Sølvkorn 18
The orbit of an irrational rotation

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Theorem 1. *Let $\theta \in \mathbb{R}$, let $\pi\colon \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ be the quotient map and let $H = \pi(\mathbb{Z}\theta)$. Then the following are equivalent:*

(1) *θ is irrational.*

(2) *H is infinite.*

(3) *For all neighbourhoods U of $\pi(0) \in \mathbb{R}/\mathbb{Z}$, there exists $n \in \mathbb{Z}$ such that $\pi(n\theta) \in U \setminus \{\pi(0)\}$.*

(4) *H is dense in $\mathbb{R}/\mathbb{Z}$.*

Proof. (1) ⇒ (2): If H is finite, there exist $j < k$ in $\mathbb{Z}$ for which $\pi(j\theta) = \pi(k\theta)$, i.e., $(k - j)\theta \in \mathbb{Z}$. Thus θ is rational.

(2) ⇒ (1): If θ is rational, taking $q \in \mathbb{Z}$ for which $q\theta \in \mathbb{Z}$ yields $H = \{\pi(0), \pi(\theta), \ldots, \pi((q-1)\theta)\}$.

(2) ⇒ (3): Assuming that H is infinite, let $U \subseteq \mathbb{R}/\mathbb{Z}$ be a neighbourhood of $\pi(0)$, and let $\varepsilon > 0$ such that $\pi((-\varepsilon, \varepsilon)) \subseteq U$. Let N be an integer larger than $\frac{1}{\varepsilon}$. Now there exist distinct $m_1, \ldots, m_{N+1} \in \mathbb{Z}$ such that the points $\pi(m_1\theta), \ldots, \pi(m_{N+1}\theta) \in \mathbb{R}/\mathbb{Z}$ are all distinct. Since

$$\mathbb{R} \setminus \mathbb{Z} = \bigcup_{i=0}^{N-1} \pi([i\varepsilon, (i+1)\varepsilon))$$

there exists $0 \le i \le N-1$ and $1 \le j < k \le N+1$ such that $\pi(m_j\theta), \pi(m_k\theta) \in \pi([i\varepsilon, (i+1)\varepsilon))$ (by the pigeonhole principle). Letting $\theta_j, \theta_k \in [i\varepsilon, (i+1)\varepsilon)$ such that $\pi(m_j\theta) = \pi(\theta_j)$ and $\pi(m_k\theta) = \pi(\theta_k)$, then $|\theta_k - \theta_j| < \varepsilon$, so $\pi((m_k - m_j)\theta) = \pi(\theta_k - \theta_j) \in \pi((-\varepsilon, \varepsilon)) \subseteq U$. Moreover, $\pi(m_j\theta) \neq \pi(m_k\theta)$ implies $\pi((m_k - m_j)\theta) \neq \pi(0)$, so $n = m_k - m_j$ satisfies the requirements of (3).

(3) ⇒ (4): If H is not dense in $\mathbb{R}/\mathbb{Z}$, there exists a non-empty open set $V \subseteq \mathbb{R}/\mathbb{Z}$ for which $H \cap V = \emptyset$, implying that there is some $a \in \mathbb{R}$ and $\varepsilon > 0$ such that $(a - \varepsilon, a + \varepsilon) \subseteq \pi^{-1}(V)$ and $H \cap \pi((a - \varepsilon, a + \varepsilon)) = \emptyset$.

Let $U = \pi((-\varepsilon, \varepsilon))$. As $\pi^{-1}(U) = \bigcup_{p \in \mathbb{Z}} (p + (-\varepsilon, \varepsilon))$ is open, U is an open neighbourhood of $\pi(0)$ (indeed, any quotient homomorphism of a topological group is an open map). Assume that there exists $n \in \mathbb{Z}$ such that $\pi(n\theta) \in U \setminus \{\pi(0)\}$. Let $y \in (-\varepsilon, \varepsilon)$ such that $\pi(y) = \pi(n\theta)$. Then $y \neq 0$, since $\pi(y) \neq \pi(0)$, and $\frac{\varepsilon}{|y|} > 1$, so there exists $p \in \mathbb{Z}$ such that

$$\tfrac{a}{|y|} \in (p\tfrac{y}{|y|} - 1, p\tfrac{y}{|y|} + 1) \subseteq (p\tfrac{y}{|y|} - \tfrac{\varepsilon}{|y|}, p\tfrac{y}{|y|} + \tfrac{\varepsilon}{|y|}).$$

Hence $a \in (py - \varepsilon, py + \varepsilon)$ and $py \in (a - \varepsilon, a + \varepsilon)$, so that $H \ni \pi(pn\theta) = \pi(py) \in \pi((a - \varepsilon, a + \varepsilon))$, a contradiction. Hence $H \cap U = \{\pi(0)\}$.

(4) ⇒ (2): If H is finite, it is closed. Hence $\overline{H} = H \neq \mathbb{R}/\mathbb{Z}$. □