb{R}^+$ and $\sigma(g(x-y)) = g(\sigma(x - y)) \subseteq \mathbb{R}^+$. Since $f(x - y)$ and $g(x - y)$ are normal with positive spectrum, ((1), Theorem 11.5) then yields that $f(x - y) \geq 0$ and $g(x - y) \geq 0$.

We claim that $f(x-y)$ is an element $z \in \mathcal{A}$ that has the wanted properties, and so define $z = f(x-y)$. As $f(x - y)$ is positive and thus self-adjoint, we obtain

$$zxz^* - zyz^* = z(x - y)z^* = z(x - y)z = f(x - y)(x - y)f(x - y) = g(x - y) \geq 0.$$

Hence $zyz^* \leq zxz^*$.

As $x \not\leq y$, then $\sigma(y - x) \subsetneq \mathbb{R}^+$ by ((1), Theorem 11.5). Thus there exists a $\lambda' \in \sigma(y - x) \subseteq \mathbb{R}$ with $\lambda' < 0$ (using that $y - x \in \mathcal{A}_{\text{sa}}$). Applying the continuous functional calculus for $y - x$ to the continuous function $t \mapsto -t$, we obtain

$$\sigma(x - y) = \sigma(-(y - x)) = -\sigma(y - x) \ni -\lambda'$$

from spectral mapping ((1), Theorem 10.3). Hence by defining $\lambda'' := -\lambda'$, $\sigma(x-y)$ contains a strictly positive number λ''.

Assume now that $zxz^* = zyz^*$ or, equivalently, $g(x - y) = z(x - y)z^* = 0$. Therefore, since ((1), Theorem 10.3) yields that $h \mapsto h(x-y)$ is a *-isomorphism $C(\sigma(x-y)) \to C^*(1, x-y)$, it must hold that $g \equiv 0$ or $g(\lambda) = 0$ for all $\lambda \in \sigma(x - y)$. However, this contradicts $g(\lambda'') = (\lambda'')^3 > 0$, and thus $zxz^* \neq zyz^*$.

Exercise 3

(i)

Let $\mathcal{A}$ be a unital commutative C^*-algebra and let $x, y \in \mathcal{A}_{\text{sa}}$ be given. Show that $x \leq y$ if and only if $\varphi(x) \leq \varphi(y)$ for all $\varphi \in \Delta(\mathcal{A})$, where $\Delta(\mathcal{A})$ is the set of characters on $\mathcal{A}$.

All characters on $\mathcal{A}$ are contractive and unit-preserving algebra homomorphisms $\mathcal{A} \to \mathbb{C}$, so by Week Sheet 5, Exercise 2, they are *-homomorphisms as well, as $\mathbb{C}$ is a commutative C^*-algebra. If $x \leq y$, then $y - x = z^*z$ for some $z \in \mathcal{A}$. Therefore

$$\varphi(y) - \varphi(x) = \varphi(y - x) = \varphi(z^*z) = \varphi(z^*)\varphi(z) = \overline{\varphi(z)}\varphi(z) = |\varphi(z)|^2 \geq 0$$

for all $\varphi \in \Delta(\mathcal{A})$. If $\varphi(x) \leq \varphi(y)$ for all $\varphi \in \Delta(\mathcal{A})$, then $\Gamma(y - x)(\varphi) \geq 0$ for all $\varphi \in \Delta(\mathcal{A})$, implying that the range of $\Gamma(y - z)$ in $\mathbb{C}$ is contained in $\mathbb{R}^+$. Since $\mathcal{A}$ is commutative, then by ((1), Corollary 5.2), $\sigma(y - x) \subseteq \mathbb{R}^+$. Since $(y - x)^* = y^* - x^* = y - x$, $y - x$ is self-adjoint and in particular normal, so from ((1), Theorem 11.5) we conclude that $y - x \in \mathcal{A}^+$ and $x \leq y$.

(ii)

Let $\mathcal{A}$ be a (possibly non-commutative) unital C^*-algebra, let $I \subseteq \mathbb{R}$ be an interval, and let $x, y \in \mathcal{A}_{\text{sa}}$ be commuting elements with $\sigma(x) \subseteq I$ and $\sigma(y) \subseteq I$ and such that $x \leq y$. Let $f : I \to \mathbb{R}$ be an increasing continuous function. Show that $f(x) \leq f(y)$.

Consider the sub-C^*-algebra $\mathcal{B} = C^*(1, x, y)$ of $\mathcal{A}$; it is commutative since x and y commute and are self-adjoint. We will reduce the problem to check that the inequality holds in $\mathcal{B}$ and then extend to the case of $\mathcal{A}$ by using theorems for C^*-algebras. Since $\mathcal{B}$ is commutative and $\sigma_{\mathcal{B}}(y-x) = \sigma_A(y-x) \subseteq \mathbb{R}^+$ by ((1), Corollary 9.11) using that $x \leq y$, we obtain $y - x \in \mathcal{B}^+$ and hence (i) yields that $\varphi(x) \leq \varphi(y)$ for all $\varphi \in \Delta(\mathcal{B})$. Using ((1), Theorem 10.3) and the fact that all characters are *-homomorphisms, then because $\mathcal{B}$ is commutative and $f \in C(\sigma_{\mathcal{B}}(x))$, we obtain

$$\varphi(f(x)) = f(\varphi(x)) \leq f(\varphi(y)) = \varphi(f(y)),$$

as $f \in C(I)$ (taking f on $\varphi(x)$ makes sense because $\varphi(x) \in \sigma_{\mathcal{B}}(x) \subseteq I$; likewise for y). Therefore by (i), $f(x) \leq f(y)$ in $\mathcal{B}$.

As $\sigma_{\mathcal{A}}(f(y) - f(x)) = \sigma_{\mathcal{B}}(f(y) - f(x)) \subseteq \mathbb{R}^+$ by ((1), Corollary 9.11), and $f(x)$ and $f(y)$ are normal in $\mathcal{A}$ since the functional calculus preserves normality of elements, then ((1), Theorem 11.5) yields that $f(y) - f(x) \in \mathcal{A}^+$ and thus $f(x) \leq f(y)$ in $\mathcal{A}$.

(iii)

Show that the conclusion in (ii) does not hold if the assumption that x and y commute is omitted.

Consider the C^*-algebra $\mathcal{A} = M_2(\mathbb{C})$, let $I = \mathbb{R}^+$ and define $f : I \to \mathbb{R}$ by $f(t) = t^2$; f is continuous and increasing. Let $x, y \in M_2(\mathbb{C})$ be given by

$$x = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

x and y do not commute as is easily checked, and because the matrices are real and symmetric, they are both self-adjoint. Since the spectrum of a matrix consists exactly of its eigenvalues, it is easily seen that the spectrum of x is determined by finding the roots of its characteristic polynomial $(1-\lambda)(-\lambda) = \lambda^2 - \lambda$ – which are 0 and 1 – and similarly for y, its spectrum consists of the roots of $(2-\lambda)(1-\lambda) - 1 = \lambda^2 - 3\lambda + 1$ – which are $\frac{1}{2}(3 \pm \sqrt{5})$, both positive. Thus $\sigma(x)$ and $\sigma(y)$ are contained in I. The spectrum of $y - x$ is determined by finding the roots of the polynomial $(1-\lambda)(1-\lambda) - 1^2 = \lambda^2 - 2\lambda$ – they are 0 and 2, so $\sigma(y - x) \subseteq \mathbb{R}^+$ and since $y - x$ is self-adjoint, we obtain $y - x \in \mathcal{A}^+$ from ((1), Theorem 11.5) and hence $x \leq y$. Thus f, x and y satisfy all but one condition in (ii), namely that x and y do not commute.

As f is a polynomial, it is clear from ((1), Theorem 10.3) that $f(z) = z^2$ for any normal matrix $z \in M_2(\mathbb{C})$ with non-negative eigenvalues (as we need $\sigma(z) \subseteq I$ to define $f(z)$); thus we can define $f(x)$ and $f(y)$ in $\mathcal{A}$. As

$$x^2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad y^2 = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix},$$

then the spectrum of $y^2 - x^2$ is determined by finding the roots of the polynomial $(4-\lambda)(2-\lambda) - 3^2 = \lambda^2 - 6\lambda - 1$. As these are $3 + \sqrt{10}$ and $3 - \sqrt{10}$, then because $3 \leq \sqrt{10}$, we obtain $\sigma(y^2 - x^2) \subsetneq \mathbb{R}^+$. However, as $y^2 - x^2$ is symmetric and real, it is self-adjoint and thus normal, so ((1), Theorem 11.5) yields that $f(y) - f(x) = y^2 - x^2 \notin \mathcal{A}^+$ and hence $f(x) \not\leq f(y)$. Thus this example shows that the conclusion in (ii) does not hold if we omit the assumption that x and y commute.

(A monotone continuous function $f : I \to \mathbb{R}$, which also is monotone when applied to self- adjoint operators (in a C^*-algebra or on a Hilbert space) with spectrum contained in I, is said to be *operator monotone*. Exercise 3(iii) shows that $t \mapsto t^2$, with domain $I = \mathbb{R}^+$, is not operator monotone. In the next exercise we shall see that $t \mapsto t^{-1}$ and $t \mapsto \sqrt{t}$, with domains $I = \mathbb{R}^+ \setminus \{0\}$ and $I = \mathbb{R}^+$, respectively, are operator monotone.)

Exercise 4

Let $\mathcal{A}$ be a unital C^*-algebra.

(i)

Let $x, y \in \mathcal{A}$ be such that $0 \leq x \leq y$ and x is invertible. Show that y is invertible and that $y^{-1} \leq x^{-1}$.

Remark 1 yields the existence of the element $x^{-1/2}$, as x is positive and invertible. By Exercise 2(ii), then because x and y are positive and thus self-adjoint, it holds that $1_{\mathcal{A}} = x^{-1/2}xx^{-1/2} \leq x^{-1/2}yx^{-1/2}$. Letting $z := x^{-1/2}yx^{-1/2}$, $z - 1_{\mathcal{A}}$ is therefore positive. As z is self-adjoint, then using spectral mapping for z with the continuous function $f : \sigma(z) \to \mathbb{C}$ given by $f(t) = t - 1$ ((1), Theorem 10.3) yields

$$\mathbb{R}_+ \supseteq \sigma(z - 1_{\mathcal{A}}) = \sigma(f(z)) = f(\sigma(z)) = \sigma(z) - 1.$$

Thus $\sigma(z) \subseteq \mathbb{R}_+ + 1$, and hence $0 \notin \sigma(z)$, so z is invertible. As $y = x^{1/2}zx^{1/2}$, y is invertible as well.

As $0 \notin \sigma(z)$ and z is self-adjoint, we can define a continuous functional $g : \sigma(z) \to \mathbb{C}$ by $g(t) = 1 - t^{-1}$, and it follows from spectral mapping for z ((1), Theorem 10.3) that

$$\sigma(1_{\mathcal{A}} - z^{-1}) = \sigma(g(z)) = g(\sigma(z)) = \{1 - t^{-1} \,|\, t \in \sigma(z)\} \subseteq \{1 - (1 + \lambda)^{-1} \,|\, \lambda \geq 0\} \subseteq \mathbb{R}_+.$$

As $(z^{-1})^* = (z^*)^{-1} = z^{-1}$, it follows that $1_{\mathcal{A}} - z^{-1}$ is self-adjoint, and thus from ((1), Theorem 11.5), we obtain $1_{\mathcal{A}} - z^{-1} \in \mathcal{A}^+$. Therefore $z^{-1} \leq 1_{\mathcal{A}}$. Note that $z^{-1} = x^{1/2}y^{-1}x^{1/2}$; applying Exercise 2(ii) once again now yields

$$y^{-1} = x^{-1/2}z^{-1}x^{-1/2} \leq x^{-1/2}1_{\mathcal{A}}x^{-1/2} = x^{-1}.$$

(ii)

Let $x \in \mathcal{A}$ and write $x = a + ib$ with $a, b \in \mathcal{A}_{\text{sa}}$. Suppose that a is positive and invertible. Show that x is invertible.

Using Remark 1, we consider the element $z = a^{-1/2}ba^{-1/2}$. It is clearly self-adjoint by Remark 1 and the fact that $b \in \mathcal{A}_{\text{sa}}$, and thus $\sigma(z) \subseteq \mathbb{R}$ by ((1), Proposition 8.2). Defining a function $f : \sigma(z) \to \mathbb{C}$ by $f(t) = 1 + it$, then clearly f is continuous, and using spectral mapping ((1), Theorem 10.3), it follows that

$$\sigma(1_{\mathcal{A}} + iz) = \sigma(f(z)) = f(\sigma(z)) \subseteq 1 + i\mathbb{R}.$$

Since $0 \notin 1 + i\mathbb{R}$ (indeed, the only $\lambda \in \mathbb{C}$ such that $1 + i\lambda = 0$ is $\lambda = i$), $1_{\mathcal{A}} + iz$ is invertible. Because $a^{1/2}$ is invertible (using Remark 1), $a^{1/2}(1_{\mathcal{A}} + iz)a^{1/2}$ is invertible as well, but

$$a^{1/2}(1_{\mathcal{A}} + iz)a^{1/2} = a^{1/2}a^{1/2} + ia^{1/2}za^{1/2} = a + ib = x.$$

Hence x is invertible.

(iii)

Let $x, y \in \mathcal{A}$ be such that $0 \leq x \leq y$. Show that $\sqrt{x} \leq \sqrt{y}$.

We will be using ((1), Theorem 11.5) a lot throughout the proof; the constant references greatly decreased the author's enjoyment of the neat ideas used in the proof, and hence they are omitted for this exercise only. To sum it up, the theorem states that for a self-adjoint element, having positive spectrum is equivalent to being positive, and this is a fact we will use over and over.

First of all, we obtain that since $y - x \in \mathcal{A}^+$ and $x \in \mathcal{A}^+$, then $y - x$ and x are self-adjoint with spectrum in $\mathbb{R}^+$, so ((1), Corollary 11.4) implies that $\sigma(y) = \sigma((y - x) + x) \subseteq \mathbb{R}^+$, and therefore since $y^* = (y - x)^* + x^* = y - x + x = y$, y is self-adjoint and thus positive. Therefore we obtain positive square roots $\sqrt{y}$ and of course $\sqrt{x}$.

Let $t > 0$ be a positive real number, and consider the elements $u, v \in \mathcal{A}$ given by

$$u = t \cdot 1_{\mathcal{A}} + \sqrt{y} - \sqrt{x}, \quad v = t \cdot 1_{\mathcal{A}} + \sqrt{y} + \sqrt{x}.$$

From ((1), Exercise 8.2), vu can be written uniquely as $vu = a + ib$ with $a, b \in \mathcal{A}_{\mathrm{sa}}$.

It follows from ((1), Corollary 11.4) that $\sigma(\sqrt{y}+\sqrt{x}) \subseteq \mathbb{R}^+$. Defining the map $f : \sigma(\sqrt{y}+\sqrt{x}) \to \mathbb{C}$ by $f(z) = z + t$, then because f is continuous and $\sqrt{y}+\sqrt{x}$ is self-adjoint and thus normal, the functional calculus for $\sqrt{y}+\sqrt{x}$ applies to f and hence

$$\sigma(v) = \sigma(f(\sqrt{y}+\sqrt{x})) = f(\sigma(\sqrt{y}+\sqrt{x})) \subseteq t + \mathbb{R}^+$$

by spectral mapping ((1), Theorem 10.3). As 0 is not contained in $t + \mathbb{R}^+$ (consisting of all $x \in \mathbb{R}$ with $x \geq t > 0$), it follows that v is invertible.

As ((1), Exercise 8.2) yields that $a = \frac{1}{2}(uv + (uv)^*)$, and u and v are self-adjoint, it follows since

$$\begin{aligned} uv &= (t \cdot 1_{\mathcal{A}} + \sqrt{y} - \sqrt{x})(t \cdot 1_{\mathcal{A}} + \sqrt{y} + \sqrt{x}) \\ &= t^2 \cdot 1_{\mathcal{A}} + t(\sqrt{y}+\sqrt{x}) + t(\sqrt{y}-\sqrt{x}) + (\sqrt{y}-\sqrt{x})(\sqrt{y}+\sqrt{x}) \\ &= t^2 \cdot 1_{\mathcal{A}} + 2t\sqrt{y} + y - x - \sqrt{x}\sqrt{y} + \sqrt{y}\sqrt{x} \end{aligned}$$

and

$$\begin{aligned} vu &= t^2 \cdot 1_{\mathcal{A}} + t(\sqrt{y}+\sqrt{x}) + t(\sqrt{y}-\sqrt{x}) + (\sqrt{y}+\sqrt{x})(\sqrt{y}-\sqrt{x}) \\ &= t^2 \cdot 1_{\mathcal{A}} + 2t\sqrt{y} + y - x + \sqrt{x}\sqrt{y} - \sqrt{y}\sqrt{x} \end{aligned}$$

that

$$a = \frac{1}{2}(uv + (uv)^*) = \frac{1}{2}(uv + vu) = \frac{1}{2}(2t^2 \cdot 1_{\mathcal{A}} + 4t\sqrt{y} + 2(y - x)) = t^2 \cdot 1_{\mathcal{A}} + 2t\sqrt{y} + (y - x).$$

Using the continuous functional calculus for $\sqrt{y}$, it follows that $\sigma(2t\sqrt{y}) = 2t\sigma(\sqrt{y}) \subseteq \mathbb{R}^+$ and since $x \leq y$, we obtain $\sigma(y - x) \subseteq \mathbb{R}^+$. As $2t\sqrt{y}$ and $y - x$ are both self-adjoint (the latter because it is positive), it follows from ((1), Corollary 11.4) that $\sigma(2t\sqrt{y} + (y - x)) \subseteq \mathbb{R}^+$; since $2t\sqrt{y} + (y - x)$ is self-adjoint, we obtain $2t\sqrt{y} + (y - x) \in \mathcal{A}^+$. Thus $a - t^2 \cdot 1_{\mathcal{A}} \in \mathcal{A}^+$, implying for a that

$$\mathbb{R}^+ \supseteq \sigma(a - t^2 \cdot 1_{\mathcal{A}}) = \sigma(a) - t^2$$

using spectral mapping with the continuous function $w \mapsto w - t^2$, $w \in \sigma(a)$. Thus $\sigma(a) \subseteq t^2 + \mathbb{R}_+$, so a is positive, being self-adjoint; in particular, $0 \notin \sigma(a)$, so a is also invertible. We now obtain from (ii) that vu is invertible, and hence $u = v^{-1}(vu)$ is invertible as well.

As $u = \sqrt{y} - \sqrt{x} + t \cdot 1_{\mathcal{A}}$ is invertible, it follows that $-t \notin \sigma(\sqrt{y} - \sqrt{x})$. Because $\sqrt{y} - \sqrt{x}$ is self-adjoint, $\sigma(\sqrt{y} - \sqrt{x}) \subseteq \mathbb{R}$, but since $t > 0$ above was arbitrary, it contains no negative numbers. Thus $\sigma(\sqrt{y} - \sqrt{x}) \subseteq \mathbb{R}^+$; hence $\sqrt{y} - \sqrt{x} \in \mathcal{A}^+$ and $\sqrt{x} \leq \sqrt{y}$.

Exercise 5

Let $\mathcal{H}$ be a Hilbert space and let $P, Q \in B(\mathcal{H})$ be orthogonal projections, i.e. elements satisfying $P = P^* = P^2$ and $Q = Q^* = Q^2$. Show that the following conditions are equivalent:

- *(i) $P \leq Q$,*
- *(ii) $\langle P\xi, \xi\rangle \leq \langle Q\xi, \xi\rangle$ for all $\xi \in \mathcal{H}$,*
- *(iii) $P(\mathcal{H}) \subseteq Q(\mathcal{H})$,*
- *(iv) $Q - P$ is an orthogonal projection.*

(i)⇒(ii). Since $Q - P = T^*T$ for some $T \in B(\mathcal{H})$, then

$$\langle Q\xi, \xi\rangle - \langle P\xi, \xi\rangle = \langle (Q - P)\xi, \xi\rangle = \langle T^*T\xi, \xi\rangle = \langle T\xi, T\xi\rangle \geq 0$$

for all $\xi \in \mathcal{H}$.

(ii)⇒(iii). We will prove that $QP = P$ or $(1 - Q)P = 0$; then it follows that $P\xi = QP\xi \in Q(\mathcal{H})$ for all $\xi \in \mathcal{H}$. For $\xi \in \mathcal{H}$, note that

$$\langle (I - Q)\xi, \xi\rangle = \langle \xi, \xi\rangle - \langle Q\xi, \xi\rangle \leq \langle \xi, \xi\rangle - \langle P\xi, \xi\rangle = \langle (I - P)\xi, \xi\rangle.$$

In particular, it holds with ξ replaced by $P\xi$. Since $(I-Q)^* = I-Q$ and

$$(I-Q)^2 = I^2 + Q^2 - IQ - QI = I + Q - 2Q = I - Q,$$

$I-Q$ is an orthogonal projection, and we obtain that

$$\begin{aligned}
\|(I-Q)P\xi\|^2 &= \langle (I-Q)P\xi, (I-Q)P\xi\rangle \\
&= \langle (I-Q)^*(I-Q)P\xi, P\xi\rangle \\
&= \langle (I-Q)P\xi, P\xi\rangle \\
&\leq \langle (I-P)P\xi, P\xi\rangle \\
&= \langle P\xi - P^2\xi, P\xi\rangle \\
&= 0.
\end{aligned}$$

Thus $(I-Q)P = 0$.

(iii)⇒(iv). For any $x \in \mathcal{H}$, $P(x) \in Q(\mathcal{H})$, so there exists $y \in \mathcal{H}$ such that $Px = Qy$, implying $QPx = Q^2y = Qy = Px$ and hence $QP = P$. This in turn implies $PQ = P^*Q^* = (QP)^* = P^* = P$, so we obtain $(Q-P)^* = Q^* - P^* = Q - P$ and

$$(Q-P)^2 = Q^2 + P^2 - PQ - QP = Q + P - 2P = Q - P,$$

and thus $Q-P$ is an orthogonal projection.

(iv)⇒(i). Since $Q - P = (Q-P)^2 = (Q-P)(Q-P) = (Q-P)^*(Q-P)$, it follows that $Q-P$ is positive in $B(\mathcal{H})$ and $P \leq Q$.

References

[1] K. Zhu, *An Introduction to Operator Algebras*, Studies in Advanced Mathematics, CRC Press, 1993.