Sølvkorn 10
Topological groups

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All of the following is inspired by the exercises concerning topological groups in [1] in sections 22 to 26.

For any topological space X, we define the *closure* of a subset A to be the intersection of all closed sets in X containing A and denote it by $\overline{A}$.

Theorem 1. *For any subset A of a topological space X, then $x \in \overline{A}$ if and only if every neighbourhood of x intersects A.*

Proof. We prove the contrapositive of the above statement. If $x \notin \overline{A}$, we want to find a neighbourhood of x that doesn't intersect A, but it's pretty clear that $X \setminus \overline{A}$ satisfies this condition. If U is a neighbourhood of a point x and U doesn't intersect A, then $X \setminus U$ is a closed set that doesn't contain x and contains A. Thus $\overline{A} \subseteq X \setminus U$, and since $x \notin X \setminus U$, we have $x \notin \overline{A}$. □

Note that the above also holds if X has a basis and we replace "neighbourhood of x" by "basis element containing x". Indeed, every basis element containing x is also a neighbourhood of x, and if every basis element containing x intersects A, then if U is a neighbourhood of x, there exists a basis element B such that $x \in B \subseteq U$. Since $B \cap A \neq \emptyset$, then $U \cap A \neq \emptyset$.

Theorem 2. *Let $A \subseteq X$ and $B \subseteq Y$, where X and Y are topological spaces. Then $\overline{A} \times \overline{B} = \overline{A \times B}$.*

Proof. Since $(X \times Y) \setminus (\overline{A} \times \overline{B}) = (X \setminus \overline{A}) \times (Y \setminus \overline{B})$ is open in $X \times Y$, $\overline{A} \times \overline{B}$ is closed in $X \times Y$. Alas, since $A \times B \subseteq \overline{A} \times \overline{B}$, we obtain $\overline{A} \times \overline{B} \supseteq \overline{A \times B}$.

Let $a \in \overline{A}$ and $b \in \overline{B}$; we aim to show that $(a, b) \in \overline{A \times B}$. If $U \times V$ is a basis element of $X \times Y$ containing (a, b), U is open in X and intersecting A, and V is open in Y intersecting B. Alas $U \times V$ intersects $A \times B$, so by the remark following Theorem 1, we have $(a, b) \in \overline{A \times B}$. □

Theorem 3. *Let X and Y be topological spaces. If $f : X \to Y$ is continuous, then $f(\overline{A}) \subseteq \overline{f(A)}$ for any subset $A \subseteq X$.*

Proof. Let $x \in \overline{A}$ and let V be a neighbourhood of $f(x)$; we want to show that V intersects $f(A)$ so that by Theorem 1, we have $f(x) \in \overline{f(A)}$. $f^{-1}(V)$ is a neighbourhood of x, so by Theorem 1, it intersects A in some point y. Then $f(y) \in V \cap f(A)$, and we are done. □

To ease notation, the binary operation of a group G is called the *product*. For a group G, we will denote the product map $G \times G \to G$ by p_G (or p) and the inversion map $G \to G$ by i_G (or i).

Definition 4. Let G be a group equipped with a topology. G is called a *topological group* if the product map p and inversion map i are continuous.

Lemma 5. *For any two topological spaces X and Y and $\alpha \in X$, $\beta \in Y$, as well as continuous maps $f : X \to X$, $g : Y \to Y$, the maps $(x, y) \mapsto (f(x), y)$ and $(x, y) \mapsto (x, g(y))$ are continuous.*

Proof. Obvious. □

Lemma 6. *If G is a group equipped with a topology, and the map $c : (x, y) \mapsto xy^{-1}$ is continuous, then G is a topological group.*

Proof. Let e denote the neutral element of G. The map $x \mapsto c(e, x)$ is continuous and equal to i. Now the map $(x, y) \mapsto c(x, i(y))$ is continuous and equal to p. □

We could make use of the following lemma, and it would most likely ease the burden of showing that a topological space is a topological group, but we won't, as it encourages showing that two quite different maps are continuous. G denotes a topological group from here onward.

1 Topological groups

Theorem 7. *Let H be a subspace of G. If H is also a subgroup of G, then both H and $\overline{H}$ are topological groups.*

Proof. The subspace topology and the product topology on $H \times H$ are the same. The restrictions of the product and inversion maps are still continuous: letting ι and ι^2 denote the (continuous) inclusion mappings $H \to G$ and $H \times H \to G \times G$, we have $p|_{H\times H} = p \circ \iota^2$ and $i|_H = i \circ \iota$. Note that this method generalizes easily, so that any group contained in G is topological.

Now we consider $\overline{H}$. It is clear that the neutral element $e \in G$ is contained in $\overline{H}$. By Theorem 2 and 3, $p(\overline{H} \times \overline{H}) = p(\overline{H \times H}) \subseteq \overline{p(H \times H)} \subseteq \overline{H}$, since H is a subgroup, so that the product is stable, and $i(\overline{H}) \subseteq \overline{i(H)} = \overline{H}$, so that any inverse of an element in $\overline{H}$ is contained in $\overline{H}$. Alas $\overline{H}$ is a group and thus a topological group. □

Theorem 8. *For any pair x, y of points of a topological group G, there exists a homeomorphism of G onto itself that carries x to y.*

Proof. We first show for any element $\alpha \in G$ that the maps $v_\alpha, w_\alpha : G \to G$ defined by $v_\alpha(g) = \alpha g$ and $w_\alpha(g) = g\alpha$ are homeomorphisms of G. v_α and w_α are obviously bijections with the inverses $v_\alpha^{-1}(g) = \alpha^{-1}g$ and $w_\alpha^{-1}(g) = g\alpha^{-1}$. It now follows that v_α and w_α as well as their inverses are continuous, by Lemma 5 and continuity of the product map.

Now define the map $f : G \to G$ by $f(g) = ygx^{-1}$. It's a homeomorphism since $f = v_y \circ w_{x^{-1}}$ and it carries x to y. □

Note that we actually only needed v_α since the map $v_y \circ v_{x^{-1}}$ has the same wanted properties.

Definition 9. Any topological space that satisfies the condition of Theorem 8 is called a *homogeneous space*.

From here onward, we let G/H denote the set of *left* cosets of a subgroup H in G. Letting π denote the surjective *canonical mapping* $G \to G/H$ that carries elements of G to the left cosets of H containing them, we give G/H the *quotient topology*; that is, a subset $U \subseteq G/H$ is open if $\pi^{-1}(U)$ (the union of the elements in U) is open in G. Alternatively, a subset $U \subseteq G/H$ is closed if $\pi^{-1}(U)$ is closed in G, which follows from the general equation

$$f^{-1}(Y \setminus B) = X \setminus f^{-1}(B),$$

where f is a map $X \to Y$ and $B \subseteq Y$. We call G/H a *quotient space* of G.

Theorem 10. *$\pi : G \to G/H$ is an open map.*

Proof. Let A be open in G. Then

$$\pi^{-1}(\pi(A)) = \bigcup_{a \in A} aH = \bigcup_{h \in H} Ah = \bigcup_{h \in H} w_h(A)$$

and is therefore a union of open sets, w_h being a homeomorphism, so $\pi(A)$ is open in G/H. □

Theorem 11. *G/H is a homogeneous space.*

Proof. Let $x, y \in G$ and consider the left cosets xH and yH. For $\alpha \in G$, the homeomorphism v_α from the proof of Theorem 8 induces the mapping $gH \mapsto v_\alpha(g)H$ from G/H onto itself. The map is well-defined: indeed, if $g_1H = g_2H$, then $\alpha g_1 H = \alpha g_2 H$. It is also clearly a bijection, with the inverse $gH \mapsto v_{\alpha^{-1}}(g)H$, and furthermore, it is continuous.

Indeed, denote the mapping by P_α. For any open set $U \subseteq G/H$, we will need to check if $\mathbb{U} := \pi^{-1}(P_\alpha^{-1}(U))$ is open so that $P_\alpha^{-1}(U)$ will be open. We have

$$g \in \mathbb{U} \Leftrightarrow v_\alpha(g)H \in U \Leftrightarrow \pi(v_\alpha(g)) \in U \Leftrightarrow g \in v_\alpha^{-1}(\pi^{-1}(U)),$$

and since $\pi^{-1}(U)$ is open and v_α is a homeomorphism, $\mathbb{U}$ is open. The same method applies to show that the inverse of P_α is continuous, and so P_α is a homeomorphism from G/H onto itself. The wanted mapping that makes G/H homogeneous is thus $P_y \circ P_{x^{-1}}$. □

Corollary 12. *If H is closed in G, G/H is T_1.*

Proof. $\{H\}$ is closed in G/H per the alternative formulation of the quotient topology, since H is closed in G. Let $gH \in G/H$ and consider a homeomorphism P that carries gH to H in G/H per the preceding theorem. Then $\{gH\}$ is equal to $P^{-1}(\{H\})$ and is therefore closed. □

We define $\pi^2 : G \times G \to G/H \times G/H$ by $\pi^2(g_1, g_2) = (\pi(g_1), \pi(g_2))$. π^2 is an open map. Indeed, if A is open in $G \times G$, A is a union of sets $U_i \times V_i$, where U_i and V_i are open in G. Then

$$\pi^2(A) = \pi^2\left(\bigcup_i U_i \times V_i\right) = \bigcup_i \pi^2(U_i \times V_i) = \bigcup_i \pi(U_i) \times \pi(V_i),$$

which is open in $G/H \times G/H$.

Theorem 13. *If H is normal, G/H is a topological group.*

Proof. It is well-known that normality of H induces a well-defined product on G/H by defining

$$(g_1H)(g_2H) := (g_1g_2)H.$$

Associativity follows from associativity of G, H is the neutral element and the inverse of xH is $x^{-1}H$. Let's show that the product is indeed well-defined: if $g_1H = g_1'H$ and $g_2H = g_2'H$, then $g_1'^{-1}g_1 \in H$ and $g_2'^{-1}g_2 \in H$. Since H is normal, $g_2'^{-1}(g_1'^{-1}g_1)g_2' \in H$, and so

$$(g_1'g_2')^{-1}(g_1g_2) = (g_2'^{-1}(g_1'^{-1}g_1)g_2')(g_2'^{-1}g_2) \in H,$$

since H is a subgroup. Therefore $(g_1g_2)H \subseteq (g_1'g_2')H$, and since $(g_1g_2)H$ is non-empty, we have $(g_1g_2)H = (g_1'g_2')H$, as the left cosets of H are a partition of G. The inversion is well-defined too: if $g_1H = g_2H$, then $g_2^{-1}g_1 \in H$; since H is normal, then $g_1g_2^{-1} = g_2(g_2^{-1}g_1)g_2^{-1} \in H$, so $g_1^{-1}H \subseteq g_2^{-1}H$, and thus $g_1^{-1}H = g_2^{-1}H$.

It remains to show that the product and inversion are continuous. Let A be open in G/H. Then $\pi^{-1}(A)$ is open in G. Let $p_{G/H}$ and $i_{G/H}$ denote the product and inversion maps on G/H respectively; then

$$g \in \pi^{-1}(i_{G/H}^{-1}(A)) \Leftrightarrow i_{G/H}(\pi(g)) = \pi(i_G(g)) \in A \Leftrightarrow g \in i_G^{-1}(\pi^{-1}(A)).$$

Alas $\pi^{-1}(i_{G/H}^{-1}(A))$ is open in G, so $i_{G/H}$ is continuous. Since

$$\begin{aligned}
\pi^2(p_G^{-1}(\pi^{-1}(A))) &= \{(g_1H, g_2H) \mid (g_1, g_2) \in p_G^{-1}(\pi^{-1}(A))\} \\
&= \{(g_1H, g_2H) \mid (g_1g_2)H \in A)\} \\
&= p_{G/H}^{-1}(A),
\end{aligned}$$

and π^2 is open, $p_{G/H}$ is continuous as well. □

2 Assumption of T_1 property

For any two subsets A, B of a topological group G, we let AB denote the set of all products ab, $a \in A$, $b \in B$, and A^{-1} the set of all points a^{-1}, $a \in A$. If A is open, then A^{-1} is open as well (the inversion being a homeomorphism).

Definition 14. A neighbourhood V of the neutral element e is said to be *symmetric* if $V = V^{-1}$.

Lemma 15. *For any neighbourhood U of e, there is a symmetric neighbourhood V of e such that $VV \subseteq U$.*

Proof. *(1)* Any neighbourhood W of e contains a symmetric neighbourhood of e, namely $W \cap W^{-1}$.

(2) Any neighbourhood W of e contains a neighbourhood Z of e such that $ZZ \subseteq W$. Since $p^{-1}(W)$ is open and contains (e, e), there now exist open sets $Z_1, Z_2 \subseteq G$ such that $(e, e) \in Z_1 \times Z_2 \subseteq p^{-1}(W)$. Let $Z = Z_1 \cap Z_2$. Z is open, and $ZZ \subseteq Z_1Z_2 \subseteq W$.

(3) Now we prove the lemma. First choose a neighbourhood Z of e such that $ZZ \subseteq U$. Choose a symmetric neighbourhood V of e contained in Z. Then $VV \subseteq ZZ \subseteq U$. □

Definition 16. A T_1 topological space X is *regular* if for each pair consisting of a point $x \in X$ and a closed set $C \subseteq X$ disjoint from x, there exist disjoint open sets containing x and C, respectively.

It's obvious that regular spaces are Hausdorff.

Theorem 17. *Assume that G is T_1. Then G is regular.*

Proof. Let $x \in G$ and C be a closed set not containing x. The T_1 assumption secures that $W := G \setminus Cx^{-1}$ is a neighbourhood of e. By the preceding lemma, there exists a symmetric neighbourhood V of e such that $VV \subseteq W$. Assume that $Vx \cap VC$ is non-empty. There exist $v_1, v_2 \in V$, $c \in C$ such that $v_1 x = v_2 c$. Then $v_2^{-1} v_1 = cx^{-1}$. Since V is symmetric, $v_2^{-1} v_1 \in VV \subseteq W$, so $cx^{-1} \subseteq Cx^{-1} \in W$, a contradiction.

Because w_α from Theorem 8 is a homeomorphism and V is open containing e, Vx and $VC = \bigcup_{c \in C} Vc$ are disjoint open sets containing x and C, respectively. Thus G is regular. □

If H is a closed normal subgroup of G, G/H becomes a regular topological group by Corollary 12 and the preceding theorem. Regularity isn't a consequence of G/H becoming a group, however.

Theorem 18. *Let H be a closed subgroup of G. Then G/H with the quotient topology is a regular topological space.*

Proof. G/H is T_1 by Corollary 12. Let $gH \in G/H$ and let $A \in G/H$ be closed in G/H, not containing gH. Then gH and $\mathbb{A} = \pi^{-1}(A)$ are closed in G. The set $W := G \setminus \mathbb{A}Hg^{-1}$ is now a neighbourhood of e. By Lemma 15, there exists a symmetric neighbourhood V of e such that $VV \subseteq W$.

Thus $V(gH) = \bigcup_{\alpha \in gH} w_\alpha(V)$ and $V\mathbb{A} = \bigcup_{\alpha \in \mathbb{A}} w_\alpha(V)$ are open sets that contain gH and AH, respectively. Since π is an open map, we have that $\pi(VgH) = \{(vg)H \mid v \in V\}$ and $\pi(V\mathbb{A}) = \{(va)H \mid v \in V, a \in \mathbb{A}\}$ are disjoint open sets in G/H containing gH and A; indeed, if they weren't disjoint, there would exist $v_1, v_2 \in V$, $a \in \mathbb{A}$ such that $(v_1 g)H = (v_2 a)H$, which would imply that $a^{-1} v_2^{-1} v_1 g = h$ for some $h \in H$ or $W \supseteq VV \ni v_2^{-1} v_1 = ahg^{-1} \in \mathbb{A}Hg^{-1}$, a contradiction. □

3 Connectedness and compactness

Recall that a *component* of a topological space X is an equivalence class determined by the equivalence relation $\sim$ and $x \sim y$ if there is a connected subspace of X containing x and y.

Theorem 19. *Let G be a topological group and let C be the component of G containing the neutral element e. C is a normal subgroup of G.*

Proof. If A is a connected subspace with a point x in common with C, then assuming there is a point in A not contained in C, A must intersect some other component of G in a point y. But then $x \sim y$, a contradiction since they lie in different equivalence classes. Alas $A \subseteq C$.

Let $f, g \in C$. Then fC and C are connected; having the point f in common, $fC \subseteq C$, and so $fg \in C$. Likewise, $f^{-1}C \subseteq C$, so $f^{-1} \in C$. Since $e \in C$, C is a subgroup. Let $x \in G$. Now xCx^{-1} is connected containing e. Thus $xCx^{-1} \subseteq C$ and therefore C is normal. □

Theorem 20. *Let G be a T_1 topological group with subspaces A, B. If A is closed and B is compact, AB is closed.*

Proof. The T_1 assumption is for general niceness (since G is then Hausdorff). Let c be in the closure of AB. Then there is a net $(c_\alpha)_\alpha$ in AB such that $c_\alpha \to c$. For all α, we have $c_\alpha = a_\alpha b_\alpha$ for some $a_\alpha \in A$ and $b_\alpha \in B$. Since B is compact, there exists a subnet $(b_\beta)_\beta$ that converges, so that $b_\beta \to b$ for some $b \in B$. This implies that $a_\beta = c_\alpha b_\alpha^{-1} \to cb^{-1}$ by continuity of the product. Since A is closed, then $a = cb^{-1}$, so $c = ab \in AB$, and we are done. □

Corollary 21. *Let H be a compact subgroup of a T_1 topological group G. Then $\pi : G \to G/H$, the surjective canonical mapping, is closed.*

Proof. Let $A \in G$ be closed; we aim to show that $\pi(A) = \{aH \mid a \in A\}$ is closed in G/H, but this is apparent since $\pi^{-1}(\pi(A)) = \bigcup_{a\in A} aH = AH$ is closed by the preceding theorem. □

Theorem 22. *Let H be a compact subgroup of a topological group G. If G/H is compact when equipped with the quotient topology, then G is compact.*

Proof. Let $\{G_i\}_{i\in I}$ be an open covering of G. For all $\alpha \in G/H$, $\pi^{-1}(\{\alpha\})$ is compact in G, since $\alpha = aH$ for some $a \in G$, and $\pi^{-1}(\alpha)$ is the image of H under the homeomorphism v_a. Thus for all $\alpha \in G/H$, $\pi^{-1}(\{\alpha\})$ can be covered by a finite subcollection of $\{G_i\}$, say $\{G_{i_j^\alpha} \mid i_j^\alpha \in I,\ j = 1, \dots, n_\alpha\}$. Now, for all $\alpha \in G/H$ let

$$G_\alpha = \bigcup_{j=1}^{n_\alpha} G_{i_j^\alpha}, \quad F_\alpha = (G/H) \setminus (\pi(G \setminus G_\alpha)).$$

G_α is open, so since π is a closed map, F_α is open as well. Secondly, because $\pi^{-1}(\alpha) \subseteq G_\alpha$, then $\pi^{-1}(\alpha)$ is not contained in $G \setminus G_\alpha$. Thus $\alpha \notin \pi(G \setminus G_\alpha)$ and $\alpha \in F_\alpha$. Thus $\{F_\alpha\}_{\alpha\in G/H}$ is an open covering of G/H. G/H is compact, so it is covered by a finite subcollection $\{F_{\alpha_i} \mid \alpha_i \in G/H,\ i = 1, \dots, n\}$. Now,

$$\pi^{-1}(F_\alpha) = \pi^{-1}((G/H)\setminus(\pi(G\setminus G_\alpha))) = G\setminus\pi^{-1}(\pi(G\setminus G_\alpha)) \subseteq G\setminus(G\setminus G_\alpha) = G_\alpha,$$

since $A \subseteq f^{-1}(f(A))$ for any map $f : A \to B$. Therefore

$$G = \pi^{-1}(G/H) = \pi^{-1}\left(\bigcup_{i=1}^{n} F_{\alpha_i}\right) = \bigcup_{i=1}^{n} \pi^{-1}(F_{\alpha_i}) \subseteq \bigcup_{i=1}^{n} G_{\alpha_i}.$$

Thus we have found a finite subcover of G, and G is compact. □

References

[1] James R. Munkres, *Topology.* Prentice Hall, Upper Saddle River, Second Edition, 2000.