Introduction to K-theory

Assignment 1

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Let $\langle \cdot, \cdot \rangle$ be the standard inner product on $\mathbb{C}^n$,

$$\langle z, w \rangle = \sum_{i=1}^{n} z_i \overline{w_i}.$$

(i)

Show that taking the conjugate transpose of a matrix (i.e. $T \mapsto T^*$) defines an involution on $M_n(\mathbb{C})$ with

$$\langle Tz, w \rangle = \langle z, T^* w \rangle, \quad z, w \in \mathbb{C}^n.$$

Lemma 1. *$(TU)^T = U^T T^T$, $(TU)^* = U^* T^*$ and $(T^*)^* = T$ for all $T, U \in M_n(\mathbb{C})$.*

Proof. Since $\langle z, \overline{w} \rangle = \langle w, \overline{z} \rangle$ for all $z, w \in \mathbb{C}^n$, we have

$$(TU)_{ij}^T = (TU)_{ji} = \langle T_{j*}, \overline{U_{*i}} \rangle = \langle U_{*i}, \overline{T_{j*}} \rangle = \langle U_{i*}^T, \overline{T_{*j}^T} \rangle = (U^T T^T)_{ij}.$$

Since

$$(\overline{T}\,\overline{U})_{ij} = \sum_{k=1}^{n} \overline{T_{i*}}\,\overline{U_{*j}} = \overline{\sum_{k=1}^{n} T_{i*} U_{*j}} = (\overline{TU})_{ij},$$

we obtain $(TU)^* = U^* T^*$. Finally, $(T^*)_{ij}^* = \overline{T_{ji}^*} = T_{ij}$. □

For $a, b \in \mathbb{C}$ and $T, U \in M_n(\mathbb{C})$, we have

$$(aT + bU)_{ij}^* = \overline{(aT + bU)_{ij}^T} = \overline{(aT + bU)_{ji}} = \overline{a}\overline{T_{ji}} + \overline{b}\,\overline{U_{ji}} = \overline{a}T_{ij}^* + \overline{b}U_{ij}^* = (\overline{a}T^* + \overline{b}U^*)_{ij},$$

so with Lemma 1, the conjugate transpose defines an involution on $M_n(\mathbb{C})$. Furthermore,

$$\begin{aligned}
\langle Tz, w \rangle &= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} T_{ij} z_j \right) \overline{w_i} = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \overline{\overline{T_{ij}} w_i} z_j \right) = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} \overline{\overline{T_{ij}} w_i} z_j \right) \\
&= \sum_{j=1}^{n} z_j \overline{\left(\sum_{i=1}^{n} \overline{T_{ij}} w_i \right)} = \sum_{j=1}^{n} z_j \overline{\left(\sum_{i=1}^{n} \overline{T_{ji}^T} w_i \right)} = \sum_{j=1}^{n} z_j \overline{\left(\sum_{i=1}^{n} T_{ji}^* w_i \right)} = \sum_{j=1}^{n} z_j \overline{(T^* w)_j} \\
&= \langle z, T^* w \rangle
\end{aligned}$$

for all $z, w \in \mathbb{C}^n$. This proves what was wanted.

(ii)

Show that for all $T \in M_n(\mathbb{C})$ the matrix $1 + TT^*$ is invertible.

Let $T \in M_n(\mathbb{C})$. Assume that v is an eigenvector of TT^* with eigenvalue λ. Then $v \neq 0$ and

$$\lambda = \frac{1}{\|v\|^2} \lambda \langle v, v \rangle = \frac{1}{\|v\|^2} \langle \lambda v, v \rangle = \frac{1}{\|v\|^2} \langle TT^* v, v \rangle = \frac{1}{\|v\|^2} \langle T^* v, T^* v \rangle \geq 0.$$

Hence TT^* does not have negative eigenvalues, and in particular, -1 is not an eigenvalue of TT^*. Thus the matrix $1 + TT^* = TT^* - (-1)1$ is invertible.

(iii)

Let $E \subseteq \mathbb{C}^n$ be a subspace. Show that for any $u \in \mathbb{C}^n$ there is a unique $v \in E$ such that

$$\|u - v\| = \inf_{w \in E} \|u - w\|$$

and that $p_E : u \mapsto v$ is a projection, i.e. that $p_E \in \mathrm{Proj}(M_n(\mathbb{C}))$.

Let $m = \inf_{w\in E} \|u - w\|$. If $\dim E = 0$, then the zero vector $0 \in E$ clearly satisfies the condition uniquely for all $u \in \mathbb{C}^n$ and the corresponding zero map $p_0 : \mathbb{C}^n \to \mathbb{C}^n$ is clearly linear, idempotent and self-adjoint and therefore a projection.

Assume that $\dim E > 0$ and choose an orthonormal basis $(a_i)_{i=1}^k$ for E. Let $u \in \mathbb{C}^n$. We first prove uniqueness of such a $v \in E$. If $v, v' \in E$ satisfy $\|u - v\| = \|u - v'\| = m$, then $w := \frac{1}{2}(v + v') \in E$, so $\|u - \frac{1}{2}(v + v')\| \geq m$. By applying the parallelogram law to $u - v$ and $u - v'$, we obtain

$$4m^2 = 2\|u - v\|^2 + 2\|u - v'\|^2 = \|2u - (v + v')\|^2 + \|v - v'\|^2 = 4\|w\|^2 + \|v - v'\|^2 \geq 4m^2 + \|v - v'\|^2,$$

yielding $\|v - v'\|^2 \leq 0$ and thus $v = v'$. Now for existence, define $v = \sum_{i=1}^k \langle u, a_i\rangle a_i$. Clearly, $v \in E$ and since

$$\langle u - v, a_i\rangle = \langle u, a_i\rangle - \langle v, a_i\rangle = \langle u, a_i\rangle - \langle u, a_i\rangle = 0, \tag{1}$$

we obtain $u - v \in E^{\perp}$. Since $\|u - w\|^2 = \|u - v\|^2 + \|v - w\|^2 \geq \|u - v\|^2$ for all $w \in E$ by Pythagoras' theorem (because $v - w \in E$, E being a subspace), we then obtain $\|u - v\| \leq \|u - w\|$ for all $w \in E$ and thus $\|u - v\| \leq m$. However, we clearly also have $\|u - v\| \geq m$, so $\|u - v\| = m$. Thus v satisfies the condition uniquely, and $p_E : \mathbb{C}^n \to \mathbb{C}^n$ defined by $p_E(u) = v$ is a well-defined map. Note also that (1) yields

$$\langle x, w\rangle = \langle p_E(x), w\rangle, \quad x \in \mathbb{C}^n,\ w \in E. \tag{2}$$

We now prove that p_E is a projection. Let $u, z \in \mathbb{C}^n$. Then for $\lambda \in \mathbb{C}$,

$$\langle \lambda p_E(u) + p_E(z), w\rangle = \lambda\langle p_E(u), w\rangle + \langle p_E(z), w\rangle = \lambda\langle u, w\rangle + \langle z, w\rangle = \langle \lambda u + z, w\rangle = \langle p_E(\lambda u + z), w\rangle$$

for all $w \in E$, using (2), thus proving that $\lambda p_E(u) + p_E(z) = p_E(\lambda u + z)$ by properties of the inner product on the vector space E, and therefore that $p_E : \mathbb{C}^n \to \mathbb{C}^n$ is linear and $p_E \in M_n(\mathbb{C})$. Additionally,

$$\langle p_E(p_E(u)), w\rangle = \langle p_E(u), w\rangle$$

for all $w \in E$, so $p_E^2 = p_E$, and since $p_E(x) \in E$ for all $x \in \mathbb{C}^n$, we obtain

$$\langle x, p_E^*(u)\rangle = \langle p_E(x), u\rangle = \overline{\langle u, p_E(x)\rangle} = \overline{\langle p_E(u), p_E(x)\rangle} = \langle p_E(x), p_E(u)\rangle = \langle x, p_E(u)\rangle$$

for all $x \in \mathbb{C}^n$, implying that $p_E^* = p_E$, using (2) in both calculations. Thus p_E is a projection.