Sølvkorn 15
The continuous functional calculus for arbitrary C^*-algebras

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This nugget will continue where [3] left off, regarding the properties of the continuous functional calculus. We will try to construct a valid extension of this calculus to any arbitrary C^*-algebras, and not just unital ones as covered in the above source.

In order to find a way to discuss the continuous functional calculus for a non-unital C^*-algebra, then to even consider it we cannot escape from the fact that we need to be able to work with a unit. This of course implies that the unitization has to be part of the discussion, and once we have brought it up we will try to wring loose of the claws that this extra structure has to grab us with. We know beforehand that this is no suicide mission, so we will succeed (no need to wet the bed here).

1 The unitization

Throughout this section, $\mathcal{A}$ denotes a non-unital C^*-algebra.

In order to construct the unitization for $\mathcal{A}$, we define a map $L_a\colon \mathcal{A} \to \mathcal{A}$ for any $a \in \mathcal{A}$ given by $L_a b = ab$ for $b \in \mathcal{A}$. This map is clearly well-defined, linear and bounded with $\|L_a\| \le \|a\|$. Hence L_a belongs to the Banach algebra $B(\mathcal{A})$ for all $a \in \mathcal{A}$. Let $\tilde{\mathcal{A}}$ be the subset of $B(\mathcal{A})$ given by

$$\tilde{\mathcal{A}} = \{L_a + \lambda \mathbb{1} \,|\, a \in \mathcal{A},\ \lambda \in \mathbb{C}\},$$

where $\mathbb{1}$ denotes the identity operator $\mathcal{A} \to \mathcal{A}$, and define a map $\Omega\colon \mathcal{A} \to \tilde{\mathcal{A}}$ by $\Omega(a) = L_a$.

Lemma 1. *The map Ω is an isometric algebra homomorphism such that $\Omega(a) \neq \mathbb{1}$ for all $a \in \mathcal{A}$. $\tilde{\mathcal{A}}$ is a subalgebra of $B(\mathcal{A})$.*

Proof. For $a, b, c \in \mathcal{A}$ and $\lambda \in \mathbb{C}$, we have

(i) $\Omega(a+b)(c) = L_{a+b}c = (a+b)c = ac + bc = L_a c + L_b c = (L_a + L_b)c = (\Omega(a) + \Omega(b))(c)$,
(ii) $\Omega(\lambda a)(c) = L_{\lambda a}c = (\lambda a)c = \lambda(ac) = (\lambda L_a)c = (\lambda\Omega(a))(c)$,
(iii) $\Omega(ab)(c) = L_{ab}c = (ab)c = a(bc) = L_a(L_b c) = \Omega(a)(\Omega(b)(c)) = (\Omega(a)\Omega(b))(c)$.

To see that Ω is an isometry, note that $\|\Omega(a)\| \le \|a\|$ and

$$\|a\|^2 = \|a^*\|^2 = \|aa^*\| = \|\Omega(a)(a^*)\| \le \|\Omega(a)\|\|a^*\| = \|\Omega(a)\|\|a\|.$$

If there were an $a \in \mathcal{A}$ such that $\Omega(a) = \mathbb{1}$, then for any $b \in \mathcal{A}$ we would have

$$ab = \Omega(a)(b) = \mathbb{1}b = b.$$

Hence a is a left unit for $\mathcal{A}$. The above equality also implies $ab^* = b^*$ or $ba^* = b$ for all $b \in \mathcal{A}$, so a^* is a right unit for $\mathcal{A}$. Thus $a = aa^* = a^*$, implying that a is a unit, contradicting the assumption that $\mathcal{A}$ is non-unital. The final statement now follows from the fact that Ω and the map $\mathbb{C} \to \tilde{\mathcal{A}}$ given by $\lambda \mapsto \lambda\mathbb{1}$ are algebra homomorphisms. □

Corollary 2. *The map $\mathcal{A} \times \mathbb{C} \to \tilde{\mathcal{A}}$ given by $(a, \lambda) \mapsto L_a + \lambda\mathbb{1}$ is a linear isomorphism.*

Proof. Surjectivity and linearity is clear. If $a_1, a_2 \in \mathcal{A}$ and $\lambda_1, \lambda_2 \in \mathbb{C}$ satisfy $L_{a_1} + \lambda_1\mathbb{1} = L_{a_2} + \lambda_2\mathbb{1}$, then $L_{a_1 - a_2} = (\lambda_2 - \lambda_1)\mathbb{1}$. If $\eta = \lambda_2 - \lambda_1$ were a non-zero number, we would have $L_{\eta^{-1}(a_1-a_2)} = \mathbb{1}$, contradicting the above lemma. Hence $\lambda_1 = \lambda_2$, so

$$\|a_1 - a_2\| = \|L_{a_1 - a_2}\| = \|(\lambda_2 - \lambda_1)\mathbb{1}\| = 0$$

and hence $a_1 = a_2$. □

The above corollary in turn yields that the map ${}^*\colon \tilde{\mathcal{A}} \to \tilde{\mathcal{A}}$ given by $(L_a + \lambda\mathbb{1})^* = L_{a^*} + \overline{\lambda}\mathbb{1}$ is well-defined, and one easily shows that it satisfies the properties of an involution. Hence $\tilde{\mathcal{A}}$ becomes a normed *-algebra.

Proposition 3. *For all $s \in \tilde{\mathcal{A}}$ we have $\|s^*s\| = \|s\|^2$.*

Proof. Let $a, x \in \mathcal{A}$ and $\lambda \in \mathbb{C}$. For any $b \in \mathcal{A}$ we have

$$(x^*a^* + \overline{\lambda}x^*)b = x^*(a^*b + \overline{\lambda}b) = x^*(L_a + \lambda\mathbb{1})^*b. \tag{†}$$

This in turn implies

$$\begin{aligned}
\|(L_a + \lambda\mathbb{1})x\|^2 &= \|ax + \lambda x\|^2 \\
&= \|(ax + \lambda x)^*(ax + \lambda x)\| \\
&= \|(x^*a^* + \overline{\lambda}x^*)(ax + \lambda x)\| \\
&= \|x^*(L_a + \lambda\mathbb{1})^*(L_a + \lambda\mathbb{1})x\| \\
&\le \|x^*\|\|(L_a + \lambda\mathbb{1})^*(L_a + \lambda\mathbb{1})x\| \le \|(L_a + \lambda\mathbb{1})^*(L_a + \lambda\mathbb{1})\|\|x\|^2,
\end{aligned}$$

using (†) at the fourth equality. Therefore

$$\|L_a + \lambda\mathbb{1}\|^2 \le \|(L_a + \lambda\mathbb{1})^*(L_a + \lambda\mathbb{1})\|.$$

The above inequality implies

$$\|L_a + \lambda\mathbb{1}\|^2 \le \|(L_a + \lambda\mathbb{1})^*\|\|L_a + \lambda\mathbb{1}\|,$$

so $\|L_a + \lambda\mathbb{1}\| \le \|(L_a + \lambda\mathbb{1})^*\| = \|L_{a^*} + \overline{\lambda}\mathbb{1}\|$. Replacing a by a^* and λ by $\overline{\lambda}$, we see that

$$\|L_a + \lambda\mathbb{1}\| = \|(L_a + \lambda\mathbb{1})^*\|.$$

This finally tells us that

$$\|L_a + \lambda\mathbb{1}\|^2 \le \|(L_a + \lambda\mathbb{1})^*(L_a + \lambda\mathbb{1})\| \le \|(L_a + \lambda\mathbb{1})^*\|\|L_a + \lambda\mathbb{1}\| = \|L_a + \lambda\mathbb{1}\|^2,$$

completing the proof. □

Proposition 4. *Let $\mathfrak{X}$ be a Banach space with closed subspaces $\mathfrak{Y}$ and $\mathfrak{Z}$. If $\mathfrak{Z}$ is finite-dimensional, then $\mathfrak{Y} + \mathfrak{Z}$ is a closed subspace of $\mathfrak{X}$.*

Proof. Recall that the quotient space $\mathfrak{X}/\mathfrak{Y}$ is a Banach space and that the quotient map $\pi\colon \mathfrak{X} \to \mathfrak{X}/\mathfrak{Y}$ is a linear contraction. Then $\pi(\mathfrak{Z})$ is a finite-dimensional subspace of $\mathfrak{X}/\mathfrak{Y}$, so it must be closed. Therefore $\pi^{-1}(\pi(\mathfrak{Z}))$ is closed as well by continuity, but

$$\begin{aligned}
x \in \pi^{-1}(\pi(\mathfrak{Z})) &\Leftrightarrow \pi(x) = \pi(z) \text{ for some } z \in \mathfrak{Z} \\
&\Leftrightarrow x - z = y \text{ for some } z \in \mathfrak{Z} \text{ and } y \in \mathfrak{Y} \\
&\Leftrightarrow x \in \mathfrak{Y} + \mathfrak{Z}.
\end{aligned}$$

Hence $\mathfrak{Y} + \mathfrak{Z}$ is closed. □

Corollary 5. *$\tilde{\mathcal{A}}$ is a closed subset of $B(\mathcal{A})$, making it a unital C^*-algebra.*

Proof. Applying Proposition 4 to $\mathfrak{X} = B(\mathcal{A})$, $\mathfrak{Y} = \Omega(\mathcal{A})$ and $\mathfrak{Z} = \mathbb{C}\mathbb{1}$ yields that $\tilde{\mathcal{A}}$ is closed. Hence $\tilde{\mathcal{A}}$ is a Banach *-algebra satisfying the C^*-identity (Proposition 3), so it is a C^*-algebra with unit $\mathbb{1}$. □

By Corollary 2, there is a linear isomorphism $\mathcal{A} \times \mathbb{C} \to \tilde{\mathcal{A}}$. Using this isomorphism, we can define a multiplication, involution and norm on $\mathcal{A} \times \mathbb{C}$ such that it becomes a unital C^*-algebra in which $\mathcal{A}$ is isometrically embedded by means of the map $\mathcal{A} \to \mathcal{A} \times \mathbb{C}$ given by $a \mapsto (a, 0)$. The multiplicative unit of $\mathcal{A} \times \mathbb{C}$ is the element $(0, 1)$. The C^*-algebra $\mathcal{A} \times \mathbb{C}$ is called the *unitization* of $\mathcal{A}$, and to honour the subset of $B(\mathcal{A})$ that made it possible, we will denote it by $\tilde{\mathcal{A}}$. The *-algebra operations in $\tilde{\mathcal{A}}$ are therefore

(i) $\mu_1(a_1, \lambda_1) + \mu_2(a_2, \lambda_2) = (\mu_1 a_1 + \mu_2 a_2, \mu_1\lambda_1 + \mu_2\lambda_2)$,
(ii) $(a_1, \lambda_1)(a_2, \lambda_2) = (a_1 a_2 + \lambda_1 a_2 + \lambda_2 a_1, \lambda_1\lambda_2)$ and
(iii) $(a_1, \lambda_1)^* = (a_1^*, \overline{\lambda_1})$

for $a_1, a_2 \in \mathcal{A}$ and $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{C}$. The norm is given by

$$\|(a, \lambda)\| = \sup\{\|ax + \lambda x\| \mid x \in \mathcal{A},\ \|x\| \le 1\}.$$

2 The continuous functional calculus, part I

We first take some time to construct the continuous functional calculus for unital C^*-algebras. If $\mathcal{A}$ is a unital C^*-algebra, the spectrum $\sigma(a)$ of an element $a \in \mathcal{A}$ consists of all the $\lambda \in \mathbb{C}$ such that $\lambda 1_{\mathcal{A}} - a$ is not invertible. The space of non-zero multiplicative linear functionals $\mathcal{A} \to \mathbb{C}$ (also called *characters*) is denoted by $\Delta(\mathcal{A})$. Any such is bounded with norm 1 and hence $\Delta(\mathcal{A})$ is a weak*-compact subset of $\mathcal{A}^*$ by Alaoglu's theorem.

The *Gelfand transform* for the unital C^*-algebra $\mathcal{A}$ is the contractive algebra homomorphism $\Gamma\colon \mathcal{A} \to C(\Delta(\mathcal{A}))$ given by

$$\Gamma(a)(\varphi) = \varphi(a), \quad a \in \mathcal{A},\ \varphi \in \Delta(\mathcal{A}).$$

If $\mathcal{A}$ is also commutative, we have that the range of $\Gamma(a)$ for any $a \in \mathcal{A}$ is in fact $\sigma(a)$. If $\mathcal{A}$ is a commutative unital C^*-algebra, Γ is an isometric *-isomorphism.

Theorem 6. *Let $\mathcal{A}$ be a unital C^*-algebra and let $a \in \mathcal{A}$ be normal. Then $C^*(1_{\mathcal{A}}, a)$ is commutative, and the character space $\mathfrak{M}$ of $C^*(1_{\mathcal{A}}, a)$ equipped with the weak* topology is homeomorphic to $\sigma(a)$ by the map $\Psi\colon \varphi \mapsto \varphi(a)$. Therefore the Gelfand transform becomes an isometric *-isomorphism $\Gamma\colon C^*(1_{\mathcal{A}}, a) \to C(\sigma(a))$.*

Proof. Omitted. See [3, Theorem 10.2]. □

Definition 7. If $\mathcal{A}$ is a unital C^*-algebra and $a \in \mathcal{A}$ is normal, the inverse unital *-homomorphism $\Gamma^{-1}\colon C(\sigma(a)) \to C^*(1_{\mathcal{A}}, a)$ is called the *continuous functional calculus* for a. For $f \in C(\sigma(a))$, we define $f(a) = \Gamma^{-1}(f)$.

It is common practice, but let us mention it anyway: if a is a normal element of a unital C^*-algebra $\mathcal{A}$ and $f \in C(\Omega)$ for some subset $\sigma(a) \subseteq \Omega \subseteq \mathbb{C}$, we define $f(a) = f|_{\sigma(a)}(a)$.

Theorem 8 (Properties of the continuous functional calculus for unital C^*-algebras)**.** *Let $\mathcal{A}$ be a unital C^*-algebra and let $a \in \mathcal{A}$. Then for all $\lambda, \mu \in \mathbb{C}$ and $f, g \in C(\sigma(a))$ we have*

(i) $(\lambda f + \mu g)(a) = \lambda f(a) + \mu g(a)$.
(ii) $(fg)(a) = f(a)g(a)$.
(iii) $\overline{f}(a) = f(a)^*$.
(iv) *If* 1 *denotes the identity map* $\sigma(a) \to \sigma(a)$*, then* $1(a) = a$.
(v) *If* 1 *denotes the constant function* $z \mapsto 1$ *for* $z \in \sigma(a)$*, then* $1(a) = 1_{\mathcal{A}}$.
(vi) *If* $P\colon z \mapsto p(z, \overline{z})$ *is a complex polynomial in* z *and* $\overline{z}$ *with no constant term, then* $P(a) = p(a, a^*)$.
(vii) *If* Ω *is a subset of* $\mathbb{C}$ *such that* $\sigma(a) \subseteq \Omega$ *and* $h \in C(\Omega)$*, then*

$$\|h(a)\| = \sup_{z \in \sigma(a)} |h(z)| \leq \sup_{z \in \Omega} |h(z)|.$$

(viii) $\sigma(f(a)) = f(\sigma(a))$.
(ix) *If* $h \in C(f(\sigma(a))$*, then* $(h \circ f)(a) = h(f(a))$.
(x) *If* Φ *is a unital* **-homomorphism of* $\mathcal{A}$ *into another unital* C^**-algebra* $\mathcal{B}$*, then* $\Phi(f(a)) = f(\Phi(a))$.

Proof. (i), (ii) and (iii) and (v) follow directly from the fact that the continuous functional calculus is a unital *-homomorphism. As for (iv), note that $\Gamma(a)$ is the identity map on $\sigma(a)$; any $z \in \sigma(a)$ corresponds uniquely to a character $\varphi \in \Delta(C^*(1_{\mathcal{A}}, a))$ such that $\varphi(a) = \Psi(\varphi) = z$. Under the identification of Theorem 6, we have

$$\Gamma(a)(z) = \Gamma(a)(\varphi) = \varphi(a) = z.$$

(vi) then follows from (i)-(v). (vii) is clear, since the continuous functional calculus is an isometry.

Since Γ^{-1} is a *-isomorphism, $\sigma(f) = \sigma(f(a))$. $f - \lambda$ is invertible if and only if $f(z) - \lambda \neq 0$ for all $z \in \sigma(a)$; hence $\lambda \in \sigma(f)$ if and only if $f(z) = \lambda$ for some $z \in \sigma(a)$ or $\lambda \in f(\sigma(a))$, so we conclude

$$\sigma(f(a)) = \sigma(f) = f(\sigma(a)),$$

and hence (viii).

To prove (ix), note that $g \circ f \in C(\sigma(a))$, so $(g \circ f)(a)$ is well-defined, and that $f(a) = \Gamma^{-1}(f)$ is normal in $\mathcal{A}$ since f is. Hence $g(f(a)) \in \mathcal{A}$ is well-defined, as $g \in C(\sigma(f(a))$ by (viii). Take a sequence $(q_n)_{n\geq 1}$ of complex polynomials in z and $\overline{z}$ such that $q_n \to g$ uniformly on $f(\sigma(a)) = \sigma(f(a))$ by Stone-Weierstrass' theorem. Then $q_n \circ f \to g \circ f$ uniformly on $\sigma(a)$, so

$$q_n(f(a)) = (q_n \circ f)(a) \to (g \circ f)(a)$$

in norm. To comprehend the first equality above, note that if $q_n(z) = \sum \lambda_{ij} z^i \overline{z}^j$ then we have

$$(q_n \circ f)(z) = \sum \lambda_{ij} f(z)^i \overline{f}(z)^j.$$

The continuous functional calculus is a *-homomorphism and thus maps this function to

$$(q_n \circ f)(a) = \sum \lambda_{ij} f(a)^i (f(a)^*)^j = q_n(f(a)).$$

Noting that $q_n(f(a)) \to g(f(a))$ in norm as well, (ix) follows.

For (x), note that $b = \Phi(a)$ is normal. As $\sigma(b) \subseteq \sigma(a)$, we see that $f(b)$ is a well-defined element of $C^*(1_{\mathcal{B}}, b) \subseteq \mathcal{B}$. Let $(p_n)_{n\geq 1}$ be a sequence of complex polynomials in z and $\overline{z}$ such that $p_n \to f$ uniformly on $\sigma(a)$. Then $p_n(a) \to f(a)$ and $p_n(b) \to f(b)$ in norm, so we have

$$\Phi(f(a)) = \Phi\left(\lim_{n\to\infty} p_n(a)\right) = \lim_{n\to\infty} \Phi(p_n(a)) = \lim_{n\to\infty} p_n(b) = f(b),$$

since Φ is a unital *-homomorphism and hence continuous. □

We will see later that $\mathcal{B}$ in (x) does not have to be unital, as long as we modify the assumptions a little. Oh, and about that...

Remark 1. If we remove the condition in (x) that Φ is unital, we run into some problems. For instance, let $p \in B(\ell^2(\mathbb{N}))$ be a finite rank projection and define $\Phi\colon \mathbb{C} \to B(\ell^2)$ by $\Phi(\lambda) = \lambda p$. Then p isn't invertible, so that $0 \in \sigma(p)$, but clearly $0 \notin \sigma(1) = \{1\}$. Hence the equation $f(\Phi(1)) = \Phi(f(1))$ for $f \in C(\{1\})$ doesn't make sense, since $f(\Phi(1)) = f(p)$ only defines an element in $B(\ell^2(\mathbb{N}))$ if $f \in C(\sigma(p))$.

In order to find a solution to this problem, let us look at the spectra first.

Lemma 9. *If $\mathcal{A}$ and $\mathcal{B}$ are unital C^*-algebras with $\mathcal{A} \subseteq \mathcal{B}$ and $1_{\mathcal{A}} \neq 1_{\mathcal{B}}$, then*

$$\sigma_{\mathcal{A}}(a) \cup \{0\} = \sigma_{\mathcal{B}}(a)$$

for all $a \in \mathcal{A}$, where $\sigma_{\mathcal{A}}(a)$ and $\sigma_{\mathcal{B}}(a)$ denote the spectrum of a in $\mathcal{A}$ and $\mathcal{B}$ respectively.

Proof. It is clear that $1_{\mathcal{A}}\mathcal{B}1_{\mathcal{A}}$ is a C^*-subalgebra of $\mathcal{B}$ with unit $1_{\mathcal{A}}$. Applying [4, Lemma A.12], we find $\sigma_{\mathcal{B}}(a) \cup \{0\} = \sigma_{1_{\mathcal{A}}\mathcal{B}1_{\mathcal{A}}}(a)$. Since $\mathcal{A} \subseteq 1_{\mathcal{A}}\mathcal{B}1_{\mathcal{A}}$ also has unit $1_{\mathcal{A}}$, the result follows from [3, Corollary 9.11]. □

Remark 2. If $\Phi\colon \mathcal{A} \to \mathcal{B}$ is a *-homomorphism of unital C^*-algebras that satisfies $\Phi(1_{\mathcal{A}}) \neq 1_{\mathcal{B}}$, we know that $\mathcal{B}_1 = \overline{\Phi(\mathcal{A})}$ is a unital C^*-subalgebra of $\mathcal{B}$ with unit $1_{\mathcal{B}_1} = \Phi(1_{\mathcal{A}})$. Then the above lemma shows that

$$\sigma_{\mathcal{B}}(\Phi(a)) = \sigma_{\mathcal{B}_1}(\Phi(a)) \cup \{0\} \subseteq \sigma_{\mathcal{A}}(a) \cup \{0\},$$

if we view Φ as a *-homomorphism $\mathcal{A} \to \mathcal{B}_1$. Hence the aforementioned problem of getting the equation to make sense can at least partly be fixed by demanding that $f \in C(\sigma(a) \cup \{0\})$.

There is still a problem, though: With the example from before, let us now consider the function $f(x) = x + 1$. Once again using the element $z = 1 \in \mathbb{C}$, then $f(z) = 2$ and thus $\Phi(f(z)) = 2p$. However, $f(\Phi(z)) = p + 1_{B(\ell^2(\mathbb{N}))}$, so equality still doesn't hold!

The next two results should clear up everything in a flash.

Lemma 10. *If $g \in C(\Omega)$ where Ω is a compact subset of $\mathbb{C}$ satisfies $g(0) = 0$, then there is a sequence of complex polynomials $(p_n(z))_{n\geq 1}$ in z and $\overline{z}$ without constant term, i.e., $p_n(0) = 0$ for all $n \geq 1$, such that $p_n \to g$ uniformly on Ω.*

Proof. Let $(q_n(z))_{n\geq 1}$ be a sequence of complex polynomials in z and $\overline{z}$ such that $q_n \to g$ uniformly on Ω, as made possible by the Stone-Weierstrass Theorem. Define polynomials $p_n(z) = q_n(z) - q_n(0)$, and note that for all $z \in \Omega$, we have

$$|p_n(z) - g(z)| \leq |q_n(z) - g(z)| + |g(0) - q_n(0)| \leq 2 \sup_{z\in\Omega} |q_n(z) - g(z)|,$$

so that $p_n \to g$ uniformly on Ω. □

Theorem 11. *If Φ is a *-homomorphism of a unital C^*-algebra $\mathcal{A}$ into another unital C^*-algebra $\mathcal{B}$ such that $\Phi(1_{\mathcal{A}}) \neq 1_{\mathcal{B}}$, $a \in \mathcal{A}$ is normal and $f \in C(\sigma(a) \cup \{0\})$, then $\Phi(f(a)) = f(\Phi(a))$ if and only if $f(0) = 0$.*

Proof. First of all, note that $b = \Phi(a)$ is normal. Since $\sigma(b) \subseteq \sigma(a) \cup \{0\}$ by Remark 2, $f(b)$ is a well-defined element of $\mathcal{B}$. Define $\lambda = f(0)$ and take a sequence $(p_n)_{n\geq 1}$ of complex polynomials in z and $\overline{z}$ without constant term such that $p_n \to f - \lambda$ uniformly on $\sigma(a) \cup \{0\}$, using Lemma 10. Then $p_n(a) \to f(a) - \lambda 1_{\mathcal{A}}$, so

$$\Phi(p_n(a)) \to \Phi(f(a)) - \lambda\Phi(1_{\mathcal{A}})$$

by continuity (see Corollary 16; no cheating!). However, we also have $p_n(b) \to f(b) - \lambda 1_{\mathcal{B}}$. Since the p_n's have no constant term, it follows that $p_n(b) = \Phi(p_n(a))$, so continuity along with the above convergence tells us that

$$\Phi(f(a)) - \lambda\Phi(1_{\mathcal{A}}) = f(b) - \lambda 1_{\mathcal{B}}.$$

The desired result immediately follows. □

The question is now: what do elements of $C^*(a)$ correspond to under the continuous functional calculus? The next proposition answers this, with the aid of a new set of continuous functions on the spectrum.

Definition 12. Let $\mathcal{A}$ be a unital C^*-algebra and $a \in \mathcal{A}$. We define

$$\mathcal{C}(a) = \{f \colon \sigma(a) \to \mathbb{C} \mid \exists g \in C(\sigma(a) \cup \{0\}) : g|_{\sigma(a)} = f \text{ and } g(0) = 0\} \subseteq C(\sigma(a)).$$

Proposition 13. *Let $\mathcal{A}$ be a unital C^*-algebra and $a \in \mathcal{A}$ be normal. Then the image of $\mathcal{C} = \mathcal{C}(a)$ under the continuous functional calculus is $C^*(a)$.*

Proof. Note first that $\mathcal{C}$ is closed in $C(\sigma(a))$; if $f \in C(\sigma(a))$ such that $f_n \to f$ uniformly with $(f_n)_{n\geq 1} \subseteq \mathcal{C}$, let $g_n \in C(\sigma(a) \cup \{0\})$ such that $g_n|_{\sigma(a)} = f_n$ and $g_n(0) = 0$ for all $n \geq 1$. Then $\|g_n - g_m\|_\infty \leq \|f_n - f_m\|_\infty$, so $(g_n)_{n\geq 1}$ is a Cauchy sequence and thus it converges to some $g \in C(\sigma(a) \cup \{0\})$ uniformly. It is clear that $g(0) = 0$ and that $|f_n(z) - g(z)| = |g_n(z) - g(z)| \leq \|g_n - g\|_\infty$ for all n and $z \in \sigma(a)$, so that $f_n \to g$ uniformly on $\sigma(a)$. Hence $f = g|_{\sigma(a)} \in \mathcal{C}$.

Let $f \in \mathcal{C}$, take $g \in C(\sigma(a) \cup \{0\})$ such that $g|_{\sigma(a)} = f$ and $g(0) = 0$. Lemma 10 yields a sequence of $(p_n(z))_{n\geq 1}$ complex polynomials in z and $\overline{z}$ without constant term such that $p_n \to g$ uniformly on $\sigma(a) \cup \{0\}$. Therefore $p_n(a) \to f(a)$. Since $p_n(a)$ is a polynomial in a and a^* without constant term, so $p_n(a) \in C^*(a)$, so $f(a) \in C^*(a)$ because $C^*(a)$ is closed.

If $b \in C^*(a)$, there exist complex polynomials $p_n(z)$ in z and $\overline{z}$ without constant term such that $p_n(a) \to b$. Since $\|p_n - p_m\|_\infty = \|p_n(a) - p_m(a)\|$ (the continuous functional calculus is an isometry), it follows that $(p_n)_{n\geq 1}$ converges uniformly to some function $f \in C(\sigma(a))$. Since $p_n \in \mathcal{C}$ and $\mathcal{C}$ is closed, it follows that $f \in \mathcal{C}$. Since $\|f(a) - p_n(a)\| = \|f - p_n\|_\infty$, it follows that $p_n(a) \to f(a)$, so that $b = f(a)$ for this $f \in \mathcal{C}$. □

3 The continuous functional calculus, part II

In this section, assume that $\mathcal{A}$ is a non-unital C^*-algebra and that $a \in \mathcal{A}$ is a normal element. Let $\iota\mathcal{A}$ be the image of $\mathcal{A}$ in $\tilde{\mathcal{A}}$ under the isometric inclusion *-homomorphism. The *spectrum* $\sigma(a)$ of $a \in \mathcal{A}$ is defined to be the spectrum of $\tilde{a} \in \tilde{\mathcal{A}}$. Note that we automatically obtain $0 \in \sigma(a)$ for all $a \in \mathcal{A}$.

Proposition 14. *For any normal element a of a non-unital C^*-algebra, we have*

(i) *a is self-adjoint if and only if $\sigma(a) \subseteq \mathbb{R}$.*

(ii) *a is positive if and only if $\sigma(a) \subseteq \mathbb{R}_+$.*
(iii) *a is a projection if and only if $\sigma(a) \subseteq \{0,1\}$.*
(iv) $\|a\| = \sup_{z\in\sigma(a)} |z|$.

Proof. We have that a is self-adjoint iff $\tilde{a}$ is self-adjoint iff $\sigma(a) = \sigma(\tilde{a}) \subseteq \mathbb{R}$ and that a is a projection iff $\tilde{a}$ is a projection iff $\sigma(a) \subseteq \{0,1\}$ by [3, Theorem 10.4]; hence (i) and (iii). (iv) follows from [3, Theorem 8.1]. If a is positive, then $\tilde{a}$ is positive and $\sigma(a) = \sigma(\tilde{a}) \subseteq \mathbb{R}_+$ by [3, Theorem 11.5]. We save the converse for later, i.e. that a is positive if $\tilde{a}$ is positive. □

The above proposition ensures that all we know about spectra of special elements in a C^*-algebra still holds whether the C^*-algebra in question has a unit or not.

Proposition 15. *For any *-homomorphism $\varphi\colon \mathcal{A} \to \mathcal{B}$ of arbitrary C^*-algebras and any $a \in \mathcal{A}$, we have*

$$\{0\} \cup \sigma(\varphi(a)) \subseteq \{0\} \cup \sigma(a).$$

Proof. Assume first that $\mathcal{A}$ and $\mathcal{B}$ are unital. If $\lambda \neq 0$ and $a - \lambda 1_{\mathcal{A}}$ is invertible with inverse x, then

$$(\varphi(a) - \lambda 1_{\mathcal{B}}) \left(\varphi(x) + \frac{1}{\lambda}\varphi(1_{\mathcal{A}}) - \frac{1}{\lambda} 1_{\mathcal{B}} \right) = 1_{\mathcal{B}},$$

so the result holds. (Note that if $\varphi(1_{\mathcal{A}}) = 1_{\mathcal{B}}$, then clearly $\sigma(\varphi(a)) \subseteq \sigma(a)$.)

Assume that $\mathcal{A}$ is unital and that $\mathcal{B}$ isn't. If $\lambda \neq 0$ and $a - \lambda 1_{\mathcal{A}}$ is invertible with inverse x, then

$$(\varphi(a), -\lambda) \left(\varphi(x) + \frac{1}{\lambda}\varphi(1_{\mathcal{A}}), -\frac{1}{\lambda} \right) = (0,1),$$

so $(\varphi(a), -\lambda) = \widetilde{\varphi(a)} - \lambda 1_{\tilde{\mathcal{B}}}$ is invertible in $\tilde{\mathcal{B}}$.

Assume penultimately that $\mathcal{A}$ is non-unital and that $\mathcal{B}$ is. If $\lambda \neq 0$ and $(a, -\lambda)$ is invertible in $\tilde{\mathcal{A}}$ with inverse (x, μ), then $ax - \lambda x + \mu a = 0$ and $-\lambda\mu = 1$, so that

$$(\varphi(a) - \lambda 1_{\mathcal{B}})\,(\varphi(x) + \mu 1_{\mathcal{B}}) = 1_{\mathcal{B}},$$

so $\varphi(a) - \lambda 1_{\mathcal{B}}$ is invertible in $\mathcal{B}$.

Lastly, assume that $\mathcal{A}$ and $\mathcal{B}$ are both non-unital. If $\lambda \neq 0$ and $(a, -\lambda)$ is invertible in $\tilde{\mathcal{A}}$ with inverse (x, μ), then

$$(\varphi(a), -\lambda)(\varphi(x), \mu) = (0,1),$$

completing the proof. □

Corollary 16. *Any *-homomorphism $\varphi\colon \mathcal{A} \to \mathcal{B}$ of arbitrary C^*-algebras is contractive.*

Proof. We have

$$\|\varphi(x)\|^2 = \|\varphi(x^*x)\| = \sup_{z\in\sigma(\varphi(x^*x))} |z| \leq \sup_{z\in\sigma(x^*x)\cup\{0\}} |z| = \sup_{z\in\sigma(x^*x)} |z| = \|x^*x\| = \|x\|^2$$

by Propositions 14 and 15. □

If $a \in \mathcal{A}$, we define $\mathcal{C}(a) = \mathcal{C}(\tilde{a})$ (see Definition 12). As $0 \in \sigma(a)$, we have

$$\mathcal{C}(a) = \{ f \in C(\sigma(a)) \mid f(0) = 0 \}.$$

Let $\pi\colon \tilde{\mathcal{A}} \to \mathbb{C}$ be the unital *-homomorphism given by $(a, \lambda) \mapsto \lambda$, and note that $\ker \pi = \iota\mathcal{A}$. By Theorem 8 we have

$$\pi(f(\tilde{a})) = f(\pi(\tilde{a})) = f(0)$$

for all $f \in C(\sigma(a))$. Hence $f(\tilde{a}) \in \iota\mathcal{A}$ if and only if $f(0) = 0$, i.e., if $f \in \mathcal{C}(a)$. This yields the following nice result:

Proposition 17. *If a is a normal element of a non-unital C^*-algebra $\mathcal{A}$, there is a *-isomorphism $F_a\colon \mathcal{C}(a) \to C^*(a)$ such that the identity map $1\colon \sigma(a) \to \sigma(a)$ is mapped to a itself, and*

$$\widetilde{F_a(f)} = f(\tilde{a})$$

for all $f \in \mathcal{C}(a)$.

Proof. By Proposition 13, restricting the continuous functional calculus yields a *-homomorphism $\varphi_a\colon \mathcal{C}(a) \to C^*(\tilde{a})$ such that 1 is mapped to $\tilde{a}$. It is easy to see that the *-isomorphism $\iota^{-1}\colon \iota\mathcal{A} \to \mathcal{A}$ restricts to a *-isomorphism of $C^*(\tilde{a})$ onto $C^*(a)$. Composing yields the desired isomorphism, as $f(\tilde{a}) = \varphi_a(f) = \iota(F_a(f))$. □

Definition 18. If $a \in \mathcal{A}$ is normal and $f \in \mathcal{C}(a)$, we define $f(a) = F_a(f) \in C^*(a)$, where F_a is the *-isomorphism of Proposition 17. The *-isomorphism itself is called the *continuous functional calculus* for a.

As before (but now a little more general): if a is a normal element of *any* C^*-algebra $\mathcal{A}$, then if $f \in C(\Omega)$ for some subset $\sigma(a) \subseteq \Omega \subseteq \mathbb{C}$ with $f(0) = 0$, we define $f(a) = f|_{\sigma(a)}(a)$.

Theorem 19 (Properties of the continuous functional calculus for non-unital C^*-algebras)**.** *Let $\mathcal{A}$ be a non-unital C^*-algebra. If $a \in \mathcal{A}$ is normal, then for all $\lambda, \mu \in \mathbb{C}$ and $f, g \in \mathcal{C}(a)$ we have*

(i) $(\lambda f + \mu g)(a) = \lambda f(a) + \mu g(a)$.
(ii) $(fg)(a) = f(a)g(a)$.
(iii) $\overline{f}(a) = f(a)^*$.
(iv) *If 1 denotes the identity map $\sigma(a) \to \sigma(a)$, then $1(a) = a$.*
(v) *If $P\colon z \mapsto p(z, \bar{z})$ is a complex polynomial in z and $\overline{z}$ with no constant term, then $P(a) = p(a, a^*)$.*
(vi) *If Ω is a subset of $\mathbb{C}$ such that $\sigma(a) \subseteq \Omega$, then*

$$\|f(a)\| = \sup_{z\in\sigma(a)} |f(z)| \leq \sup_{z\in\Omega} |f(z)|.$$

(vii) $\sigma(f(a)) = f(\sigma(a))$.
(viii) *If $h \in \mathcal{C}(f(a))$, then $h \circ f \in \mathcal{C}(a)$ and $(h \circ f)(a) = h(f(a))$.*
(ix) *If Φ is a *-homomorphism of $\mathcal{A}$ into another C^*-algebra $\mathcal{B}$, then $\Phi(f(a)) = f(\Phi(a))$.*

Proof. (i)-(iii) follows from the continuous functional calculus being a *-homomorphism. (iv) is immediate from the definition, and (v) follows accordingly. To see (vi), note that Theorem 8(vii) yields

$$\|f(a)\| = \|f(\tilde{a})\| = \sup_{z\in\sigma(\tilde{a})} |f(z)| = \sup_{z\in\sigma(a)} |f(z)|.$$

(vii) is easy, as Theorem 8(viii) yields

$$\sigma(f(a)) = \sigma(\widetilde{f(a)}) = \sigma(f(\tilde{a})) \overset{8(\text{viii})}{=} f(\sigma(\tilde{a})) = f(\sigma(a)).$$

To prove (viii), take $F \in C(\sigma(a) \cup \{0\})$ such that $F|_{\sigma(a)} = f$ and $F(0) = 0$ and $H \in C(\sigma(f(a)) \cup \{0\})$ such that $H|_{\sigma(f(a))} = h$ and $H(0) = 0$. As $F(\sigma(a) \cup \{0\}) = f(\sigma(a)) \cup \{0\} = \sigma(f(a)) \cup \{0\}$ by (vii), we have that $H \circ F \in C(\sigma(a) \cup \{0\})$, that $(H \circ F)(z) = H(f(z)) = (h \circ f)(z)$ for all $z \in \sigma(a)$ and $H(F(0)) = 0$, we have $h \circ f \in \mathcal{C}(a)$. Theorem 8(ix) then yields

$$\widetilde{(h \circ f)(a)} = (h \circ f)(\tilde{a}) \overset{8(\text{ix})}{=} h(f(\tilde{a})) = h(\widetilde{f(a)}) = \widetilde{h(f(a))}.$$

For (ix), assume first that $\mathcal{B}$ is unital and define a *-homomorphism $\tilde{\Phi}\colon \tilde{\mathcal{A}} \to \mathcal{B}$ by $\tilde{\Phi}(a, \lambda) = \Phi(a) + \lambda 1_{\mathcal{B}}$. Theorem 8(x) now yields

$$\Phi(f(a)) = \tilde{\Phi}(\widetilde{f(a)}) = \tilde{\Phi}(f(\tilde{a})) \overset{8(\text{x})}{=} f(\tilde{\Phi}(\tilde{a})) = f(\Phi(a)).$$

If $\mathcal{B}$ is non-unital, we instead define a *-homomorphism $\tilde{\Phi}\colon \tilde{\mathcal{A}} \to \tilde{\mathcal{B}}$ by $\tilde{\Phi}(a, \lambda) = (\Phi(a), \lambda)$. Then

$$\widetilde{\Phi(f(a))} = \tilde{\Phi}(\widetilde{f(a)}) = \tilde{\Phi}(f(\tilde{a})) \overset{8(\text{x})}{=} f(\tilde{\Phi}(\tilde{a})) = f(\widetilde{\Phi(a)}) = \widetilde{f(\Phi(a))},$$

and hence we obtain the wanted equality. □

Thus all the usual tricks that one may perform in order to construct specific "nice" elements of a C^*-algebra still work.

Last part of the proof of Proposition 14. Since $\sigma(a) = \sigma(\tilde{a}) \subseteq \mathbb{R}_+$, let $f(t) = \sqrt{t}$ for $t \in \sigma(a)$. As $f(0) = 0$, we have $f \in \mathcal{C}(a)$ and $f(a) \in \mathcal{A}$. Since $f(a)$ is self-adjoint as $\overline{f} = f$ and $f(t)^2 = \mathbb{1}(t)$ for all $t \in \sigma(\tilde{a})$ we conclude that $f(a)^* f(a) = \mathbb{1}(a) = a$, i.e. that a is positive. □

As promised, we provide a proof of Theorem 8(x) in the case where $\mathcal{B}$ is not unital.

Theorem 20. *Let $\mathcal{A}$ be a unital C^*-algebra, and let $a \in \mathcal{A}$ be normal. If $\mathcal{B}$ is a non-unital C^*-algebra and $\Phi\colon \mathcal{A} \to \mathcal{B}$ is a *-homomorphism, then*

$$\Phi(f(a)) = f(\Phi(a))$$

for all $f \in C(\sigma(a) \cup \{0\})$ such that $f(0) = 0$.

Proof. Let f be as above, and define $\tilde{\Phi}\colon \mathcal{A} \to \tilde{\mathcal{B}}$ by $\tilde{\Phi}(x) = (\Phi(x), 0)$. Then $\tilde{\Phi}$ is a *-homomorphism and Theorem 11 thus yields

$$\widetilde{\Phi(f(a))} = \tilde{\Phi}(f(a)) = f(\tilde{\Phi}(a)) = f(\widetilde{\Phi(a)}) = \widetilde{f(\Phi(a))}.$$

This implies the wanted result. □

4 For your eyes only, only for you

We provide a summary of what we have proved so far, enabling us to use the continuous functional calculus for any C^*-algebra, unital or not.

Theorem 21 (Properties of the continuous functional calculus for arbitrary C^*-algebras)**.** *Let $\mathcal{A}$ be an arbitrary C^*-algebra, let $a \in \mathcal{A}$ be normal and define a *-subalgebra $\mathcal{C}(a)$ of $C(\sigma(a))$ given by*

$$\mathcal{C}(a) = \{f \in C(\sigma(a)) \mid \exists g \in C(\sigma(a) \cup \{0\}) : g|_{\sigma(a)} = f \text{ and } g(0) = 0\}.$$

Then there exists a *-isomorphism $\mathcal{C}(a) \to C^*(a)$, $f \mapsto f(a)$ such that for all $\lambda, \mu \in \mathbb{C}$ and $f, g \in \mathcal{C}(a)$, we have

(i) $(\lambda f + \mu g)(a) = \lambda f(a) + \mu g(a)$.
(ii) $(fg)(a) = f(a)g(a)$.
(iii) $\overline{f}(a) = f(a)^*$.
(iv) *If $\mathbb{1}$ denotes the identity map $\sigma(a) \to \sigma(a)$, then $\mathbb{1}(a) = a$.*
(v) *If $P\colon z \mapsto p(z, \tilde{z})$ is a complex polynomial in z and $\overline{z}$ with no constant term, then $P(a) = p(a, a^*)$.*
(vi) *If Ω is a subset of $\mathbb{C}$ such that $\sigma(a) \subseteq \Omega$, then*

$$\|f(a)\| = \sup_{z \in \sigma(a)} |f(z)| \leq \sup_{z \in \Omega} |f(z)|.$$

(vii) $\sigma(f(a)) = f(\sigma(a))$.
(viii) *If $h \in C(f(\sigma(a)))$ has an continuous extension to $f(\sigma(a)) \cup \{0\}$ such that $h(0) = 0$, then $(h \circ f)(a) = h(f(a))$.*
(ix) *If Φ is a *-homomorphism of $\mathcal{A}$ into another C^*-algebra $\mathcal{B}$, then $\Phi(h(a)) = h(\Phi(a))$ for all $h \in C(\sigma(a) \cup \{0\})$ such that $h(0) = 0$.*

Proof. This follows from Theorems 8, 11, 19 and 20. □

5 Examples

We provide some nice applications to emphasize that we do not need to know whether a C^*-algebra is unital or not in order to use the continuous functional calculus. Recall that the spectrum of an element a in a C^*-algebra $\mathcal{A}$ is contained in $\{z \in \mathbb{C} \mid |z| \leq \|a\|\}$.

Proposition 22. *Let $\mathcal{A}$ and $\mathcal{B}$ be arbitrary C^*-algebras and let $\varphi\colon \mathcal{A} \to \mathcal{B}$ be a *-homomorphism. Then if $b \in \varphi(\mathcal{A})$ and b is self-adjoint, there exists a self-adjoint element $a \in \mathcal{A}$ such that $\varphi(a) = b$ and $\|a\| = \|b\|$.*

Proof. Take $x \in \mathcal{A}$ such that $\varphi(x) = b$ and define $y = \frac{1}{2}(x + x^*)$. Then y is self-adjoint and

$$\varphi(y) = \frac{1}{2}(\varphi(x) + \varphi(x)^*) = \frac{1}{2}(b + b) = b.$$

Define $f\colon \mathbb{R} \to \mathbb{R}$ by

$$f(z) = \begin{cases} -\|b\| & z \leq -\|b\| \\ z & -\|b\| \leq z \leq \|b\| \\ \|b\| & z \geq \|b\| \end{cases}$$

Clearly f is continuous and $f(0) = 0$. We have that $\sigma(b) \subseteq \mathbb{R}$ and that $f|_{\sigma(b)}$ is the identity map on $\sigma(b)$, so we conclude $f(b) = b$. Likewise $\sigma(y) \subseteq \mathbb{R}$, so we can define $a = f(y) \in \mathcal{A}$. Since $\sigma(a) = \sigma(f(y)) = f(\sigma(y)) \subseteq \mathbb{R}$, a is self-adjoint and

$$\|a\| = \sup_{z \in \sigma(y)} |f(z)| \leq \|b\|.$$

As

$$\varphi(a) = \varphi(f(y)) = f(\varphi(y)) = f(b) = b$$

and $\|b\| = \|\varphi(a)\| \leq \|a\|$, we obtain the desired result. □

Theorem 23. *Let $\varphi\colon \mathcal{A} \to \mathcal{B}$ be a *-homomorphism of arbitrary C^*-algebras. Then the image $\varphi(\mathcal{A})$ is a C^*-subalgebra of $\mathcal{B}$.*

Proof. Assume that $y \in \mathcal{B}$ and that $\varphi(x_n) \to y$ for some sequence $(x_n)_{n\geq 1}$ in $\mathcal{A}$. Writing $y = y' + iy''$ and $x_n = x_n' + ix_n''$, where $y', y'', x_n', x_n'' \in \mathcal{A}$ are self-adjoint, then $\varphi(x_n') \to y'$ and $\varphi(x_n'') \to y''$. Hence it is enough to show that if $\varphi(a_n) \to b$ for self-adjoint $b \in \mathcal{B}$ and a sequence $(a_n)_{n\geq 1} \subseteq \mathcal{A}$ of self-adjoint elements, then $b = \varphi(a)$ for some $a \in \mathcal{A}$. Taking $N_1 \geq 1$ such that $\|\varphi(a_n) - \varphi(a_m)\| < \frac{1}{2}$ for $n, m \geq N_1$ and inductively taking $N_{n+1} \geq N_n$ for $n \geq 1$ such that $\|\varphi(a_n) - \varphi(a_m)\| < \frac{1}{2^2}$ for $n, m \geq N_{n+1}$ and so on, we now define $c_n = a_{N_n}$ for all $n \geq 1$. Then $(c_n)_{n\geq 1}$ is a sequence of self-adjoint elements satisfying $\varphi(c_n) \to b$ and

$$\|\varphi(c_{n+1}) - \varphi(c_n)\| < \frac{1}{2^n}, \quad n \geq 1.$$

For all $n \geq 1$, define $f_n\colon \mathbb{R} \to \mathbb{R}$ by

$$f_n(z) = \begin{cases} -\frac{1}{2^n} & z \leq -\frac{1}{2^n} \\ z & -\frac{1}{2^n} \leq z \leq \frac{1}{2^n} \\ \frac{1}{2^n} & z \geq \frac{1}{2^n} \end{cases}$$

Since $\sigma(\varphi(c_{n+1}) - \varphi(c_n)) \subseteq \mathbb{R}$ with $\|\varphi(c_{n+1}) - \varphi(c_n)\| < \frac{1}{2^n}$, f_n is the identity map when restricted to $\sigma(\varphi(c_{n+1}) - \varphi(c_n))$, so that the fact that f_n is continuous and satisfies $f_n(0) = 0$ yields

$$\varphi(c_{n+1}) - \varphi(c_n) = f_n(\varphi(c_{n+1}) - \varphi(c_n)) = f_n(\varphi(c_{n+1} - c_n)) = \varphi(f_n(c_{n+1} - c_n)).$$

Moreover, $\|f_n(c_{n+1} - c_n)\| \leq \frac{1}{2^n}$. Hence $\sum_{n=1}^{\infty} f_n(c_{n+1} - c_n)$ converges in $\mathcal{A}$, and

$$\begin{aligned} \varphi\left(c_1 + \sum_{n=1}^{\infty} f_n(c_{n+1} - c_n)\right) &= \varphi(c_1) + \varphi\left(\sum_{n=1}^{\infty} f_n(c_{n+1} - c_n)\right) \\ &= \varphi(c_1) + \sum_{n=1}^{\infty} \varphi(f_n(c_{n+1} - c_n)) \\ &= \varphi(c_1) + \sum_{n=1}^{\infty} (\varphi(c_{n+1}) - \varphi(c_n)) = \lim_{n\to\infty} \varphi(c_n) = b, \end{aligned}$$

by continuity of φ. Hence the result follows. □

Proposition 24. *If $\varphi\colon \mathcal{A} \to \mathcal{B}$ is an unital injective *-homomorphism of unital C^*-algebras, then it is an isometry and*

$$\sigma(\Phi(a)) = \sigma(a)$$

for all $a \in \mathcal{A}$.

Proof. Let $a \in \mathcal{A}$ be normal and define $b = \Phi(a)$. By Proposition 15, it follows that

$$\sigma(b) \subseteq \sigma(a).$$

Assume that there exists $\lambda \in \sigma(a)$ such that $\lambda \notin \sigma(b)$. Then Urysohn's lemma provides a non-zero continuous function $f\colon \sigma(a) \to [0,1]$ such that $f|_{\sigma(b)} = 0$ and $f(\lambda) = 1$. In particular, f is non-zero in $C(\sigma(a))$. Hence the continuous functional calculus yields $f(a) \neq 0$, but

$$\Phi(f(a)) = f(b) = 0,$$

since f is the zero function on $\sigma(b)$, which contradicts the assumption that Φ is injective. Hence the sets are equal. By Proposition 23, hence $\Phi(\mathcal{A})$ is a unital C^*-subalgebra of $\mathcal{B}$ that is isomorphic to $\mathcal{A}$, so [3, Corollary 9.11] tells us that $\sigma_{\mathcal{A}}(a) = \sigma_{\Phi(\mathcal{A})}(a) = \sigma_{\mathcal{B}}(a)$. □

Lemma 25. *Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras with $\mathcal{A} \subseteq \mathcal{B}$ and $a \in \mathcal{A}$. Let $\sigma_{\mathcal{A}}(a)$ and $\sigma_{\mathcal{B}}(a)$ denote the spectrum of a in $\mathcal{A}$ and $\mathcal{B}$ respectively. If $\mathcal{A}$ and $\mathcal{B}$ are unital with $1_{\mathcal{A}} = 1_{\mathcal{B}}$, then $\sigma_{\mathcal{A}}(a) = \sigma_{\mathcal{B}}(a)$. If not,*

$$\sigma_{\mathcal{A}}(a) \cup \{0\} = \sigma_{\mathcal{B}}(a).$$

In any case, $\sigma_{\mathcal{A}}(a) \cup \{0\} = \sigma_{\mathcal{B}}(a) \cup \{0\}$.

Proof. Assume first that $\mathcal{A}$ and $\mathcal{B}$ are unital. If $1_{\mathcal{A}} = 1_{\mathcal{B}}$, the result follows from [3, Corollary 9.11]; if $1_{\mathcal{A}} \neq 1_{\mathcal{B}}$, then the claim follows from Lemma 9.

If $\mathcal{B}$ is unital but $\mathcal{A}$ isn't, then $\mathcal{A} + \mathbb{C}1_{\mathcal{B}}$ is a unital C^*-algebra that is isomorphic to the unitization $\tilde{\mathcal{A}}$. Then

$$\sigma_{\mathcal{A}}(a) \cup \{0\} = \sigma_{\mathcal{A}}(a) = \sigma_{\tilde{\mathcal{A}}}(\tilde{a}) = \sigma_{\mathcal{A}+\mathbb{C}1_{\mathcal{B}}}(a) = \sigma_{\mathcal{B}}(a)$$

by [3, Corollary 9.11] and Proposition 24.

If $\mathcal{A}$ is unital and $\mathcal{B}$ isn't, note that the map $\mathcal{A} \to \widetilde{1_{\mathcal{A}}}\tilde{\mathcal{B}}\widetilde{1_{\mathcal{A}}}$ given by $a \mapsto \tilde{a}$ is a unital *-isomorphism. Hence we get from Lemma 9 and Proposition 24 that

$$\sigma_{\mathcal{B}}(a) = \sigma_{\tilde{\mathcal{B}}}(\tilde{a}) = \sigma_{\widetilde{1_{\mathcal{A}}}\tilde{\mathcal{B}}\widetilde{1_{\mathcal{A}}}}(\tilde{a}) \cup \{0\} = \sigma_{\mathcal{A}}(a) \cup \{0\}.$$

Finally, if both $\mathcal{A}$ and $\mathcal{B}$ are non-unital, then the subset $\mathcal{C} = \{(a, \lambda) \mid a \in \mathcal{A},\ \lambda \in \mathbb{C}\}$ of $\tilde{\mathcal{B}}$ is a C^*-subalgebra that is isomorphic to $\tilde{\mathcal{A}}$, yielding $\sigma_{\mathcal{A}}(a) = \sigma_{\tilde{\mathcal{A}}}(\tilde{a}) = \sigma_{\mathcal{C}}(\tilde{a}) = \sigma_{\tilde{\mathcal{B}}}(a) = \sigma_{\mathcal{B}}(a)$ by Proposition 24. □

Corollary 26. *Let $\Phi\colon \mathcal{A} \to \mathcal{B}$ be a injective *-homomorphism of C^*-algebras. Then Φ is an isometry and*

$$\sigma(a) \cup \{0\} = \sigma(\Phi(a)) \cup \{0\}$$

for all $a \in \mathcal{A}$. In particular, if $a \in \mathcal{A}$ and $\Phi(a)$ is self-adjoint (resp. positive, a projection), then a is self-adjoint (resp. positive, a projection).

Proof. Let $a \in \mathcal{A}$. Since $\Phi(\mathcal{A})$ is a C^*-subalgebra of $\mathcal{B}$ that is isomorphic to $\mathcal{A}$, Lemma 25 tells us that

$$\sigma_{\mathcal{A}}(a) \cup \{0\} = \sigma_{\Phi(\mathcal{A})}(\Phi(a)) \cup \{0\} = \sigma(\Phi(a)) \cup \{0\}.$$

Therefore

$$\|a\|^2 = \|a^*a\| = \sup_{z \in \sigma(a^*a)} |z| = \sup_{z \in \sigma(\Phi(a)^*\Phi(a))} |z| = \|\Phi(a)^*\Phi(a)\| = \|\Phi(a)\|^2$$

by Proposition 14 and [3, Theorem 8.1]. The rest follows from Proposition 14 as well as [3, Theorems 10.4 and 11.5]. □

Theorem 27. *Any self-adjoint element x of any C^*-algebra $\mathcal{A}$ is a difference of positive elements x^+ and x^- in $\mathcal{A}$ such that $x^+x^- = x^-x^+ = 0$ and $\|x\| = \max\{\|x^+\|, \|x^-\|\}$.*

Proof. The proof of [3, Theorem 11.2] adjusts easily. □

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