Introduction to K-theory

Assignment 2

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Let $n \geq 1$ and recall that the group $U(n) \subseteq GL_n(\mathbb{C})$ is the group of *unitary* $n \times n$ matrices, i.e. matrices P such that $PP^* = P^*P = I_n$.

(a)

Show that for any $A \in GL_n(\mathbb{C})$ the matrix A^*A is positive, invertible, and that $A^*A = R^{-1}DR$ for some real diagonal matrix $D = (d_{ij})$ with $d_{ii} > 0$ and some $R \in U(n)$ (the spectral theorem).

$B := A^*A$ is clearly invertible, as A and thus A^* are invertible, and its inverse is the matrix $A^{-1}(A^{-1})^*$. It is also clear that B is normal, since B is self-adjoint; indeed, $B^* = (A^*A)^* = A^*(A^*)^* = A^*A = B$, so $BB^* = BB = B^*B$. For any eigenvalue λ of B with corresponding eigenvector v, then

$$\lambda = \lambda\langle v, v\rangle = \langle \lambda v, v\rangle = \langle A^*Av, v\rangle = \langle Av, Av\rangle > 0.$$

The strict inequality follows from the fact that $Av \neq 0$ because A is invertible and $v \neq 0$. The spectrum of B consists of all eigenvalues of B, and thus the spectrum of B is a subset of a set of real, positive numbers, and therefore B is positive.

B has at least one eigenvalue by the fundamental theorem of algebra. For each eigenvalue λ_i of B, choose an orthonormal basis $\mathfrak{S}_i$ for the corresponding eigenspaces $\mathfrak{G}_i$. We proceed to prove that $\mathfrak{S} = \bigcup_i \mathfrak{S}_i$ is an orthonormal basis for $\mathbb{C}^n$, i.e. all vectors are orthogonal, have length 1 and constitute a basis for $\mathbb{C}^n$. Let $\mathfrak{J} = \text{span}\mathfrak{S}$. Then for $w \in \mathfrak{J}$, Bw is a linear combination of vectors of $\mathfrak{S}$ and thus $w \mapsto Bw$ maps $\mathfrak{J}$ into $\mathfrak{J}$. Also for $w \in \mathfrak{J}$,

$$\langle Bz, w\rangle = \langle z, B^*w\rangle = \langle z, Bw\rangle = \overline{\lambda}\langle z, w\rangle = 0$$

for all $z \in \mathfrak{J}^\perp$, meaning that $Bz \in \mathfrak{J}^\perp$ for all $z \in \mathfrak{J}^\perp$. Considering the restricted linear operator $z \mapsto Bz$, $z \in \mathfrak{J}^\perp$, then this linear map would have at least one eigenvalue if it were that $\mathfrak{J}^\perp \neq \{0\}$ (using the fundamental theorem of algebra); but then an eigenvector of B would be contained in $\mathfrak{J}^\perp$, and since it is also contained in $\mathfrak{J}$, $\mathfrak{J} \cap \mathfrak{J}^\perp \neq \{0\}$, a contradiction. Hence $\mathfrak{J}^\perp = \{0\}$, so $\mathfrak{J} = \mathbb{C}^n$.

Now, $\mathfrak{S}$ is orthonormal; all vectors in $\mathfrak{S}$ have length 1 and pick $z \in \mathfrak{S}_i$, $w \in \mathfrak{S}_j$ with $i \neq j$. Then

$$(\lambda_i - \lambda_j)\langle z, w\rangle = \langle \lambda_i z, w\rangle - \langle z, \lambda_j w\rangle = \langle Bz, w\rangle - \langle z, Bw\rangle = \langle z, Bw\rangle - \langle z, Bw\rangle = 0,$$

because all eigenvalues are real and B is self-adjoint. Since $\lambda_i \neq \lambda_j$, then $\langle z, w\rangle = 0$, so z and w are orthogonal. Thus $\mathfrak{S}$ is an orthonormal basis for $\mathbb{C}^n$. Letting $(z_i)_1^n$ denote the n elements of $\mathfrak{S}$ with corresponding eigenvalues $(\alpha_i)_1^n$, then defining S to be the matrix $[z_i]_1^n$ (the columns being the vectors (z_i)), we obtain an invertible matrix S with $(S_{ki})_{k=1}^n = z_i$ for all $i = 1, \dots, n$. From this, we deduce that S is unitary, since

$$(S^*S)_{ij} = \sum_{k=1}^n S^*_{ik}S_{kj} = \sum_{k=1}^n \overline{S_{ki}}S_{kj} = \langle z_j, z_i\rangle = \begin{cases} 0 & j \neq i \\ 1 & j = i \end{cases} \tag{1}$$

from $(z_i)_{i=1}^n$ being an orthonormal set. Thus $S^*S = 1_n$. $SS^* = 1_n$ follows from the fact that S is injective and therefore bijective by the rank-nullity theorem. Letting $D = (d_{ij})$ denote the diagonal matrix with diagonal entries $d_{ii} = \alpha_i > 0$, then since $Se_i = z_i$, e_i being the i'th element of the standard basis, we obtain

$$SDS^{-1}z_i = SDe_i = S(\alpha_i e_i) = \alpha_i z_i = Bz_i$$

for all i, so $SDS^{-1} = B$, by (z_i) being a basis. Defining $R = S^*$, then R is unitary and we finally obtain $B = R^{-1}DR$ for a diagonal matrix D with positive diagonal entries and a unitary matrix R. The properties of R and D in the case of A will be denoted as follows; whenever some $R \in U(n)$ and $D = (d_{ij})$ with $d_{ii} > 0$ satisfy $A^*A = R^{-1}DR$, we will write $(R, D) \in ❀(A)$.

(b)

Let $D_t = (d_{ij}^{-t/2})$ for $t \in \mathbb{R}$. Show that $A_t = AR^{-1}D_tR$ defines a path of invertible matrices, which is independent of the choices of R and D.

The map $\mathbb{R} \to M_n(\mathbb{C})$ given by $t \mapsto D_t$ is continuous since its entries consist of continuous functions of t. As a result, the map $t \mapsto A_t$ is continuous as well because its entries are sums and products of continuous functions of t (for topological spaces $\mathcal{X}$ and C^*-algebras $\mathcal{A}$, a map $\mathcal{X} \to M_n(\mathcal{A})$ is continuous if and only if its entry maps $\mathcal{X} \to \mathcal{A}$ are continuous).

For $t \in \mathbb{R}$, the diagonal matrix D_t is invertible since $d_{ij}^{-t/2} > 0$ for all diagonal entries $d_{ij} > 0$ in D (by (a)), making their product, and thus the determinant, non-zero. Since A and R are invertible, it follows that A_t is invertible. Thus $t \mapsto A_t$ is a path of invertible matrices.

Assume now that $(T, D') \in ❀(A)$ with $\beta_i = d'_{ii}$, $i = 1, \ldots, n$, being the positive diagonal entries of D', and define D'_t in the same way as for D_t. Then $T^{-1}D'T = A^*A = R^{-1}DR$. From (1) in (a), we can deduce that the columns of $T^{-1} = T^*$ constitute an orthonormal basis $(w_i)_1^n$ of $\mathbb{C}^n$ with $T^{-1}e_i = w_i$, since

$$\sum_{k=1}^{n} (T^*)_{ki}\overline{(T^*)_{kj}} = \sum_{k=1}^{n} T_{jk}(T^*)_{ki} = (TT^*)_{ji} = \begin{cases} 0 & j \neq i \\ 1 & j = i \end{cases},$$

and thus $A^*Aw_i = T^{-1}D'Tw_i = T^{-1}D'e_i = T^{-1}(\beta_i e_i) = \beta_i w_i$, implying that $(\beta_i)_1^n$ are eigenvalues of A^*A with corresponding eigenvectors $(w_i)_1^n$. For $i = 1, \ldots, n$, then using the proof from (a), $\beta_i = \lambda_j$ and $w_i \in \mathfrak{G}_j$ for some j. Thus we can express w_i as a linear combination

$$w_i = \sum_{\substack{1 \leq k \leq n \\ z_k \in \mathfrak{G}_j}} \mu_k z_k, \quad \mu_k \in \mathbb{C}.$$

Then from the equation $R^{-1}e_i = z_i$, we obtain

$$\begin{aligned} R^{-1}D_tRw_i &= R^{-1}D_tR\left(\sum_{\substack{1 \leq k \leq n \\ z_k \in \mathfrak{G}_j}} \mu_k z_k\right) = R^{-1}D_t\left(\sum_{\substack{1 \leq k \leq n \\ z_k \in \mathfrak{G}_j}} \mu_k e_k\right) \\ &\overset{!!}{=} R^{-1}\left(\sum_{\substack{1 \leq k \leq n \\ z_k \in \mathfrak{G}_j}} \lambda_j^{-t/2}\mu_k e_k\right) = \sum_{\substack{1 \leq k \leq n \\ z_k \in \mathfrak{G}_j}} \lambda_j^{-t/2}\mu_k z_k = \lambda_j^{-t/2} \sum_{\substack{1 \leq k \leq n \\ z_k \in \mathfrak{G}_j}} \mu_k z_k = \lambda_j^{-t/2} w_i. \end{aligned}$$

This requires some explanation. At the third equality sign, diagonal elements of D_t are only multiplied onto *some* elements of the standard basis. Which ones? Well, all the elements of the standard basis that "correspond" to the eigenvectors z_k of $\mathfrak{G}_j$ by the equation $Rz_k = e_k$, and we know that the diagonal elements of D corresponding to these z_k are all λ_j. Since

$$T^{-1}D'_tTw_i = T^{-1}D'_te_i = T^{-1}(\beta_i^{-t/2}e_i) = \lambda_j^{-t/2}w_i,$$

$R^{-1}D_tR$ and $T^{-1}D'_tT$ agree on basis elements of $\mathbb{C}^n$ for all $t \in \mathbb{R}$, and so must be equal. Hence the path does not depend on choices of R and D.

(c)

Show that A_1 is unitary, $A_0 = A$ and $A \mapsto A_1$ defines a map $GL_n(\mathbb{C}) \to U(n)$ which is homotopy inverse to the inclusion $U(n) \subseteq GL_n(\mathbb{C})$.

Let $A \in GL_n(\mathbb{C})$ and $(R, D) \in \text{❀}(A)$. Then $A^* = R^{-1}DRA^{-1}$ and therefore

$$A_1 A_1^* = (AR^{-1}D_1R)(R^*(D_1)^*(R^{-1})^*A^*) = AR^{-1}D_1D_1RA^* = AR^{-1}D^{-1}RA^* = I_n$$

and

$$A_1^*A_1 = R^{-1}D_1R(A^*A)R^{-1}D_1R = R^{-1}D_1R(R^{-1}DR)R^{-1}D_1R = R^{-1}D_1D_{-2}D_1R = I_n.$$

Indeed, everything neatly follows from the fact that

1. since R is unitary, we have $R^{-1} = R^*$ and $(R^{-1})^* = R$;
2. since D_t is real and diagonal, $(D_t)^* = D_t$;
3. $D_sD_t = D_{s+t}$, since a product of two diagonal matrices is the diagonal matrix with entries equal to the products of the respective diagonal entries; thus $D_s^{-1} = D_{-s}$ and $D_2 = D^{-1}$.

Thus A_1 is unitary. It is also clear that $D_0 = I_n$, so $A_0 = AR^{-1}I_nR = AR^{-1}R = A$.

We now define a map $f : GL_n(\mathbb{C}) \to U(n)$ by

$$f(A) = A_1 := AR^{-1}D_1R,$$

for $(R, D) \in \text{❀}(A)$, with D_t as in (b); (b) yields that this map does not depend on choices of R and D, and therefore it is well-defined. Let $g : U(n) \to GL_n(\mathbb{C})$ denote the inclusion.

Let $P : GL_n(\mathbb{C}) \times [0, 1] \to GL_n(\mathbb{C})$ and $Q : U(n) \times [0, 1] \to U(n)$ be defined by $P(A, t) = A_t$ and $Q(A, t) = A$. It is clear from (b) that P is well-defined. P is continuous[1], and it is immediate that $P(A, 0) = A_0 = A$ and $P(A, 1) = A_1 = g(f(A))$, so $gf \simeq 1_{GL_n(\mathbb{C})}$.

Q is clearly continuous since it is a projection. For $A \in U(n)$, $A_t = AR^{-1}D_tR$ does not depend of choices of $(R, D) \in \text{❀}(A)$, using (b). Since $(A, I_n) \in \text{❀}(A)$ because $I_n = A^*A = A^{-1}I_nA$ and the diagonal elements of I_n are positive, then $A_t = AA^{-1}I_nA = A$ for all unitary matrices A since $(I_n)_t = I_n$ for all $t \in \mathbb{R}$. This implies $f(A) = A$, so $Q(A, 0) = A$ and $Q(A, 1) = A = f(A) = f(g(A))$ for all $A \in U(n)$. Thus $fg \simeq 1_{U(n)}$. Hence $U(n)$ and $GL_n(\mathbb{C})$ are homotopy equivalent spaces.

[1] I have not been able to prove continuity of P.