KomAn exercise set 1

Rasmus Sylvester Bryder

February 11th, 2010

Exercise 1.7

Let $f : G \to \mathbb{C}$ be holomorphic in a domain G and assume that $|f|$ is constant. We wish to prove that f itself is constant.

f can be written in the form $f = u + iv$ where $u = \mathrm{Re} f$ and $v = \mathrm{Im} f$; u and v are then functions of two real variables x, y such that $x + iy \in G$. Assume that $|f(z)| = k$, $k \geq 0$ for all $z \in G$. We have for any $z = x + iy \in G$ that

$$k^2 = |f(z)|^2 = |u(x,y) + iv(x,y)|^2 = (u(x,y))^2 + (v(x,y))^2,$$

or $u^2 + v^2 = k^2$. If $k = 0$, then u and v must be 0 for all $z \in G$, so f will be constant (equal to 0). From here on we will assume that $u^2 + y^2 = k$, $k > 0$. / u and v are differentiable for all x, y such that $x + iy \in G$ because f is holomorphic in G (CB 1.6); then so are u^2 and v^2 and the sum of them.

Since $u^2 + v^2$ is a function of two real variables x, y, we get by partial differentation with respect to x and y that $\partial/\partial x(u^2 + v^2) = \partial/\partial y(u^2 + v^2) = 0$, $u^2 + v^2$ being constant.

From this we get $\partial/\partial x(u^2) + \partial/\partial x(v^2) = 0$ and $\partial/\partial y(u^2) + \partial/\partial y(v^2) = 0$ from standard operations with derivatives.

These all being real-valued functions differentiated, we get that

1. $u\partial u/\partial x + v\partial v/\partial x = 0$ (for fixed y and all x such that $x + iy \in G$) and
2. $u\partial u/\partial y + v\partial v/\partial y = 0$ (for fixed x and all y such that $x + iy \in G$),

since $(f^2)' = 2ff'$ for any real-valued differentiable function f (using the differential quotient of the composition of differentiable functions). Thus the above equations hold for all $z = x + iy \in G$.

Because f is holomorphic in G, we get from the Cauchy-Riemann equations that $v\partial v/\partial x = -v\partial u/\partial y$ and $v\partial v/\partial y = v\partial u/\partial x$, giving us the linear system of equations

$$u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} = 0 \quad \text{and} \quad v\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial y} = 0.$$

We wish to solve this with regards to $(\partial u/\partial x, \partial u/\partial y)$. The determinant of the system is thus $u^2 + v^2 = k^2 > 0$, and Cramer's formulae tell that

$$\frac{\partial u}{\partial x} = \frac{\begin{vmatrix} 0 & -v \\ 0 & u \end{vmatrix}}{u^2 + v^2} = 0 \quad \text{and} \quad \frac{\partial u}{\partial y} = \frac{\begin{vmatrix} u & 0 \\ v & 0 \end{vmatrix}}{u^2 + v^2} = 0.$$

Because of the Cauchy-Riemann equations (since f is holomorphic on G), we get that $\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$ as well.

Alas we get from the mean value theorem that u is constant on every vertical and horizontal line segment in G; same goes for v. Since G is a domain, u and v are constant in G, so $f = u + iv$ is constant as well.

Exercise 1.10

We wish to prove the addition formulae

$$\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$$
$$\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$$

plus the formula $(\cos z)^2 + (\sin z)^2 = 1$ for all $z_1, z_2, z \in \mathbb{C}$.

Let $z_1, z_2 \in \mathbb{C}$. With Eulers formulae we get

$$\begin{aligned}
&\sin z_1 \cos z_2 + \cos z_1 \sin z_2 \\
= \; & \frac{e^{iz_1} - e^{-iz_1}}{2i} \cdot \frac{e^{iz_2} + e^{-iz_2}}{2} + \frac{e^{iz_1} + e^{-iz_1}}{2} \cdot \frac{e^{iz_2} - e^{-iz_2}}{2i} \\
= \; & \frac{(e^{iz_1} - e^{-iz_1})(e^{iz_2} + e^{-iz_2}) + (e^{iz_1} + e^{-iz_1})(e^{iz_2} - e^{-iz_2})}{4i} \\
= \; & \frac{(e^{iz_1}e^{iz_2} - e^{-iz_1}e^{iz_2} + e^{iz_1}e^{-iz_2} - e^{-iz_1}e^{-iz_2})}{4i} \\
& + \frac{(e^{iz_1}e^{iz_2} + e^{-iz_1}e^{iz_2} - e^{iz_1}e^{-iz_2} - e^{-iz_1}e^{-iz_2})}{4i} \\
= \; & \frac{2e^{iz_1}e^{iz_2} - 2e^{-iz_1}e^{-iz_2}}{4i} = \frac{e^{i(z_1+z_2)} - e^{-i(z_1+z_2)}}{2i} = \sin(z_1 + z_2),
\end{aligned}$$

and

$$\begin{aligned}
&\cos z_1 \cos z_2 - \sin z_1 \sin z_2 \\
= \; & \frac{e^{iz_1} + e^{-iz_1}}{2} \cdot \frac{e^{iz_2} + e^{-iz_2}}{2} - \frac{e^{iz_1} - e^{-iz_1}}{2i} \cdot \frac{e^{iz_2} - e^{-iz_2}}{2i} \\
= \; & \frac{(e^{iz_1} + e^{-iz_1})(e^{iz_2} + e^{-iz_2}) + (e^{iz_1} - e^{-iz_1})(e^{iz_2} - e^{-iz_2})}{4} \\
= \; & \frac{(e^{iz_1}e^{iz_2} + e^{-iz_1}e^{iz_2} + e^{iz_1}e^{-iz_2} + e^{-iz_1}e^{-iz_2})}{4} \\
& + \frac{(e^{iz_1}e^{iz_2} - e^{-iz_1}e^{iz_2} - e^{iz_1}e^{-iz_2} + e^{-iz_1}e^{-iz_2})}{4} \\
= \; & \frac{2e^{iz_1}e^{iz_2} + 2e^{-iz_1}e^{-iz_2}}{4} = \frac{e^{i(z_1+z_2)} + e^{-i(z_1+z_2)}}{2} = \cos(z_1 + z_2).
\end{aligned}$$

Letting $z \in \mathbb{C}$, we get that

$$(\cos z)^2 + (\sin z)^2 = (\cos z + i \sin z)(\cos z - i \sin z) = e^{iz}e^{-iz} = e^0 = 1.$$