Sølvkorn 12
Funny prime lemmas

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Lemma 1. *For a prime $p \in \mathbb{N}$, $\sqrt{p}$ is irrational.*

Proof. We will use Euclid's lemma for prime numbers, namely that if $p \mid ab$ for $a, b \in \mathbb{Z}$ then $p \mid a$ or $p \mid b$. Assume that $\sqrt{p}$ is rational and write $\sqrt{p} = \frac{r}{s}$ where the fraction is irreducible. Then $s^2 p = r^2$. Now, $p \mid r^2$ so $p \mid r$ by Euclid's lemma. Let $t \in \mathbb{N}$ such that $r = pt$. Then $s^2 p = p^2 t^2$, so $s^2 = pt^2$. Now, $p \mid s^2$ so $p \mid s$ by Euclid's lemma. This contradicts that the original fraction was irreducible. □

Lemma 2. *For different primes $p, q \in \mathbb{N}$, $\sqrt{pq}$ is irrational.*

Proof. We will use Euclid's lemma for prime numbers, namely that if $p \mid ab$ for $a, b \in \mathbb{Z}$ then $p \mid a$ or $p \mid b$. Assume that $\sqrt{pq}$ is rational and write $\sqrt{pq} = \frac{r}{s}$ where the fraction is irreducible. Then $s^2 pq = r^2$. Now, $p \mid r^2$ and $q \mid r^2$ so $p \mid r$ and $q \mid r$ by Euclid's lemma. Since p and q are different, $pq \mid r$: if $r = pu = qv$ for some $u, v \in \mathbb{N}$, then $q \mid u$, so $r = p(qw)$ for some $w \in \mathbb{N}$. Let $t \in \mathbb{N}$ such that $r = (pq)t$. Then $s^2 pq = p^2 q^2 t^2$, so $s^2 = pqt^2$. Now, $p \mid s^2$ and $q \mid s^2$ so $p \mid s$ and $q \mid s$ by Euclid's lemma. This again implies $pq \mid s$, but since $pq \mid r$, this contradicts that the original fraction was irreducible. □