Sølvkorn 16
The dual of $L^1(G)$ for a locally compact group G

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It happens often that the dual of the Banach space $L^1(X)$ (where X is some measure space) is in fact isomorphic to the well-known Banach space $L^\infty(X)$ of essentially bounded, Borel-measurable, complex functions. For instance, it is always true when X is σ-finite. It isn't true in general, however, and when working with locally compact (Hausdorff) groups G with a fixed left Haar measure it *really* is a pity, simply because the structures of the spaces $L^1(G)$ and $L^\infty(G)$ are quite essential to understanding whichever properties G may have. Nonetheless, it turns out that there is a way to work around this problem. Using that the duality holds for σ-finite measure spaces, we can modify the definition of $L^\infty(G)$ slightly, enlarging the space and obtaining the duality for all locally compact groups.

Let G be a locally compact group with fixed left Haar measure μ, and consider the Banach space $L^1(G)$ of Borel-measurable functions on G that are integrable with respect to μ (identified modulo null sets), equipped with the usual norm:

$$\|f\|_1 = \int |f(t)| \,\mathrm{d}\mu(t).$$

Recall that G contains an open, closed and σ-compact subgroup H [1, Proposition 2.4]. Letting $G/H = \{sH \mid s \in G\}$, then since G is the union of the sets of G/H, we can choose a *transversal* for H, i.e., a subset $Y \subseteq G$ that satisfies $G = \bigcup_{y\in Y} yH$ and for which $y_1, y_2 \in Y$ with $y_1 \neq y_2$ implies $y_1H \neq y_2H$. Indeed, the map $G \mapsto G/H$ given by $s \mapsto sH$ is surjective, so there exists an injective function $\alpha\colon G/H \to G$ by the axiom of choice, the image of which we denote by Y. By definition, α satisfies $\alpha(sH)H = sH$ for all $s \in G$, so if $y_1, y_2 \in Y$ there exist s_1H, s_2H such that $\alpha(s_1H) = y_1$ and $\alpha(s_2H) = y_2$. If $y_1H = y_2H$, then

$$s_1H = \alpha(s_1H)H = y_1H = y_2H = \alpha(s_2H)H = s_2H,$$

so $y_1 = y_2$. For $s \in G$, then $s \in sH = \alpha(sH)H$, so Y is a transversal.

In the sequel, let H be a fixed open, closed and σ-compact subgroup and let Y be a fixed transversal for H.

Proposition 1. *Let $E \subseteq G$ be a Borel set. Let $I = \{y \in Y \mid E \cap yH \neq \emptyset\}$. Then $E \subseteq \bigcup_{y\in I} yH$. If I is countable, then $\mu(E) = \sum_{y\in I} \mu(E \cap yH)$; if I is uncountable, then $\mu(E) = \infty$.*

Proof. See [1, Proposition 2.22]. □

A set $E \subseteq G$ is *locally Borel* if $E \cap F$ is Borel for all Borel sets $F \subseteq G$ with finite measure. A locally Borel set $E \subseteq G$ is *locally null* if and only if $\mu(E\cap F) = 0$ for all Borel sets $F \subseteq G$ with finite measure. A function $f\colon G \to \mathbb{C}$ is *locally measurable* if $f^{-1}(B)$ is locally Borel for all Borel sets $B \subseteq \mathbb{C}$.

Proposition 2. *Let H and Y be as before.*

(i) *A subset $E \subseteq G$ is locally Borel if and only if $E \cap yH$ is Borel for all $y \in Y$.*
(ii) *A subset $E \subseteq G$ is locally null if and only if $\mu(E \cap yH) = 0$ for all $y \in Y$.*
(iii) *A function $f\colon G \to \mathbb{C}$ is locally measurable if and only if $f|_{yH}$ is measurable for all $y \in Y$.*

Proof. Since H is σ-compact there exist a sequence $(H_n)_{n\geq 1}$ of compact sets such that $H = \bigcup_{n\geq 1} H_n$. All the sets H_n have finite measure, so yH_n has finite measure for all $y \in Y$ and $n \geq 1$. Let $F \subseteq G$

be an arbitrary Borel set with finite measure. By the preceding proposition the set $I \subseteq Y$ of $y \in I$ such that $F \cap yH \neq \emptyset$ is countable and $F \subseteq \bigcup_{y \in I} yH$.

(i) If E is locally Borel, then $E \cap yH_n$ is Borel for all $y \in Y$ since $\mu(yH_n) = \mu(H_n) < \infty$, so that $E \cap yH = \bigcup_{n \geq 1} E \cap yH_n$ is Borel. Conversely, if $E \cap yH$ is Borel for all $y \in Y$, then

$$E \cap F = \left(\bigcup_{y \in I} E \cap yH \right) \cap F$$

is Borel, so E is locally Borel.

(ii) If E is locally null, then $\mu(E \cap yH) = \sum_{n \geq 1} \mu(E \cap yH_n) = 0$ for all $y \in Y$. The inequality $\mu(E \cap F) \leq \sum_{y \in I} \mu(E \cap yH)$ proves the converse.

(iii) Recall that the σ-algebra on yH consists of all Borel sets of G intersected with yH. We then have that

$$\begin{aligned}
&f \colon G \to \mathbb{C} \text{ is locally measurable} \\
&\Leftrightarrow f^{-1}(B) \text{ is locally Borel for all Borel sets } B \subseteq \mathbb{C} \\
&\Leftrightarrow f^{-1}(B) \cap yH = f|_{yH}^{-1}(B) \text{ is Borel in } G \text{ for all Borel sets } B \subseteq \mathbb{C} \text{ and all } y \in Y \\
&\Leftrightarrow f|_{yH}^{-1}(B) \text{ is Borel in } yH \text{ for all Borel sets } B \subseteq \mathbb{C} \text{ and all } y \in Y \\
&\Leftrightarrow f|_{yH} \text{ is measurable for all } y \in Y
\end{aligned}$$

by (i), completing the proof. □

It follows from the above result that sums, products, conjugates and scalar multiples of locally measurable functions are again locally measurable.

For any locally measurable function $f \colon G \to \mathbb{C}$ we define

$$\|f\|_\infty = \inf\{c \geq 0 \,|\, \text{there exists } N \subseteq G \text{ locally null such that } |f(s)| \leq c \text{ for all } s \in G \setminus N\}.$$

We now let $\mathcal{L}^\infty(G)$ denote the set of locally measurable functions $f \colon G \to \mathbb{C}$ with $\|f\|_\infty < \infty$. For a general measure space (X, ν) and any measurable function $f \colon X \to \mathbb{C}$, we define

$$\|f\|_u = \inf\{c \geq 0 \,|\, \text{there exists } N \subseteq X \ \nu\text{-null such that } |f(x)| \leq c \text{ for all } x \in X \setminus N\},$$

and $L^u(X)$ denotes the space of measurable functions $f \colon X \to \mathbb{C}$ with $\|f\|_u < \infty$ where functions are identified if they differ only on a null set. A standard result of basic measure theory states that $\| \cdot \|_u$ is a norm on $L^u(X)$ turning it into a Banach space (it is even a C^*-algebra). By virtue of the above proposition, we shall now show that the $\| \cdot \|_\infty$-norm can be expressed by means of the $\| \cdot \|_u$-norm, and here's how. Letting $f_y = f|_{yH}$ for all $y \in Y$, then:

Proposition 3. *If $f \colon G \to \mathbb{C}$ is a locally measurable function, then*

$$\|f\|_\infty = \sup_{y \in Y} \|f_y\|_u \,.$$

Proof. Assume first that $\|f\|_\infty < \infty$ and let $\varepsilon > 0$. Then there exists $c \geq 0$ such that we can find a locally null set $N \subseteq G$ for which $|f(s)| \leq c$ for all $s \in G \setminus N$, with $\|f\|_\infty > c - \varepsilon$. For each $y \in Y$, $N \cap yH$ is a null set (under the Haar measure restricted to yH) and $|f(s)| \leq c$ for all $s \in yH \setminus (N \cap yH)$. Since f is locally measurable, f_y is measurable by Proposition 2, so we see that $\|f_y\|_u \leq c$ for all $y \in Y$. Since ε was arbitrary, we conclude that $\sup_{y \in Y} \|f_y\|_u \leq \|f\|_\infty < \infty$.

Assume that $M = \sup_{y \in Y} \|f_y\|_u < \infty$ and let $\varepsilon > 0$. For all $y \in Y$, there exists $c_y \geq 0$ such that there is a null set $N_y \subseteq yH$ for which $|f(s)| \leq c_y$ for all $s \in yH \setminus N_y$, with $\|f_y\|_u > c_y - \varepsilon$. In particular it holds for all $y \in Y$ that $|f(s)| < M + \varepsilon$ for all $s \in yH \setminus N_y$. Defining $N = \bigcup_{y \in Y} N_y$, then N is a locally null set, and $|f(s)| < M + \varepsilon$ for all $s \in G \setminus N$. Therefore $\|f\|_\infty \leq M + \varepsilon < \infty$, and the proof is complete. □

It follows from the above result that $\mathcal{L}^\infty(G)$ is a vector space and that $\|\cdot\|_\infty$ is a norm on $\mathcal{L}^\infty(G)$ if we identify locally measurable functions that differ only on a locally null set (or functions whose difference under $\|\cdot\|_\infty$ is 0). The new space obtained by this identification is denoted by $L^\infty(G)$. As a side note, $L^\infty(G)$ is in fact a unital C^*-algebra: the map $L^\infty(G) \to \prod_{y\in Y} L^u(yH)$ given by $f \mapsto (f_y)_{y\in Y}$ is an isometric *-isomorphism.

Note that the definition of $\|\cdot\|_\infty$ entails that if $f\colon G \to \mathbb{C}$ is a locally measurable function with $f \in L^\infty(G)$, then there exists a sequence of locally null nets $(N_n)_{n\geq 1}$ such that for any given $n \geq 1$ we have $|f(s)| < \|f\|_\infty + \frac{1}{n}$ for $s \in G \setminus N_n$. Then

$$\{s \in G \,|\, |f(s)| > \|f\|_\infty\} = \bigcup_{n\geq 1} \left\{ s \in G \,\middle|\, |f(s)| > \|f\|_\infty + \frac{1}{n} \right\} \subseteq \bigcup_{n\geq 1} N_n.$$

By Proposition 2, the set $\{s \in G \,|\, |f(s)| > \|f\|_\infty\}$ is locally Borel and hence locally null, as it is contained in a locally null set.

We now race for the prize, namely that the dual of $L^1(G)$ is isometrically isomorphic to our brand new $L^\infty(G)$:

Theorem 4. *Let G be a locally compact group with fixed left Haar measure μ. Define a map $\Phi\colon L^\infty(G) \to (L^1(G))^*$ by*

$$\Phi(f)(g) = \int fg \,\mathrm{d}\mu, \quad g \in L^1(G).$$

Then Φ is an isometric isomorphism.

Proof. We first check that Φ is well-defined. Let $f \in L^\infty(G)$, $g \in L^1(G)$ and define $h = fg$. Then h is locally measurable. Let $B \subseteq \mathbb{C}$ be a Borel set. Note that since g is integrable, the set $G_n = \{s \in G \,|\, |g(s)| > \frac{1}{n}\}$ is Borel and has finite measure for all $n \geq 1$. If $f(s)g(s) \in B \setminus \{0\}$ for some $s \in G$, then $g(s) \neq 0$, so that $s \in G_n$ for some $n \geq 1$, implying $h^{-1}(B \setminus \{0\}) \subseteq G_n$. Since $B \setminus \{0\}$ is Borel and h is locally measurable, $h^{-1}(B \setminus \{0\})$ is locally Borel and hence Borel, as it is contained in a Borel set G_n of finite measure. In particular, this holds for $B = \mathbb{C}$, so $h^{-1}(\mathbb{C} \setminus \{0\})$ is Borel. Hence $G \setminus h^{-1}(\mathbb{C} \setminus \{0\}) = h^{-1}(\{0\})$ is Borel, so we conclude that $h^{-1}(B)$ is Borel for all Borel sets $B \subseteq \mathbb{C}$. Therefore fg is measurable.

Defining $G = \bigcup_{n\geq 1} G_n$, then because $F = \{s \in G \,|\, |f(s)| > \|f\|_\infty\}$ is locally null, we have

$$\mu(F \cap G) \leq \sum_{n\geq 1} \mu(F \cap G_n) = 0,$$

and therefore

$$\int |fg| \,\mathrm{d}\mu \leq \|f\|_\infty \int |g| \,\mathrm{d}\mu = \|f\|_\infty \|g\|_1.$$

Hence fg is integrable, and it clearly follows that Φ is a well-defined, contractive linear map. To prove that it is isometric, it therefore remains to prove that $\|\Phi(f)\| \geq \|f\|_\infty$ for all $f \in L^\infty(G)$. If $\|f\|_\infty = 0$, this is clear, so we can assume that $\|f\|_\infty > 0$. Let $\varepsilon > 0$ with $\varepsilon < \|f\|_\infty$. Then $E = \{s \in G \,|\, |f(s)| > \|f\|_\infty - \varepsilon\}$ is locally Borel, but not locally null (otherwise, we would have $|f(s)| \leq \|f\|_\infty - \varepsilon$ for all s except on a locally null set and hence $\|f\|_\infty \leq \|f\|_\infty - \varepsilon$, which is just plain stupid), so there exists a Borel set $F \subseteq G$ with finite measure such that $\mu(E \cap F) > 0$. Define

$$g(s) = \frac{1_{E\cap F}}{\mu(E \cap F)} \chi(s), \quad s \in G,$$

where $\chi(s) = \frac{|f(s)|}{f(s)}$ if $f(s) \neq 0$ and $\chi(s) = 0$ otherwise. Then $g \in L^1(G)$, $\|g\|_1 = 1$ and

$$\Phi(f)(g) = \int fg \,\mathrm{d}\mu = \frac{1}{\mu(E \cap F)} \int 1_{E\cap F} |f| \,\mathrm{d}\mu \geq \frac{1}{\mu(E \cap F)} \int 1_{E\cap F} (\|f\|_\infty - \varepsilon) \,\mathrm{d}\mu = \|f\|_\infty - \varepsilon.$$

Hence $\|\Phi(f)\| \geq \|f\|_\infty - \varepsilon$, so since ε was arbitrary, we obtain the reverse inequality.

Finally, let $\varphi \in (L^1(G))^*$ and let $y \in Y$. Then we can define a linear, contractive inclusion $\iota_y \colon L^1(yH) \to L^1(G)$ defined by

$$\iota_y(g) = \begin{cases} g(s) & \text{if } s \in yH \\ 0 & \text{if } s \notin yH, \end{cases} \qquad g \in L^1(yH).$$

Then $\varphi \circ \iota_y \in (L^1(yH))^*$. As yH is σ-compact, the measure space (yH, μ) is σ-finite, and hence by [2, Theorem 6.16] there exists $f_y \in L^u(yH)$ with $\|f_y\|_u = \|\varphi \circ \iota_y\|$ such that

$$\varphi(\iota_y(g)) = \int_{yH} f_y g \,\mathrm{d}\mu, \quad g \in L^1(yH).$$

Now we can define a locally measurable function $f \colon G \to \mathbb{C}$ by $f(s) = f_y(s)$ for $s \in yH$. Then by Proposition 3,

$$\|f\|_\infty = \sup_{y\in Y} \|f_y\|_u \le \|\varphi\| < \infty,$$

so $f \in L^\infty(G)$. For $g \in L^1(G)$, the set $G_n = \{s \in G \,|\, |g(s)| > \frac{1}{n}\}$ is Borel and has finite measure, so by Proposition 1 there is a countable subset $I_n \subseteq Y$ such that $G_n \subseteq \bigcup_{y \in I_n} yH$. Letting $I = \bigcup_{n\ge1} I_n$, then I is countable and $\{s \in G \,|\, |g(s)| > 0\} \subseteq \bigcup_{y\in I} yH$. For all $y \in I$ we let $g_y = g|_{yH}$, so that

$$g = \sum_{y \in I} \iota_y(g_y)$$

pointwise. By Lebesgue's dominated convergence theorem [2, Theorem 1.34], the equality also holds in $L^1(G)$, and hence

$$\varphi(g) = \sum_{y\in I} \varphi(\iota_y(g_y)) = \sum_{y \in I} \int 1_{yH} f g \,\mathrm{d}\mu = \int \sum_{y\in I} 1_{yH} f g \,\mathrm{d}\mu = \int f g \,\mathrm{d}\mu,$$

once again by the dominated convergence theorem. Hence $\Phi(f) = \varphi$, so Φ is surjective. □

References

[1] G. B. Folland. *A Course in Abstract Harmonic Analysis*, CRC Press, 1995.

[2] W. Rudin. *Real and Complex Analysis*, Third Edition, WCB/McGraw-Hill, 1987.