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Periodic and cyclic integers

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The number 142 857 has always sparked my curiosity: it is well-known (and otherwise easily verified) that

$$
\begin{aligned}
2 \cdot 142\,857 &= 285\,714, \\
3 \cdot 142\,857 &= 428\,571, \\
4 \cdot 142\,857 &= 571\,428, \\
5 \cdot 142\,857 &= 714\,285, \\
6 \cdot 142\,857 &= 857\,142,
\end{aligned}
$$

and noted that each of these multiples are obtained by shifting the digits of 142 857 a number of places to the left, but other than the fact that $\frac{1}{7} = 0.142\,857\,142\,857\,142\,857\ldots$, it was never clear to me why 142 857 was so special, or if there were perhaps other numbers with this relationship between its multiples and the digits (and their order) in its decimal representation.

Running some simulations, it quickly became clear that 142 857 was not *all* that special when it came to generating some multiple by shifting digits to the left; for instance,

$$
\begin{aligned}
4 \cdot 025\,641 &= 102\,564, \\
10 \cdot 025\,641 &= 256\,410, \\
16 \cdot 025\,641 &= 410\,256, \\
22 \cdot 025\,641 &= 564\,102, \\
25 \cdot 025\,641 &= 641\,025.
\end{aligned}
$$

The number 142 857 is in fact the only 6-digit number a (allowing for leading zeroes) for which every multiple $a, 2a, \ldots, 6a$ arises by shifting the digits of a, but that does not make a number like 025 641 less interesting. In this small survey, we will consider positive integers for which there exists a non-trivial multiple that can be obtained by shifting the digits in the integer's original base b representation – non-trivial meaning that the factor of the multiple is not a power of b – and find a way of constructing all such numbers by very simple means.

In the following, we let $b \geq 2$ denote a fixed base. Let $n \geq 1$, and let $a = \sum_{i=0}^{n-1} a_i b^i$ be a positive integer less than b^n (where $0 \leq a_i \leq b-1$ for each i). Let $\pi_n \colon \mathbb{Z} \to \mathbb{Z}$ denote the remainder map of division by n, meaning that π_n takes values in $\{0, \ldots, n-1\}$ and that $\pi_n(i) = i$ for all $0 \leq i \leq n-1$. Then $\pi_n(\pi_n(x) + \pi_n(y)) = \pi_n(x+y)$ for $x, y \in \mathbb{Z}$, as n divides $x + y - (\pi_n(x) + \pi_n(y))$; moreover, if n divides $x - y$ and $0 \leq y \leq n-1$, then $\pi_n(x) = y$.

We will investigate numbers a of the above form for which there exists some $k \geq 2$ and $1 \leq j \leq n-1$ such that

$$ka = \sum_{i=0}^{n-1} a_{\pi_n(j+i)} b^i.$$

It is clear from the outset that the interesting cases occur only when k is *not* a power of b. For instance, if we consider $b = 10$ and $n = 5$, then $451 = 00451$ clearly divides 45 100, but this can also be done with any number strictly smaller than $b^{n-1} = 10^4$, and this gets in the way of our exploring the intriguing cases. In the following we discern between these trivial cases and the ones that are not, and only consider trivial cases if it makes for less verbose statements of the results. Moreover, we will mostly use the standard decimal base $b = 10$ for examples, devoting most of the examples to finding all numbers that satisfy this property "non-trivially" in base 10, for smaller values of n.

Definition 1. Let Σ_n denote the set of positive integers of the form $a = \sum_{i=0}^{n-1} a_i b^i$ satisfying the above condition. If $a \in \Sigma_n$ with k and j as above, a is said to be *n-periodic* (in base b), and the pair (k, j) is said to be an *enabling pair* for a.

If there exists an enabling pair (k, j) for a such that k is not a power of b, we say that a is *properly n-periodic*. The set of properly n-periodic numbers is denoted by Σ_n^*. A *periodic (resp. properly periodic) number* is a number that is n-periodic (resp. properly n-periodic) for some $n \geq 1$.

Remark 1. If $b = 10$ and $a \in \Sigma_n$ has no leading zeros (i.e., $a \geq 10^{n-1}$) and $(k, n-1)$ is an enabling pair for a, then a is k-transposable in the sense of Kahan [1]. More generally, if $b \geq 2$ is arbitrary and $(k, n-j)$ is an enabling pair for a in base b, then a is a k-transposable, j-shift integer (expressed in base 10) in the sense of Ludington [2]. Ludington gave necessary and sufficient conditions for the existence of k-transposable, j shift-integers in any base, and proved that k-transposable, j-shift integers exist in bases 3, 4 and 6 for all $j \geq 2$, and all other bases ≥ 5 for all $j \geq 1$.

Lemma 1. *Let $n \geq 1$. If $a \in \Sigma_n$ and (k, j) is an enabling pair for a, then*

(i) $b \leq a < b^n/2$.

(ii) $k \leq \frac{b^n-1}{a}$.

(iii) $b-1$ *divides* $(k-1)a$.

Proof. It is clear that ka cannot be larger than or equal to b^n which proves (ii) and the latter half of (i), and $n = 1$ implies $k = 1$ as well, so that $a \geq b$. Finally, since π_n is a bijection of $\{0, \ldots, n-1\}$, modulo $b-1$ we have

$$a = \sum_{i=0}^{n-1} a_i b^i \equiv \sum_{i=0}^{n-1} a_i = \sum_{i=0}^{n-1} a_{\pi_n(j+i)} \equiv \sum_{i=0}^{n-1} a_{\pi_n(j+i)} b^i = ka,$$

proving (iii). □

Remark 2. Due to the first integer in the enabling pair not being allowed to be a power of b, one might assume that a properly n-periodic number $a \in \Sigma_n$ is not divisible by b. That is not true, however: for $b = 10$ and $n = 6$, note that $38 \cdot 020790 = 790020$ and $9 \cdot 109890 = 989010$, so that $020790, 109890 \in \Sigma_6$.

The next lemma is more consequential.

Lemma 2. *Let a be a positive integer of the form $a = \sum_{i=0}^{n-1} a_i b^i$, and let $1 \leq j \leq n-1$ and $k > 1$. If $a \in \Sigma_n$ with enabling pair (k, j), then*

$$(b^{n-j} - k)a = \left(\sum_{i=j}^{n-1} a_i b^{i-j}\right)(b^n - 1) \tag{0.1}$$

$$(b^j k - 1)a = \left(\sum_{i=0}^{j-1} a_i b^i\right)(b^n - 1) \tag{0.2}$$

Conversely, if one of the equations hold, then $a \in \Sigma_n$ with enabling pair (k, j).

Proof. As $\pi_n(j+i) = j+i$ for $0 \leq i \leq n-1-j$, and $0 \leq i-(n-j) \leq n-1$ and $n = j+i-(i-(n-j))$ for $n-j \leq i \leq n-1$, we see that $\pi_n(j+i) = i-(n-j)$ for $n-j \leq i \leq n-1$. Hence

$$\begin{aligned}
\sum_{i=0}^{n-1} a_{\pi_n(j+i)} b^i &= \sum_{i=0}^{n-j-1} a_{\pi_n(j+i)} b^i + \sum_{i=n-j}^{n-1} a_{\pi_n(j+i)} b^i \\
&= \sum_{i=0}^{n-j-1} a_{j+i} b^i + \sum_{i=n-j}^{n-1} a_{i-(n-j)} b^i \\
&= b^{-j} \sum_{i=j}^{n-1} a_i b^i + b^{n-j} \sum_{i=0}^{j-1} a_i b^i \\
&= b^{-j} \sum_{i=j}^{n-1} a_i b^i + b^{n-j} \left(a - \sum_{i=j}^{n-1} a_i b^i\right),
\end{aligned}$$

so that if we define $s = \sum_{i=0}^{n-1} a_{\pi_n(j+i)} b^i$, then

$$b^{n-j}a - s = (b^{n-j} - b^{-j}) \sum_{i=j}^{n-1} a_i b^i = (b^n - 1) \sum_{i=j}^{n-1} a_i b^{i-j}.$$

This yields

$$\begin{aligned}\left(\sum_{i=0}^{j-1} a_i b^i\right)(b^n - 1) &= \left(a - b^j \sum_{i=j}^{n-1} a_i b^{i-j}\right)(b^n - 1) \\ &= a(b^n - 1) - b^j(b^{n-j}a - s) = b^j s - a.\end{aligned}$$

If $s = ka$, the equations follow immediately. Conversely, if one of the equations hold, then we either have $b^{n-j}a - s = (b^{n-j} - k)a$ or $b^j s - a = (b^j k - 1)a$, and both imply $s = ka$. □

We may deduce some properties of properly periodic numbers straight away, using the above characterization.

Corollary 3. *Let $n \geq 1$. If $a \in \Sigma_n$ is properly n-periodic, (k, j) is an enabling pair for a, and x is an integer dividing $b^n - 1$ but either does not divide $b^j k - 1$ or does not divide $b^{n-j} - k$, then a and x are not relatively prime. In particular, a and $\sum_{i=0}^{n-1} b^i$ are not relatively prime.*

Proof. If a and x are relatively prime, then as $b^{n-j} \neq k$ and $b^j k > 1$, and x divides the right hand side of equations (0.1) and (0.2), the same equations now yield that x divides both $b^{n-j} - k$ and $b^j k - 1$.

Since $j > 0$ we have $x' = \sum_{i=0}^{n-1} b^i > b^{n-j} \geq b^{n-j} - k$, so x' does not divide $b^{n-j} - k$. As $(b-1)x' = b^n - 1$ it follows that a and x' are not relatively prime. □

We now briefly return to the kind of number that instigated this survey.

Definition 2. Let $n \geq 2$. A positive integer $a \leq b^n$ of the form $a = \sum_{i=0}^{n-1} a_i b^i$ is said to be *n-cyclic* (in base b) if for all $1 \leq k \leq n$ there exists $0 \leq j \leq n-1$ such that

$$ka = \sum_{i=0}^{n-1} a_{\pi_n(j+i)} b^i.$$

We can actually provide a characterization of n-cyclic numbers:

Proposition 4. *Let $n \geq 1$ be a positive integer. Then there exists an n-cyclic number a in base b only if $n+1$ is prime and does not divide b, in which case there is at most* one *n-cyclic number, namely*

$$a = \frac{b^n - 1}{n+1}.$$

If a is indeed n-cyclic, the sum of the digits in a equals $(b-1)\frac{n}{2}$.

Fermat's little theorem implies that p divides $b^{p-1} - 1$ if p is prime and does not divide b; hence a in the above result is an integer.

Furthermore, even if $n+1$ is prime and does not divide b, it may be that $\frac{b^n-1}{n+1}$ is not n-cyclic: for $b = 10$, note that

$$\frac{10^{10} - 1}{10+1} = 909090909$$

is not 10-cyclic, even though $10+1$ is prime.

Proof. Assume that a is n-cyclic. For all $1 \leq k \leq n$, define

$$A_k = \left\{ 0 \leq j \leq n-1 \;\middle|\; ka = \sum_{i=0}^{n-1} a_{\pi_n(j+i)} b^i \right\}.$$

Then A_k is non-empty for all $1 \leq k \leq n$ and $A_k \cap A_{k'}$ is empty for all distinct $1 \leq k, k' \leq n$. Since $\bigcup_{k=1}^{n} A_k \subseteq \{0, \ldots, n-1\}$, it follows that each A_k is a singleton and $\bigcup_{k=1}^{n} A_k = \{0, \ldots, n-1\}$. Therefore we may define a bijection $\sigma\colon \{1, \ldots, n\} \to \{0, \ldots, n-1\}$ with the property that $ka = \sum_{i=0}^{n-1} a_{\pi_n(\sigma(k)+i)} b^i$.

For each $2 \le k \le n$ there exists a positive $c_k > 0$ such that $(b^{n-\sigma(k)} - k)a = c_k(b^n - 1)$ by Lemma 2. Let d denote the greatest common divisor of a and $b^n - 1$ and define $q = \frac{b^n-1}{d}$. Since q and $\frac{a}{d}$ are relatively prime, we have

$$(b^{n-\sigma(k)} - k)\frac{a}{d} = c(k)q \equiv 0 \pmod{q}$$

and $b^{n-\sigma(k)} \equiv k \pmod{q}$. Moreover, for $2 \le k \le n$, assume that k and q are not relatively prime. Then there would exist a divisor $p > 1$ of k and q that would also divide $b^{n-\sigma(k)}$ since q divides $b^{n-\sigma(k)} - q$. Hence p would also divide b^n, but as q divides $b^n - 1$, so would p, yielding a contradiction. Hence all $2 \le k \le n$ are relatively prime to q. In particular, $n + 1 \le q$. Notice also that q cannot divide b.

Let $2 \le k \le n$ such that $\sigma(k) = n - 1$ (indeed, $k \ne 1$ since $\sigma(1) = 0$), so that $b = b^{n-\sigma(k)} \equiv k \pmod{q}$. Let $2 \le m \le n$ be smallest such that $km > n$. Because σ is a bijection and $\sigma(1) = 0$, we know that $\sigma(m) > 0$, so there exists $1 \le m' \le n$ such that $\sigma(m') = \sigma(m) - 1$. Now

$$km \equiv bm \equiv b^{n+1-\sigma(m)} \equiv m' \pmod{q}.$$

As q divides $km - m' > n - m' \ge 0$, we must have $q \le km - m'$. Since $k(m-1) \le n$ by how m was chosen (if $m = 2$ the statement holds trivially), we have

$$q \le km - m' \le n - m' + k \le 2n - 1.$$

In particular, $q \le 2n - 1 \le n^2$ and $\sqrt{q} \le n$. If q were not prime, some integer $p \le \sqrt{q} \le n$ would divide q, contradicting that all integers between 2 and n are relatively prime to q; hence q is prime. Moreover, since b has order n in the multiplicative group of integers modulo q (which has order $q - 1$ due to q being prime), n must divide $q - 1$, and since $q - 1 < 2n - 1 < 2n$ by what we have seen above, it follows that $n = q - 1$. In particular, $n + 1$ is prime and does not divide b.

Since $\frac{b^n-1}{d} = n + 1$ and $d = \frac{b^n-1}{n+1}$ divides a, note also that

$$a \le \frac{b^n - 1}{n} < 2\frac{b^n - 1}{n+1}$$

by Lemma 1. Hence $a = \frac{b^n-1}{n+1}$. Finally, if we write $a = \sum_{i=0}^{n-1} a_i b^i$ and define $s = \sum_{i=0}^{n-1} a_i$, then

$$\frac{n(n+1)}{2}a = \sum_{k=1}^{n}\sum_{i=0}^{n-1} a_{\pi_n(\sigma(k)+i)}b^i = \sum_{i=0}^{n-1} b^i \sum_{k=1}^{n} a_{\pi_n(\sigma(k)+i)} = s\frac{b^n - 1}{b - 1},$$

since σ is a bijection, so that the map $\{1,\dots,n\} \to \{0,\dots,n-1\}$ given by $k \mapsto \pi_n(\sigma(k)+i)$ attains all values in $\{0,\dots,n-1\}$. Since $(n+1)a = b^n - 1$, we have $\frac{n}{2} = \frac{s}{b-1}$ and $s = (b-1)\cdot\frac{n}{2}$. This completes the proof. $\square$

We now return to the less strictly defined numbers in Σ_n, and try to obtain a more “abstract“ criterion for an integer to be contained in that set.

Lemma 5. *If n and $1 \le a \le b^n - 1$ are positive integers, $1 \le j \le n - 1$ and $1 < k \le \min\{\frac{b^n-1}{a}, b^{n-j}\}$ are positive integers such that $b^n - 1$ divides $(b^{n-j} - k)a$ or $(b^j k - 1)a$, then $a \in \Sigma_n$ with enabling pair (k, j).*

Proof. We first consider the case when there exists an integer $c \ge 0$ such that $(b^{n-j} - k)a = c(b^n - 1)$. As

$$\begin{aligned}(a - cb^j)b^{-j}(b^n - 1) &= b^{-j}(b^n - 1)a - c(b^n - 1)\\ &= a(b^{-j}(b^n - 1) - (b^{n-j} - k))\\ &= a(k - b^{-j}) < ka \le b^n - 1,\end{aligned}$$

it follows that $a - cb^j < b^j$. Since

$$(b^{n-j} - k)(a - cb^j) = c(b^n - 1 - (b^{n-j} - k)b^j) = c(b^j k - 1) \ge 0,$$

we also have $a - cb^j \ge 0$, so c is the quotient of a after dividing by b^j. If we write $a = \sum_{i=0}^{n-1} a_i b^i$ for $a_i \in \{0,\dots,b-1\}$, then

$$a = b^j \sum_{i=j}^{n-1} a_i b^{i-j} + \sum_{i=0}^{j-1} a_i b^i.$$

Since $0 \le \sum_{i=0}^{j-1} a_i b^i \le b^j - 1$, it follows from uniqueness of quotients and remainders in Euclidean division that

$$c = \sum_{i=j}^{n-1} a_i b^{i-j}, \quad a - cb^j = \sum_{i=0}^{j-1} a_i b^i.$$

By Lemma 2 we conclude that $a \in \Sigma_n$ with enabling pair (k, j).

Next, assume the existence of an integer $d \geq 0$ such that $(b^j k - 1)a = d(b^n - 1)$. Then

$$a(b^{n-j} - k)b^j = a(b^n - kb^j) = a(b^n - 1) - a(b^j k - 1) = (a - d)(b^n - 1).$$

Since b^j and $b^n - 1$ are relatively prime and b^j divides the left hand side of the equation, b^j must divide $a - d$; notice also that the above equation implies $a - d \geq 0$. Writing $cb^j = a - d$ for a positive integer c, we have $a(b^{n-j} - k) = c(b^n - 1)$, so using the first part of the lemma we find that $a \in \Sigma_n$ with enabling pair (k, j). □

The above criterion may then finally be used to obtain a complete characterization of properly periodic numbers.

Proposition 6. *Let $n \geq 1$ and $1 \leq a \leq b^n - 1$ be positive integers. Then the following are equivalent:*

(i) *$a \in \Sigma_n^*$, i.e., a is properly n-periodic.*

(ii) *Let d be the greatest common divisor of a and $b^n - 1$. Then there exists a positive integer $1 < k \leq \frac{b^n-1}{a}$ such that $k \equiv b^j \pmod{\frac{b^n-1}{d}}$ for some $1 \leq j \leq n-1$ and k is not a power of b.*

(iii) *There exists a divisor d of $b^n - 1$ and a positive integer $1 < k \leq q = \frac{b^n-1}{d}$ such that $k \equiv b^j \pmod{q}$ for some $1 \leq j \leq n-1$, k is not a power of b, and $a = md$ for some $1 \leq m \leq \frac{q}{k}$.*

If conditions (ii) or (iii) hold, then $(n - j, k)$ is an enabling pair for a.

Proof. (i) $\Longrightarrow$ (ii): Let (k, j) be an enabling pair for a. Since $\frac{a}{d}$ and $q = \frac{b^n-1}{d}$ are relatively prime, the equations (0.1) and (0.2) imply that q divides both $(b^{n-j} - k)\frac{a}{d}$ and $(b^j k - 1)\frac{a}{d}$. In particular q divides $b^{n-j} - k$. The rest of (ii) now follows since $1 \leq n - j \leq n - 1$, $k > 1$ and k is not a power of b by definition, and $k \leq a^{-1}(b^n - 1)$ by Lemma 1.

(ii) $\Longrightarrow$ (iii): Let d be the greatest common divisor of a and $b^n - 1$, let k be a positive integer satisfying the properties of (ii) and let $m = \frac{a}{d}$. Since $k \leq \frac{b^n-1}{a} \leq \frac{b^n-1}{d}$, and m is an integer satisfying $1 \leq m = \frac{a}{d} \leq \frac{b^n-1}{kd}$ and $a = md$. Thus d and k satisfy the properties of (iii).

(iii) $\Longrightarrow$ (i): Let d, k, j and m be integers satisfying the properties of (iii). If $b^j < d$, then $k \equiv b^j \pmod{\frac{b^n-1}{d}}$ implies that k equals b^j since b^j is the remainder when dividing k by d. This is a contradiction, so $b^j \geq d > k$. Let $j_0 = n - j$ and let c be an integer such that $b^{n-j_0} - k = b^j - k = c\frac{b^n-1}{d}$, which implies $(b^{n-j_0} - k)a = (b^{n-j_0} - k)md = mc(b^n - 1)$. Hence $b^n - 1$ divides $(b^{n-j_0} - k)a$. Since $k < b^j = b^{n-j_0}$ and

$$k \leq \frac{b^n - 1}{md} = \frac{b^n - 1}{a},$$

Lemma 5 implies that $a \in \Sigma_n^*$ with enabling pair (k, j). □

Definition 3. For any divisor d of $b^n - 1$, let $\Sigma_{n,d}^*$ be the set of positive integers $1 \leq a \leq b^n - 1$ for which a and d satisfy condition (iii) of Proposition 6.

Remark 3. We have previously seen that 142 857 is 6-cyclic in base 10 and uniquely so, but 142 857 is also unique in another way, as shown by Kahan; using what we have found above, we may actually recover the main result of [1].

Assume that $a \in \Sigma_n$ satisfies $a \geq 10^{n-1}$ and that there exists $k > 1$ such that $(k, n-1)$ is an enabling pair for a (so that a is a k-transposable integer in the sense of [1]. Since $a > 10^{n-1}$, we must have $1 < k < 10$. We now use the proof of (i) $\Longrightarrow$ (ii) in Proposition 6. Let d be the greatest common divisor of a and $10^n - 1$, so that $10 \equiv k \pmod{q}$ for $q = \frac{10^n-1}{d}$. Since q divides $10 - k > 0$, we must have $q < 10$. Note first q cannot be even or equal 5, as neither 5 nor any even number divides $10^n - 1$. If $q \in \{1, 3, 9\}$, then $d = \sum_{i=0}^{n-1} x10^i$ and

$$a = \sum_{i=0}^{n-1} \frac{a}{d} x10^i$$

for some $x \in \{1, 3, 9\}$, but this cannot happen by our assumption that $a \in \Sigma_n$. Therefore there is only one possibility, namely $q = 7$, so that $d = \frac{10^n-1}{7}$ and $k = 3$. Since $a \leq \frac{10^n-1}{3}$ by Lemma 1, writing $sd = a$ for an integer $s \geq 1$ yields

$$\frac{s(10^n - 1)}{7} \leq \frac{10^n - 1}{3}$$

and $s \leq 2$. Note also that 6 divides n; indeed, if we write $n = 6m + r$ for $0 \leq r < 6$, then

$$(10^n - 1) - (10^r - 1) = 10^r(10^{6m} - 1) = 10^r(10^6 - 1)\sum_{i=0}^{m-1} 10^{6i}.$$

Since 7 divides the right hand side and $10^n - 1$, 7 must also divide $10^r - 1$, implying $r = 0$. We conclude that

$$a = \frac{10^n - 1}{7}x = 142\,857\sum_{i=0}^{m-1} 10^{6i}, \text{ or } a = \frac{2(10^n - 1)}{7} = 285\,714\sum_{i=0}^{m-1} 10^{6i}.$$

Remark 4. Let $n \geq 1$. To find Σ_n^*, we may use condition (iii) of Proposition 6 as follows. If d divides $b^n - 1$ and $q = \frac{b^n-1}{d}$, note that there is no $1 < k \leq q$ with $k \equiv b^j \pmod{q}$ for some $1 \leq j \leq n-1$ such that k is not a power of b, if either:

(i) $q = \sum_{i=0}^{m-1} xb^i$ for x dividing $b - 1$ and m divides n.

(ii) $q > b^{n-1}$.

Indeed, if $q = \sum_{i=0}^{m-1} xb^i$, then $1 < b^j < q$ for $1 \leq j \leq m-1$ and $b^m \equiv 1 \pmod{q}$, so that every $1 \leq k \leq q$ satisfying $k \equiv b^j \pmod{q}$ for some $1 \leq j \leq n-1$ must be a power of b.

If $q > b^{n-1}$, and $1 \leq j \leq n-1$ and $1 < k \leq q$ are integers such that q divides $k - b^j$, then since $k - b^j < q$ and $b^j - k < b^{n-1} < q$ we have $|k - b^j| < q$. Thus $k - b^j = 0$. This implies that we only need to consider divisors $d \geq b$ such that d is not of the form

$$\sum_{i=1}^{n/m-1} xb^{im}$$

for m dividing n and x dividing $b - 1$ – notice that this includes $\frac{b^n-1}{y}$ for all y dividing $b - 1$ – otherwise $\Sigma_{n,d} = \emptyset$.

For every divisor d of $b^n - 1$ of the above form, we now find all residue classes of b^j modulo $q = \frac{b^n-1}{d}$. If one of the residue classes modulo q is not a power of b, we let $1 < k \leq q$ be the smallest possible representative of such a residue class, in which case $md \in \Sigma_n^*$ for all $1 \leq m \leq \frac{q}{k}$ – the above proposition then states that Σ_n^* is determined by finding all such numbers, for all divisors d of $b^n - 1$.

In the following, for all divisors q of $b^n - 1$ we define

$$R_q = \{1 \leq k \leq q \,|\, k \equiv b^j \pmod{q} \text{ for some } 1 \leq j \leq n-1\}.$$

Remark 5. Let d be a divisor of $b^n - 1$ and let p be a divisor of $q = \frac{b^n-1}{d}$. If R_p and R_q both contain non-powers of b and the smallest numbers of R_q and R_p that are not powers of b coincide, then $\Sigma^*_{n,\frac{q}{p}d} \subseteq \Sigma^*_{n,d}$.

Indeed, let $a \in \Sigma^*_{n,\frac{q}{p}d}$. Then $a = m\frac{q}{p}d$ for

$$1 < k \leq \frac{b^n - 1}{\frac{q}{p}d} = p$$

and $1 \leq m \leq \frac{p}{k}$ such that $b^j \equiv k \pmod{p}$ for some $1 \leq j \leq n-1$ and k is not a power of b. In particular, $k \in R_p$, so k majorizes the smallest non-power of b in R_p, say, k', and $1 \leq m \leq \frac{p}{k'}$. Since $k' \in R_q$ by assumption, there exists $1 \leq i \leq n-1$ such that $k' \equiv b^i$ modulo q. Since

$$1 \leq m\frac{q}{p} \leq \frac{q}{k'},$$

it follows that $a \in \Sigma^*_{n,d}$.

Example 1. For $b = 8$, let us find the set of properly 5-periodic numbers. As $x = 8^5 - 1 = 7 \cdot 31 \cdot 151$, we now need to run through each divisor $d \geq 8$ and $d \notin \{\frac{8^5-1}{7}, 8^5 - 1\}$ as follows:

Case 1: $d = 31$, in which case $q = \frac{x}{d} = 7 \cdot 151$. Now $R_q = \{8, 8^2, 8^3, 925\}$, we choose k to be the smallest non-power of 8, so that $k = 925$. The largest integer smallest than $\frac{q}{k}$ is 1, so we conclude that $\Sigma^*_{5,d} = \{1\} \cdot d = \{31\}$.

Case 2: $d = 151$, in which case $q = \frac{x}{d} = 217$ and $R_q = \{8, 8^2, 78, 190\}$. We choose $k = 78$, and the largest integer smaller than $\frac{q}{k}$ is 2, so that $\Sigma^*_{5,d} = \{151, 302\}$.

Case 3: $d = 217$, in which case $q = 151$ and $R_q = \{8, 8^2, 59, 19\}$. We choose $k = 19$. The largest integer smaller than $\frac{q}{k}$ is 7, so that $\Sigma^*_{5,d} = \{1,\dots,7\}\cdot 217$.

Case 4: $d = 7\cdot 151 = 1\,057$, in which case $q = 31$ and $R_q = \{8,2,16,4\}$; choosing $k = 2$, the largest integer smaller than $\frac{q}{k}$ is 15, so that $\Sigma^*_{5,d} = \{1,\dots,15\}\cdot 1\,057$.

The union of the four sets found above constitute all of Σ^*_5. Verifying that each of the numbers is indeed 5-periodic is easy: for instance, consider $12\cdot 1057 = 12\,684$ from Case 4. In base 8, 12 684 has the base 8 representation 30614 and $25\,368 = 2\cdot 12\,684$ has base 8 representation 61430, proving that 12 684 is indeed properly 5-periodic in base 8.

For the rest of this survey we fix $b = 10$.

Example 2 (2-digit numbers in base 10 ($\leq 10^2$))**.** The set Σ^*_2 is empty by the above remark, since the only non-trivial divisors of 99 are 3, 9, 11 and 33.

Example 3 (3-digit numbers in base 10 ($\leq 10^3$))**.** We run through each divisor $10 \leq d$ of $999 = 3^3\cdot 37$, excluding 111, 333 and 999 due to Remark 4:

1. $d = 27$ ($q = 37$): As $R_q = \{10, 26\}$, we let $k = 26$. Hence $\Sigma^*_{3,27} = \{27\}$ (since $m = 1$ is the only integer $1 \leq m \leq \frac{37}{26}$).
2. $d = 37$ ($q = 27$): As $R_q = \{10, 19\}$, let $k = 19$. Since $m = 1$ is the only integer $1 \leq m \leq \frac{27}{19}$, $\Sigma^*_{3,37} = \{37\}$.

We conclude that $\Sigma^*_3 = \{27, 37\}$, and indeed one can see that $26\cdot 027 = 702$ and $19\cdot 037 = 703$.

Example 4 (4-digit numbers in base 10 ($\leq 10^4$))**.** We run through each divisor $10 \leq d$ of $9\,999 = 3^2\cdot 11\cdot 101$, excluding 101, 303, 909, 1 111, 3 333 and 9 999, due to Remark 4. If $d = 11$ ($q = 909$), then $R_q = \{10, 100, 91\}$, so we let $k = 91$. If m is an integer such that $1 \leq m \leq \frac{909}{91}$, then $m \leq 9$, and hence $\Sigma^*_{4,11} = \{1,\dots,9\}\cdot 11$. Since R_q for $q = 303$ and $q = 101$ both equal $\{10, 100, 91\}$, Remark 5 yields $\Sigma^*_{4,33} \cup \Sigma^*_{4,99} \subseteq \Sigma^*_{4,11}$. We conclude that $\Sigma^*_4 = \{11, 22, \dots, 99\}$. Indeed note that $91\cdot 0011 = 1\,001$, $91\cdot 0022 = 2\,002$, and so on.

Example 5 (5-digit numbers in base 10 ($\leq 10^5$))**.** We run through each divisor $10 \leq d \leq 10^4$ of $99\,999 = 3^2\cdot 41\cdot 271$:

1. $d = 41$ ($q = 2\,439$): As $R_q = \{10, 10^2, 10^3, 244\}$, let $k = 244$. If m is an integer such that $1 \leq m \leq \frac{2\,439}{244}$, then $m \leq 9$, and hence $\Sigma^*_{5,11} = \{1,\dots,9\}\cdot 41$.
2. $d = 123$ ($q = 813$): As $R_q = \{10, 10^2, 187, 244\}$, let $k = 187$. If $m \leq \frac{813}{187}$ is an integer, then $m \leq 4$, and therefore $\Sigma^*_{5,123} = \{1,2,3,4\}\cdot 123$.
3. $d = 369$ ($q = 271$): As $R_q = \{10, 10^2, 187, 244\}$, so $\Sigma^*_{5,369} \subseteq \Sigma^*_{5,123}$ by Remark 5.
4. $d = 271$ ($q = 369$): As $R_q = \{10, 10^2, 262, 37\}$, let $k = 37$. If $m \leq \frac{369}{37}$ is an integer, then $m \leq 9$, so $\Sigma^*_{5,271} = \{1,\dots,9\}\cdot 271$.
5. $d = 813$ ($q = 123$): As $R_q = \{10, 10^2, 16, 37\}$, let $k = 16$. If $m \leq \frac{123}{16}$ is an integer, $m \leq 7$, meaning $\Sigma^*_{5,813} = \{1,\dots,7\}\cdot 813$.
6. $d = 2\,439$ ($q = 41$): As $R_q = \{10, 18, 16, 37\}$, let $k = 16$ again; if $m \leq \frac{41}{16}$ is an integer, $m \leq 2$, so $\Sigma^*_{5,2439} = \{1,2\}\cdot 2439$.

We conclude that Σ^*_5 consists of 23 numbers, namely

$$\Sigma^*_5 = \{1,\dots,9,12\}\cdot 41 \cup \{1,\dots,9\}\cdot 271 \cup \{4,\dots,7\}\cdot 813.$$

Remark 6. The above example yields that it is not necessarily true that the non-powers of a base b in the set R_q and $R_{q'}$ coincide for divisors q and q' of $b^n - 1$, even if q divides q'.

Example 6 (6-digit numbers in base 10 ($\leq 10^6$))**.** As $999\,999 = 3^3\cdot 7\cdot 11\cdot 13\cdot 37$ has a lot of divisors (64), we go for a less verbose solution in Figure , running through each divisor d of $10^6 - 1$. For the divisor d, a row has three greyed-out cells if we can rule out non-emptyness of $\Sigma^*_{6,d}$ straight away (with Remark 4), and two greyed-out cells if we can tell that we have already found Σ^*_d via a calculation made previously (with Remark 5).

Example 7 (7-digit numbers in base 10 ($\leq 10^7$))**.** Using the fact that $10^7 - 1 = 3^2\cdot 239\cdot 4\,649$, one can show that Σ^*_7 consists of 28 numbers:

$$\Sigma^*_7 = \{239, 478\} \cup \{1,\dots,5\}\cdot 717 \cup \{2,\dots,9\}\cdot 2\,151 \cup \{1,2\}\cdot 4\,649 \cup \{1,2\}\cdot 13\,947 \cup \{1,\dots,9\}\cdot 41\,841.$$

Figure 1: Finding Σ_6^* by investigating all divisors of 999 999 one by one

Divisor d	R_q for $q = \frac{10^n-1}{d}$	Smallest $k \neq 10^j$	Largest integer $m \leq \frac{q}{k}$	$\Sigma_{6,d}^*$
1				$\emptyset$
7				$\emptyset$
11	$\{10,10^2,10^3,10^4,9091\}$	9091	9	$\{1,\dots,9\}\cdot 11$
13	$\{10,10^2,10^3,10^4,23077\}$	23077	3	$\{1,2,3\}\cdot 13$
37	$\{10,10^2,10^3,10^4,18919\}$	18919	1	$\{37\}$
$77 = 7\cdot 11$	$\{10,10^2,10^3,10^4,9091\}$			$\subseteq \Sigma_{6,11}^*$
$91 = 7\cdot 13$	$\{10,10^2,10^3,10^4,1099\}$	1099	9	$\{1,\dots,9\}\cdot 91$
$259 = 7\cdot 37$	$\{10,10^2,10^3,2278,3475\}$	2278	1	$\{259\}$
$143 = 11\cdot 13$	$\{10,10^2,10^3,3007,2098\}$	2098	3	$\{1,2,3\}\cdot 143$
$407 = 11\cdot 37$	$\{10,10^2,10^3,172,1720\}$	172	14	$\{1,\dots,14\}\cdot 407$
$481 = 13\cdot 37$	$\{10,10^2,10^3,1684,208\}$	208	9	$\{1,\dots,9\}\cdot 481$
$1001 = 7\cdot 11\cdot 13$				$\emptyset$
$2849 = 7\cdot 11\cdot 37$	$\{10,10^2,298,172,316\}$			$\subseteq \Sigma_{6,407}^*$
$3367 = 7\cdot 13\cdot 37$	$\{10,10^2,109,199,208\}$	109	2	$\{1,2\}\cdot 3367$
$5291 = 11\cdot 13\cdot 37$	$\{10,10^2,55,172,19\}$	19	9	$\{1,\dots,9\}\cdot 5291$
$37037 = 7\cdot 11\cdot 13\cdot 37$	$\{10,19,1\}$			$\subseteq \Sigma_{6,5291}^*$
3				$\emptyset$
$21 = 3\cdot 7$	$\{10,10^2,10^3,10^4,4762\}$	4762	9	$\{1,\dots,9\}\cdot 21$
$33 = 3\cdot 11$	$\{10,10^2,10^3,10^4,9091\}$			$\subseteq \Sigma_{6,11}^*$
$39 = 3\cdot 13$	$\{10,10^2,10^3,10^4,23077\}$			$\subseteq \Sigma_{6,13}^*$
$111 = 3\cdot 37$	$\{10,10^2,10^3,991,901\}$	901	9	$\{1,\dots,9\}\cdot 111$
$231 = 3\cdot 7\cdot 11$	$\{10,10^2,10^3,1342,433\}$	433	9	$\{1,\dots,9\}\cdot 231$
$273 = 3\cdot 7\cdot 13$	$\{10,10^2,10^3,2674,1099\}$			$\subseteq \Sigma_{6,91}^*$
$777 = 3\cdot 7\cdot 37$	$\{10,10^2,10^3,991,901\}$			$\subseteq \Sigma_{6,111}^*$
$429 = 3\cdot 11\cdot 13$	$\{10,10^2,10^3,676,2098\}$	676	3	$\{1,2,3\}\cdot 429$
$1221 = 3\cdot 11\cdot 37$	$\{10,10^2,181,172,82\}$	82	9	$\{1,\dots,9\}\cdot 1221$
$1443 = 3\cdot 13\cdot 37$	$\{10,10^2,307,298,208\}$			$\subseteq \Sigma_{6,481}^*$
$3003 = 3\cdot 7\cdot 11\cdot 13$				$\emptyset$
$8547 = 3\cdot 7\cdot 11\cdot 37$	$\{10,10^2,64,55,82\}$	55	2	$\{1,2\}\cdot 8547$
$10101 = 3\cdot 7\cdot 13\cdot 37$				$\emptyset$
$15873 = 3\cdot 11\cdot 13\cdot 37$	$\{10,37,55,46,19\}$			$\subseteq \Sigma_{6,5291}^*$
$111111 = 3\cdot 7\cdot 11\cdot 13\cdot 37$				$\emptyset$
$9 = 3^2$				$\emptyset$
$63 = 3^2\cdot 7$	$\{10,10^2,10^3,10^4,4762\}$			$\subseteq \Sigma_{6,21}^*$
$99 = 3^2\cdot 11$	$\{10,10^2,10^3,10^4,9091\}$			$\subseteq \Sigma_{6,11}^*$
$117 = 3^2\cdot 13$	$\{10,10^2,10^3,1453,5983\}$	1453	5	$\{1,\dots,5\}\cdot 117$
$333 = 3^2\cdot 37$	$\{10,10^2,10^3,991,901\}$			$\subseteq \Sigma_{6,111}^*$
$693 = 3^2\cdot 7\cdot 11$	$\{10,10^2,10^3,1342,433\}$			$\subseteq \Sigma_{6,231}^*$
$819 = 3^2\cdot 7\cdot 13$	$\{10,10^2,10^3,232,1099\}$	232	5	$\{1,\dots,5\}\cdot 819$
$2331 = 3^2\cdot 7\cdot 37$	$\{10,10^2,142,133,43\}$	43	9	$\{1,\dots,9\}\cdot 2331$
$1287 = 3^2\cdot 11\cdot 13$	$\{10,10^2,223,676,544\}$	223	3	$\{1,2,3\}\cdot 1287$
$3663 = 3^2\cdot 11\cdot 37$	$\{10,10^2,181,172,82\}$			$\subseteq \Sigma_{6,1221}^*$
$4329 = 3^2\cdot 13\cdot 37$	$\{10,10^2,76,67,208\}$	67	3	$\{1,2,3\}\cdot 4329$
$9009 = 3^2\cdot 7\cdot 11\cdot 13$				$\emptyset$
$25641 = 3^2\cdot 7\cdot 11\cdot 37$	$\{10,22,25,16,4\}$	4	9	$\{1,\dots,9\}\cdot 25641$
$30303 = 3^2\cdot 7\cdot 13\cdot 37$				$\emptyset$
$47619 = 3^2\cdot 11\cdot 13\cdot 37$	$\{10,16,13,4,19\}$	4	5	$\{1,\dots,5\}\cdot 47619$
$333333 = 3^2\cdot 7\cdot 11\cdot 13\cdot 37$				$\emptyset$
$27 = 3^3$	$\{10,10^2,10^3,10^4,25926\}$	25926	1	$\{27\}$
$189 = 3^3\cdot 7$	$\{10,10^2,10^3,4709,4762\}$	4709	1	$\{189\}$
$297 = 3^3\cdot 11$	$\{10,10^2,10^3,3266,2357\}$	2357	1	$\{297\}$
$351 = 3^3\cdot 13$	$\{10,10^2,10^3,1453,285\}$	285	9	$\{1,\dots,9\}\cdot 351$
$999 = 3^3\cdot 37$	$\{10,10^2,10^3,991,901\}$			$\subseteq \Sigma_{6,111}^*$
$2079 = 3^3\cdot 7\cdot 11$	$\{10,10^2,38,380,433\}$	38	12	$\{1,\dots,12\}\cdot 2079$
$2457 = 3^3\cdot 7\cdot 13$	$\{10,10^2,186,232,285\}$	186	2	$\{1,2\}\cdot 2457$
$6993 = 3^3\cdot 7\cdot 37$	$\{10,10^2,142,133,43\}$			$\subseteq \Sigma_{6,2331}^*$
$3861 = 3^3\cdot 11\cdot 13$	$\{10,10^2,223,158,26\}$	26	9	$\{1,\dots,9\}\cdot 3861$
$10989 = 3^3\cdot 11\cdot 37$	$\{10,9,90,81,82\}$	9	10	$\{1,\dots,10\}\cdot 10989$
$12987 = 3^3\cdot 13\cdot 37$	$\{10,23,76,67,54\}$	23	3	$\{1,2,3\}\cdot 12987$
$27027 = 3^3\cdot 7\cdot 11\cdot 13$	$\{10,26,1\}$			$\subseteq \Sigma_{6,3861}^*$
$76923 = 3^3\cdot 7\cdot 11\cdot 37$	$\{10,9,12,3,4\}$	3	4	$\{1,\dots,4\}\cdot 76923$
$90909 = 3^3\cdot 7\cdot 13\cdot 37$				$\emptyset$
$142857 = 3^3\cdot 11\cdot 13\cdot 37$	$\{3,2,6,4,5\}$	2	3	$\{1,2,3\}\cdot 142857$
$999999 = 3^3\cdot 7\cdot 11\cdot 13\cdot 37$				$\emptyset$

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