Introduction to operator algebra

Homework assignment # 3

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Exercise 1

Let $\mathcal{A}$ be a unital C^*-algebra such that $\mathcal{A} \neq \mathbb{C}1_{\mathcal{A}}$.

(i)

Show that $\mathcal{A}$ contains a self-adjoint element $x \notin \mathbb{C}1_{\mathcal{A}}$.

Since $\mathcal{A} \neq \mathbb{C}1_{\mathcal{A}}$, take $z \in \mathcal{A}$ with $z \notin \mathbb{C}1_{\mathcal{A}}$ and write $z = a + ib$ for $a, b \in \mathcal{A}_{\text{sa}}$. Both a and b cannot be contained in the subspace $\mathbb{C}1_{\mathcal{A}}$, as that would imply that $z \in \mathbb{C}1_{\mathcal{A}}$; therefore one of the two is not contained in $\mathbb{C}1_{\mathcal{A}}$. Call that element x; then x is self-adjoint and $x \notin \mathbb{C}1_{\mathcal{A}}$.

(ii)

Show that $\mathcal{A}$ contains two non-zero positive elements a, b such that $ab = 0$.

Let $x \in \mathcal{A} \setminus \mathbb{C}1_{\mathcal{A}}$ be self-adjoint (as found in (i)). As $\sigma(x) \subseteq \mathbb{R}$ is non-empty, there exists $\lambda \in \sigma(x)$. Assume that $\sigma(x) = \{\lambda\}$. Then by spectral mapping ((1), Theorem 5.4), it follows that

$$\sigma(x - \lambda 1_{\mathcal{A}}) = \sigma(x) - \lambda = \{0\};$$

as $x - \lambda 1_{\mathcal{A}}$ is self-adjoint, hence normal, it follows that $\|x - \lambda 1_{\mathcal{A}}\| = r(x - \lambda 1_{\mathcal{A}}) = 0$ from ((1), Theorem 8.1), so $x = \lambda 1_{\mathcal{A}} \in \mathbb{C}1_{\mathcal{A}}$, a contradiction. Therefore, $\sigma(x)$ contains at least two different points, and we take $\lambda_1, \lambda_2 \in \sigma(x)$ with $\lambda_1 < \lambda_2$.

Now define two maps $f : \sigma(x) \subseteq \mathbb{R} \to \mathbb{C}$ and $g : \sigma(x) \subseteq \mathbb{R} \to \mathbb{C}$ by

$$f(t) = \begin{cases} 0 & t \leq \frac{\lambda_1 + \lambda_2}{2} \\ \frac{2}{\lambda_2 - \lambda_1}(t - \lambda_2) + 1 & \frac{\lambda_1 + \lambda_2}{2} \leq t \leq \lambda_2 \\ 1 & t \geq \lambda_2 \end{cases}$$

and

$$g(t) = \begin{cases} 1 & t \leq \lambda_1 \\ \frac{2}{\lambda_2 - \lambda_1}(\lambda_1 - t) + 1 & \lambda_1 \leq t \leq \frac{\lambda_1 + \lambda_2}{2} \\ 0 & t \geq \frac{\lambda_1 + \lambda_2}{2}. \end{cases}$$

As is easily seen, f and g are well-defined and continuous, and $f(t)g(t) = 0$ for all $t \in \sigma(x)$. Using the functional calculus for $x \in \mathcal{A}_{\text{sa}}$, we obtain elements $f(x)$ and $g(x)$ in $C^*(1, x)$. As f and g are both not the zero function (this is what we needed two different elements in the spectrum for), it follows from ((1), Theorem 10.3) that $f(x)$ and $g(x)$ are non-zero and additionally that $\sigma(f(x)) = f(\sigma(x)) \subseteq \mathbb{R}^+$ and $\sigma(g(x)) = g(\sigma(x)) \subseteq \mathbb{R}^+$. Since $f(x)$ and $g(x)$ are also normal, it follows from ((1), Corollary 9.11 and Theorem 11.5) that $f(x), g(x) \in \mathcal{A}_+$. Finally, $f(x)g(x) = (fg)(x) = 0(x) = 0$, thereby yielding the desired result by defining $a = f(x)$ and $b = g(x)$.

Note that in the following three exercises, then because we will be working with Hilbert spaces, we need only consider sequences when discussing closures of subsets of Hilbert spaces; nets are not necessary.

Exercise 2

Let $\mathcal{A}$ be a unital C^*-algebra, let $\mathcal{H}$ be a Hilbert space, and let $\pi : \mathcal{A} \to B(\mathcal{H})$ be a unital *-homomorphism. (We say that π is a *-representation of $\mathcal{A}$ on $\mathcal{H}$.) A closed subspace $\mathcal{H}_0$ of $\mathcal{H}$ is said to be *invariant for* π if $\pi(a)\mathcal{H}_0 \subseteq \mathcal{H}_0$ for all $a \in \mathcal{A}$. We say that π is *irreducible* if the only closed invariant subspaces for π are $\{0\}$ and $\mathcal{H}$. Denote by $\pi(\mathcal{A})'$ the *commutant* of $\mathcal{A}$ relatively to $B(\mathcal{H})$, i.e.

$$\pi(\mathcal{A})' = \{T \in B(\mathcal{H}) \,|\, \forall a \in \mathcal{A} : \pi(a)T = T\pi(a)\}.$$

(i)

Suppose that $\mathcal{H}_0$ is an invariant subspace for π. Let P be the orthogonal projection onto $\mathcal{H}_0$. Show that $\pi(a)P = P\pi(a)P$ for all $a \in \mathcal{A}$ (†). Show next that $P \in \pi(\mathcal{A})'$.

Let $a \in \mathcal{A}$. Note that $\pi(a)Px \in \pi(a)\mathcal{H}_0 \subseteq \mathcal{H}_0$ for all $x \in \mathcal{H}$ by invariance. Since $Pz = z$ for any $z \in \mathcal{H}_0$, then in particular $P(\pi(a)Px) = \pi(a)Px$ for all $x \in \mathcal{H}$, yielding (†). Since π is a *-homomorphism, then from (†) we obtain

$$P\pi(a)P = P^*\pi(a^*)^*P^* = (P\pi(a^*)P)^* = (\pi(a^*)P)^* = P^*\pi(a^*)^* = P\pi(a)$$

for all $a \in \mathcal{A}$ so that $\pi(a)P = P\pi(a)P = P\pi(a)$ for all $a \in \mathcal{A}$, once again using (†). Thus $P \in \pi(\mathcal{A})'$.

(ii)

Show that $R(T) = \overline{T(\mathcal{H})}$ is an invariant subspace for π whenever $T \in \pi(A)'$.

Assume that $T \in \pi(\mathcal{A})'$. Letting $x \in R(T)$, there exists a sequence (x_n) of $T(\mathcal{H})$ with $x_n \to x$; for $n \in \mathbb{N}$, take $z_n \in \mathcal{H}$ such that $Tz_n = x_n$. For $a \in \mathcal{A}$, note that

$$\pi(a)x_n = \pi(a)(Tz_n) = T(\pi(a)z_n) \in T(\mathcal{H}) \subseteq R(T).$$

Since $\pi(a)$ is continuous, then $\pi(a)x_n \to \pi(a)x$ in $\mathcal{H}$, and since $R(T)$ is closed in $\mathcal{H}$ and contains the sequence $(\pi(a)x_n)$, $\pi(a)x \in R(T)$. Thus $\pi(a)R(T) \subseteq R(T)$ for all $a \in \mathcal{A}$. Additionally, since $R(T)$ is the closure of the subspace $T(\mathcal{H})$, it is itself a subspace, so $R(T)$ is invariant for π.

(iii)

Show that π is irreducible if and only if $\pi(\mathcal{A})' = \mathbb{C}I$.

$\pi(\mathcal{A})'$ is the centralizer of $\pi(\mathcal{A}) \subseteq B(\mathcal{H})$ (see ((1), Exercise 5.4)) and is therefore a unital Banach subalgebra of $B(\mathcal{H})$. As

$$T^*\pi(a) = T^*\pi(a^*)^* = (\pi(a^*)T)^* = (T\pi(a^*))^* = \pi(a^*)^*T^* = \pi(a)T^*$$

for all $a \in \mathcal{A}$ and $T \in \pi(\mathcal{A})'$ (the assumption that $T \in \pi(\mathcal{A})'$ being used at the third equality), $\pi(\mathcal{A})'$ is closed under the involution and is therefore a unital sub-C^*-algebra of $B(\mathcal{H})$.

Assume first that π is not irreducible. Then there exists a non-trivial closed subspace $\mathcal{H}_0$ of $\mathcal{H}$ that is invariant for π. Let P denote the orthogonal projection onto $\mathcal{H}_0$; then it follows from (i) that $P \in \pi(\mathcal{A})'$. Because $\mathcal{H}_0 \neq \{0\}$ there exists $z \in \mathcal{H}_0$ with $z \neq 0$, and because $\mathcal{H}_0 \neq \mathcal{H}$ there exists $w \in \mathcal{H} \setminus \mathcal{H}_0$. Assuming that $P = \lambda I$ for some $\lambda \in \mathbb{C}$, then since

$$|\lambda - 1|\|z\| = \|\lambda z - z\| = \|Pz - z\| = \|z - z\| = 0,$$

we obtain $\lambda = 1$ or $P = I$ since $\|z\| > 0$. This contradicts the fact that $Pw \neq w$: since $\mathcal{H} \setminus \mathcal{H}_0$ is open, then taking an ε-ball inside $\mathcal{H} \setminus \mathcal{H}_0$ around w yields $\|y - w\| \geq \varepsilon$ for all $y \in \mathcal{H}_0$, hence $\|Pw - w\| \geq \varepsilon > 0$. Thus $P \notin \mathbb{C}I$, and we conclude that $\pi(\mathcal{A})' \neq \mathbb{C}I$.

Now assume that $\pi(\mathcal{A})' \neq \mathbb{C}I$. Since $\pi(\mathcal{A})'$ is a C^*-algebra, then by Exercise 1(ii) there exists two positive operators $P, Q \in \pi(\mathcal{A})'$ satisfying $P \neq 0$, $Q \neq 0$ and $PQ = 0$. Assuming that $R(Q) = \mathcal{H}$, then for any $x \in \mathcal{H}$ there is a sequence $(x_n) \subseteq Q(\mathcal{H})$ with $x_n \to x$. Taking $z_n \in \mathcal{H}$ such that $Qz_n = x_n$

for each $n \in \mathbb{N}$, we see that $Px_n = PQz_n = 0$ and since $Px_n \to Px$ by continuity of P we obtain $Px = 0$. Since $x \in \mathcal{H}$ was arbitrary, this implies $P = 0$, a contradiction. Additionally, $R(Q) = \{0\}$ implies $Q(\mathcal{H}) = \{0\}$ or $Q = 0$, another contradiction. Hence $R(Q)$ is a non-trivial subset of $\mathcal{H}$. Since $Q \in \pi(\mathcal{A})'$, (ii) implies that $R(Q)$ is an invariant subspace under π, so we have found a non-trivial closed subspace of $\mathcal{H}$ that is invariant for π; hence π is not irreducible. This proves the biconditional.

Exercise 3

Let $\mathcal{A}$ be a unital C^*-algebra and let φ be a state on $\mathcal{A}$. Let $(\mathcal{H}_\varphi, \pi_\varphi, \xi_\varphi)$ be the associated GNS representation.

(i)

Show that ξ_φ is separating for $\pi_\varphi(\mathcal{A})'$, i.e. if $T \in \pi_\varphi(\mathcal{A})'$ and $T\xi_\varphi = 0$, then $T = 0$.

Assume that $T \in \pi_\varphi(\mathcal{A})'$, i.e. that $T\pi_\varphi(a) = \pi_\varphi(a)T$ for all $a \in \mathcal{A}$, and that $T\xi_\varphi = 0$. Let $x \in \mathcal{H}_\varphi$. Since $\pi_\varphi(\mathcal{A})\xi_\varphi$ is dense in $\mathcal{H}_\varphi$, there exists a sequence (x_n) of $\pi_\varphi(\mathcal{A})\xi_\varphi$ such that $x_n \to x$. Let $a_n \in \mathcal{A}$ such that $\pi_\varphi(a_n)\xi_\varphi = x_n$ for each $n \in \mathbb{N}$; then

$$Tx_n = T\pi_\varphi(a_n)\xi_\varphi = \pi_\varphi(a_n)T\xi_\varphi = \pi_\varphi(a_n)(0) = 0$$

using that $T \in \pi_\varphi(\mathcal{A})'$. Since $Tx_n = 0 \to 0$ and $Tx_n \to Tx$ by continuity of T, then because $\mathcal{H}_\varphi$ is Hausdorff, Tx must be 0. Since x was arbitrary, we conclude that $T = 0$.

For each $T \in \pi_\varphi(\mathcal{A})'$ let $\psi_T : \mathcal{A} \to \mathbb{C}$ be given by

$$\psi_T(x) = \langle \pi_\varphi(x)\xi_\varphi, T\xi_\varphi \rangle, \quad x \in \mathcal{A}.$$

It is easily verified that ψ_T is a bounded linear functional on $\mathcal{A}$.

(ii)

Show that ψ_T is positive if $T \in \pi_\varphi(\mathcal{A})'$ is positive.

Assume that $T \in \pi_\varphi(\mathcal{A})'$ is positive and let $x \in \mathcal{A}_+$. Recall that elements of $\mathcal{A}$ that are contained in a sub-C^*-algebra $\mathcal{B}$ and are positive in $\mathcal{A}$ are also positive in $\mathcal{B}$ since the elements are self-adjoint in $\mathcal{B}$ and the spectrum is the same (using ((1), Corollary 9.11 and Theorem 11.5)). Therefore we can pick $S \in \pi_\varphi(\mathcal{A})'$ such that $T = S^*S$, and likewise there exists $y \in \mathcal{A}$ such that $x = y^*y$. Then

$$\begin{aligned}
\psi_T(x) &= \langle \pi_\varphi(y^*y)\xi_\varphi, S^*S\xi_\varphi \rangle \\
&= \langle \pi_\varphi(y^*)\pi_\varphi(y)\xi_\varphi, S^*S\xi_\varphi \rangle \\
&= \langle S\pi_\varphi(y^*)\pi_\varphi(y)\xi_\varphi, S\xi_\varphi \rangle \\
&= \langle \pi_\varphi(y^*)\pi_\varphi(y)S\xi_\varphi, S\xi_\varphi \rangle \\
&= \langle \pi_\varphi(y)S\xi_\varphi, \pi_\varphi(y)S\xi_\varphi \rangle \\
&\geq 0
\end{aligned}$$

because $S \in \pi_\varphi(\mathcal{A})'$ and $\pi_\varphi(y^*)^* = \pi_\varphi(y)$. Hence ψ_T is positive.

(iii)

Let $S, T \in \pi_\varphi(\mathcal{A})'$. Show that $\psi_S = \psi_T$ if and only if $S = T$. Show also that $\psi_T(1_\mathcal{A}) \neq 0$ whenever $T \in \pi_\varphi(\mathcal{A})'$ is non-zero and positive.

That $S = T$ implies $\psi_S = \psi_T$ is clear from the definition. Assuming that $\psi_S = \psi_T$, then we can pick a sequence $(a_n) \in \mathcal{A}$ such that $\pi_\varphi(a_n)\xi_\varphi \to (S - T)\xi_\varphi$, since $\pi_\varphi(\mathcal{A})\xi_\varphi$ is dense in $\mathcal{H}_\varphi$. By continuity of the inner product we obtain

$$\begin{aligned}\|(S-T)\xi_\varphi\|^2 &= \langle (S-T)\xi_\varphi, (S-T)\xi_\varphi\rangle \\ &= \lim_{n\to\infty} \langle \pi_\varphi(a_n)\xi_\varphi, (S-T)\xi_\varphi\rangle \\ &= \lim_{n\to\infty} [\langle \pi_\varphi(a_n)\xi_\varphi, S\xi_\varphi\rangle - \langle \pi_\varphi(a_n)\xi_\varphi, T\xi_\varphi\rangle] \\ &= \lim_{n\to\infty} [\psi_S(a_n) - \psi_T(a_n)] \\ &= \lim_{n\to\infty} 0 = 0.\end{aligned}$$

Hence $(S-T)\xi_\varphi = 0$. Since $\pi_\varphi(\mathcal{A})'$ is a sub-C^*-algebra of $B(\mathcal{H}_\varphi)$, then $S-T \in \pi_\varphi(\mathcal{A})'$, and from (i), we then obtain $S-T=0$ or $S=T$.

Assuming that $T \in \pi_\varphi(\mathcal{A})'$, then using (i), $T \neq 0$ implies $T\xi_\varphi \neq 0$. Assuming now that $T \in \pi_\varphi(\mathcal{A})'$ is non-zero and positive, then there exists $S \in \pi_\varphi(\mathcal{A})'$ such that $T = S^*S$. S is clearly non-zero as well (otherwise T would be zero), hence $S\xi_\varphi \neq 0$. Since π_φ is unit-preserving, $\pi_\varphi(1_\mathcal{A})$ is the identity operator on $\mathcal{H}_\varphi$ and hence

$$\psi_T(1_\mathcal{A}) = \langle \pi_\varphi(1_\mathcal{A})\xi_\varphi, T\xi_\varphi\rangle = \langle \xi_\varphi, T\xi_\varphi\rangle = \langle \xi_\varphi, S^*S\xi_\varphi\rangle = \langle S\xi_\varphi, S\xi_\varphi\rangle > 0,$$

and we are done.

(iv)

Show that π_φ is irreducible if φ is a pure state on $\mathcal{A}$.

First a remark. If a projection P in a C^*-algebra $\mathcal{A}$ satisfies $P = \lambda 1_\mathcal{A}$, then $\lambda \in \{0,1\}$: it follows from the equation

$$|\lambda^2 - \lambda| = |\lambda^2 - \lambda|\|1_\mathcal{A}\| = \|\lambda^2 1_\mathcal{A} - \lambda 1_\mathcal{A}\| = \|P^2 - P\| = 0.$$

We will now prove the contrapositive of the above conditional. Assume that π_φ is not irreducible. We saw in Exercise 2(iii) that this implies that there exists an orthogonal projection $P \in \pi_\varphi(\mathcal{A})'$ onto a non-trivial closed subspace $\mathcal{H}_0$ of $\mathcal{H}_\varphi$ that is invariant for π_φ. Define $Q = I - P$. Since $\mathcal{H}_0$ is non-trivial, then $P \neq I$ and $P \neq 0$ and from this we also obtain $Q \neq I$ and $Q \neq 0$.

Note first that

$$\varphi(x) = \langle \pi_\varphi(x)\xi_\varphi, \xi_\varphi\rangle = \langle \pi_\varphi(x)\xi_\varphi, P\xi_\varphi\rangle + \langle \pi_\varphi(x)\xi_\varphi, Q\xi_\varphi\rangle = \psi_P(x) + \psi_Q(x) \tag{1}$$

for all $x \in \mathcal{A}$. Since $P = P^2 = P^*P$, P is positive, and since Q is a projection as well, it follows that $Q = Q^*Q$ and hence Q is positive. It follows from (ii) and the proof of (iii) that ψ_P are ψ_Q are positive and that $\psi_P(1_\mathcal{A}) > 0$ and $\psi_Q(1_\mathcal{A}) > 0$ since P and Q are non-zero operators.

We now construct states from ψ_P and ψ_Q. Defining $\lambda_P = \psi_P(1_\mathcal{A}) > 0$ and $\lambda_Q = \psi_Q(1_\mathcal{A}) > 0$, we obtain two states by defining $\varphi_P = \lambda_P^{-1}\psi_P$ and $\varphi_Q = \lambda_Q^{-1}\psi_Q$; indeed, considering the case of P, then φ_P is clearly linear, it holds that

$$\varphi_P(1_\mathcal{A}) = \psi_P(1_\mathcal{A})^{-1}\psi_P(1_\mathcal{A}) = 1,$$

and for positive elements $x \in \mathcal{A}$, $\varphi_P(x) = \lambda_P^{-1}\psi_P(x) \geq 0$ since ψ_P is positive and $\lambda_P^{-1} > 0$.

Note now that (1) yields $\lambda_P + \lambda_Q = \varphi(1_\mathcal{A}) = 1$ since φ is a state, and thus

$$\lambda_P = 1 - \lambda_Q < 1, \quad \lambda_Q = 1 - \lambda_P < 1.$$

(1) now yields

$$\varphi = \psi_P + \psi_Q = \lambda_P\varphi_P + \lambda_Q\varphi_Q.$$

Since $0 < \lambda_P, \lambda_Q < 1$ it only remains to prove that φ_P and φ_Q are both not equal to each other nor to φ to prove that φ is not pure. Assume for contradiction that $\varphi_P = \varphi$. Then

$$\psi_P = \lambda_P\varphi_P = \lambda_P\varphi = \psi_{\overline{\lambda_P}I} = \psi_{\lambda_P I},$$

as is easily seen from the definition. (ii) now implies that $P = \lambda_P I$, so either $\lambda_P = 0$ or $\lambda_P = 1$ (by the remark), meaning that $P = 0$ or $P = I$, a clear contradiction (indeed, we found earlier that $P \neq 0$ and $P \neq I$). Similarly it is seen that $\varphi_Q \neq \varphi$. Finally, $\varphi_P \neq \varphi_Q$ since $\varphi_P = \varphi_Q$ would imply $\varphi = \varphi_P = \varphi_Q$. Therefore, φ is a proper convex combination of two states and hence is not a pure state.

(One can show that the converse also holds: π_φ is irreducible *if and only if* φ is a pure state on $\mathcal{A}$.)

Exercise 4

Let $\mathcal{A}$ be a unital C^*-algebra, and let τ be a tracial state on $\mathcal{A}$. Consider the GNS representation $(\mathcal{H}_\tau, \pi_\tau, \xi_\tau)$ of $\mathcal{A}$ associated with τ. Recall that

$$\mathcal{L}_\tau = \{x \in \mathcal{A} \mid \tau(x^*x) = 0\},$$

that $[a] = a + \mathcal{L}_\tau$, and that

$$\pi_\tau(\mathcal{A})\xi_\tau = \mathcal{H}_\tau^\circ = \{[a] \mid a \in \mathcal{A}\}.$$

(i)

Show that $\mathcal{L}_\tau$ is a closed two-sided ideal in $\mathcal{A}$.

It follows from ((1), Proposition 14.1) that $\mathcal{L}_\tau$ is a closed left ideal in $\mathcal{A}$, and we therefore only need to prove that it is a right ideal as well. For any $x \in \mathcal{L}_\tau$ and $y \in \mathcal{A}$, note that

$$\tau((xy)^*(xy)) = \tau(y^*(x^*xy)) = \tau((x^*xy)y^*) = \tau(x^*(xyy^*)) = 0$$

following from ((1), Proposition 14.1) using that τ is a tracial state and that $x \in \mathcal{L}_\tau$. Therefore $xy \in \mathcal{L}_\tau$ and hence $\mathcal{L}_\tau$ is a right ideal.

For each $x \in \mathcal{A}$ consider the map $R_x^\circ : \mathcal{H}_\tau^\circ \to \mathcal{H}_\tau^\circ$ given by $R_x^\circ([a]) = [ax]$ for all $a \in \mathcal{A}$.

(ii)

Show that R_x° is well-defined, i.e. that for $a, b \in \mathcal{A}$ then $[a] = [b]$ implies $[ax] = [bx]$.

Fix $x \in \mathcal{A}$, let $a, b \in \mathcal{A}$ and assume that $[a] = [b]$. Then $a - b \in \mathcal{L}_\tau$. Since $\mathcal{L}_\tau$ is a right ideal, we obtain $ax - bx = (a - b)x \in \mathcal{L}_\tau$, so $[ax] = [bx]$. Hence R_x° is well-defined.

(iii)

Show that $\|R_x^\circ\| \leq \|x\|$.

First a lemma.

Lemma 1. *Let $\mathcal{A}$ be a unital C^*-algebra. For all $x, y \in \mathcal{A}$ it holds that*

(a) $x^*x \leq \|x\|^2 1_{\mathcal{A}}$.
(b) $y^*x^*xy \leq \|x\|^2 y^*y$.

Proof. *(b)* follows from *(a)* by multiplying with y^* on the left and y on the right (see Assignment #2, Exercise 2(ii)). For *(a)*, note that since x^*x is self-adjoint, it follows that $r(x^*x) = \|x^*x\| = \|x\|^2$ from ((1), Theorem 8.1). Hence $\sigma(x^*x) \subseteq [0, \|x\|^2]$, as x^*x has positive spectrum ((1), Theorem 11.5). By spectral mapping, we obtain

$$\sigma(\|x\|^2 1_{\mathcal{A}} - x^*x) = \|x\|^2 - \sigma(x^*x) \subseteq [0, \|x\|^2] \subseteq \mathbb{R}_+;$$

since $\|x\|^2 1_{\mathcal{A}} - x^*x$ is self-adjoint, ((1), Theorem 11.5) yields that $\|x\|^2 1_{\mathcal{A}} - x^*x \in \mathcal{A}_+$. □

Fix $x \in \mathcal{A}$ and let $y \in \mathcal{H}_\tau^\circ$. Then $y = [a]$ for some $a \in \mathcal{A}$, so

$$\begin{aligned}
\|R_x^\circ(y)\|^2 &= \langle R_x^\circ([a]), R_x^\circ([a])\rangle = \langle [ax],[ax]\rangle \\
&= \tau((ax)^*(ax)) = \tau((x^*a^*)(ax)) = \tau((ax)(x^*a^*)) \\
&\leq \tau(\|x^*\|^2 aa^*) \\
&= \|x^*\|^2\tau(aa^*) = \|x\|^2\tau(a^*a) \\
&= \|x\|^2\langle [a],[a]\rangle = \|x\|^2\|[a]\|^2 = \|x\|^2\|y\|^2,
\end{aligned}$$

using Lemma 1, that τ is a positive trace and that the involution is an isometry. Hence we have $\|R_x^\circ(y)\| \leq \|x\|\|y\|$ for all $y \in \mathcal{H}_\tau^\circ$, yielding $\|R_x^\circ\| \leq \|x\|$.

One can now conclude, as in the usual GNS construction, that there exists $R_x \in B(\mathcal{H}_\tau)$ such that $\|R_x\| \leq \|x\|$ and $R_x([a]) = [ax]$ for all $a \in \mathcal{A}$. Define $\rho_\tau : \mathcal{A} \to B(\mathcal{H}_\tau)$ by $\rho_\tau(x) = R_x$ for $x \in \mathcal{A}$. It is easily verified that ρ_τ is linear.

(iv)

Show that ρ_τ is an anti-representation *of $\mathcal{A}$ on $B(\mathcal{H}_\tau)$, i.e. show that $\rho_\tau(xy) = \rho_\tau(y)\rho_\tau(x)$ and $\rho_\tau(x^*) = \rho_\tau(x)^*$ for all $x, y \in \mathcal{A}$.*

Fix $x, y \in \mathcal{A}$ and let $a, z \in \mathcal{A}$. Then

$$\rho_\tau(xy)([a]) = [axy] = \rho_\tau(y)([ax]) = \rho_\tau(y)\rho_\tau(x)([a]);$$

since $\mathcal{H}_\tau^\circ$ is dense in $\mathcal{H}_\tau$ and the bounded operators $\rho_\tau(xy)$ and $\rho_\tau(y)\rho_\tau(x)$ are equal on $\mathcal{H}_\tau^\circ$, it follows by a continuity argument that $\rho_\tau(xy) = \rho_\tau(y)\rho_\tau(x)$. Now for $a, z \in \mathcal{A}$, note that

$$\begin{aligned}
\langle [a], \rho_\tau(x)^*([z])\rangle &= \langle \rho_\tau(x)([a]), [z]\rangle \\
&= \langle [ax],[z]\rangle \\
&= \tau((z^*a)x) \\
&= \tau(x(z^*a)) \\
&= \tau((zx^*)^*a) \\
&= \langle [a],[zx^*]\rangle \\
&= \langle [a], \rho_\tau(x^*)([z])\rangle,
\end{aligned}$$

using that τ is a trace. Since $\mathcal{H}_\tau^\circ$ is dense in $\mathcal{H}_\tau$, then for any $x \in \mathcal{H}$ we can pick a sequence (a_n) of $\mathcal{A}$ such that $[a_n] \to x$. By continuity of the inner product, this yields

$$\langle x, \rho_\tau(x)^*([z])\rangle = \lim_{n\to\infty}\langle [a_n], \rho_\tau(x)^*[z]\rangle = \lim_{n\to\infty}\langle [a_n], \rho_\tau(x^*)([z])\rangle = \lim_{n\to\infty}\langle x, \rho_\tau(x^*)([z])\rangle$$

for any $z \in \mathcal{A}$, and the properties of the Hilbert space $\mathcal{H}_\tau$ then gives us $\rho_\tau(x)^*([z]) = \rho_\tau(x^*)([z])$ for all $z \in \mathcal{A}$. By a dense-and-continuous argument as used before, we then obtain $\rho_\tau(x)^* = \rho_\tau(x^*)$.

(v)

Show that $\rho_\tau(x) \in \pi_\tau(\mathcal{A})'$ for all $x \in \mathcal{A}$.

Fix $x \in \mathcal{A}$ and let $a \in \mathcal{A}$. Then

$$\rho_\tau(x)\pi_\tau(a)([z]) = \rho_\tau(x)([az]) = [azx] = \pi_\tau(a)([zx]) = \pi_\tau(a)\rho_\tau(x)([z])$$

for all $z \in \mathcal{A}$. Since $\mathcal{H}_\tau^\circ$ is dense in $\mathcal{H}_\tau$ and the bounded operators $\rho_\tau(x)\pi_\tau(a)$ and $\pi_\tau(a)\rho_\tau(x)$ are equal on $\mathcal{H}_\tau^\circ$, it follows by a continuity argument that the operators are equal; since a was arbitrary, we conclude that $\rho_\tau(x) \in \pi_\tau(\mathcal{A})'$.

(vi)

Show that $\ker \pi_\tau = \ker \rho_\tau = \mathcal{L}_\tau$.

Assuming that $x \in \mathcal{A}$ satisfies $\pi_\tau(x) = 0$, it holds that $[x] = [x1_\mathcal{A}] = \pi_\tau(x)\xi_\varphi = 0 = [0]$, so $x = x - 0 \in \mathcal{L}_\tau$. If $x \in \mathcal{A}$ satisfies $\rho_\tau(x) = 0$, then likewise it holds that $[x] = [1_\mathcal{A}x] = \rho_\tau(x)\xi_\varphi = 0 = [0]$ and $x \in \mathcal{L}_\tau$. On the other hand, if $x \in \mathcal{L}_\tau$, then for all $a \in \mathcal{A}$, $ax, xa \in \mathcal{L}_\tau$ by (i) and hence $\pi_\tau(x)(y) = [xa] = [0] = 0$ and $\rho_\tau(x)(y) = [ax] = [0] = 0$. Once again, as both $\pi_\tau(x)$ and $\rho_\tau(x)$ are both equal to 0 on a dense subset of $\mathcal{H}_\tau$, then by continuity, they are equal to 0 everywhere on $\mathcal{H}_\tau$, and we conclude that $\pi_\tau(x) = 0$ and $\rho_\tau(x) = 0$. Thus $x \in \ker \pi_\tau$ and $x \in \ker \rho_\tau$, proving the above equalities.

(vii)

Show that the unique tracial state on a UHF-algebra is not a pure state.

Note first the following lemma:

Lemma 2. *Let $\mathcal{A}$ be a unital Banach algebra. If φ is a character on $\mathcal{A}$, then* $\ker \varphi$ *is a proper, closed two-sided ideal in $\mathcal{A}$.*

Proof. First of all, since $\varphi \neq 0$ we have $\ker \varphi \neq \mathcal{A}$. Since φ is continuous ((1), Proposition 4.1), $\ker \varphi$ is closed as well, and it is clear that $\ker \varphi$ is a two-sided ideal since φ is a character. □

Since UHF-algebras are simple, infinite-dimensional and have a unique tracial state, it suffices to prove that tracial states on simple C^*-algebras $\mathcal{A}$ different from $\mathbb{C}$ are not pure.

Let $\mathcal{A}$ be a simple C^*-algebra that is not isomorphic to $\mathbb{C}$, and let τ be a tracial state on $\mathcal{A}$. Furthermore, let $(\mathcal{H}_\tau, \pi_\tau, \xi_\tau)$ be the GNS representation of $\mathcal{A}$ associated with τ. We will use the constructions made in this exercise along with the results found for them and consider the homomorphisms ρ_τ as defined above. It suffices to show that $\pi_\tau(\mathcal{A})' \neq \mathbb{C}I$ (see Exercise 2); then it follows from Exercises 2(iii) and 3(iv) that τ is not pure.

Assume for contradiction that $\pi_\tau(\mathcal{A})' = \mathbb{C}I$. From (v) we then obtain that for each $x \in \mathcal{A}$, $\rho_\tau(x) = \lambda_x I$ for some $\lambda_x \in \mathbb{C}$ which is necessarily unique. Define $\varphi : \mathcal{A} \to \mathbb{C}$ by $\varphi(x) = \lambda_x$. Note now that since $\mathcal{A}$ is unital and

$$\rho_\tau(1_\mathcal{A})([a]) = R_{1_\mathcal{A}}([a]) = [a1_\mathcal{A}] = [a] = I([a])$$

for all $a \in \mathcal{A}$ then because both $\rho_\tau(1_\mathcal{A})$ and I are continuous and $\mathcal{H}_\tau^\circ$ is dense in $\mathcal{H}_\tau$, it follows that $\rho_\tau(1_\mathcal{A}) = I$. Hence $\varphi(1_\mathcal{A}) = 1$, so $\varphi \neq 0$. Note additionally that because ρ_τ is linear, then for all $x, y \in \mathcal{A}$ and $\mu, \mu' \in \mathbb{C}$ we obtain

$$\lambda_{\mu x + \mu' y} I = \rho_\tau(\mu x + \mu' y) = \mu \rho_\tau(x) + \mu' \rho_\tau(y) = (\mu \lambda_x + \mu' \lambda_y) I,$$

so $\varphi(\mu x + \mu' y) = \mu \varphi(x) + \mu' \varphi(y)$. Hence φ is linear. Additionally, by (iv) we obtain

$$\lambda_{xy} I = \rho_\tau(xy) = \rho_\tau(y)\rho_\tau(x) = (\lambda_y I)(\lambda_x I) = \lambda_x \lambda_y I$$

and hence $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in \mathcal{A}$, so φ is multiplicative and hence a character on $\mathcal{A}$. Lemma 2 now yields that $\ker \varphi$ is a proper and closed two-sided ideal in $\mathcal{A}$. Because $\mathcal{A}$ is simple and $\ker \varphi$ is proper then $\ker \varphi = \{0\}$, so φ is injective. Since characters on C^*-algebras are *-homomorphisms (see Week Sheet 5, Exercise 2) and φ clearly is surjective (it is non-zero) then $\mathcal{A} \cong \mathbb{C}$, a contradiction. Hence $\pi_\tau(\mathcal{A})' \neq \mathbb{C}I$ and therefore τ is not pure.

(One can show that a tracial state is pure if and only if it is a character. This exercise also says that a tracial state on a unital C^*-algebra $\mathcal{A}$ gives rise to a representation $\pi_\tau : \mathcal{A} \to B(\mathcal{H}_\tau)$ and an anti-representation $\rho_\tau : \mathcal{A} \to B(\mathcal{H}_\tau)$ whose images commute with each other.)

References

[1] K. Zhu, *An Introduction to Operator Algebras*, Studies in Advanced Mathematics, CRC Press, 1993.