Introduction to K-theory

Assignment 5

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Let $\mathcal{A}$ be a unital Banach algebra and $i_n : \mathcal{A} \to M_n(\mathcal{A})$ the inclusion

$$i_n(a) = \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

(a)

Show that i_n induces an isomorphism $K_0(\mathcal{A}) \to K_0(M_n(\mathcal{A}))$.

For any $k \in \mathbb{N}$, note that $M_k(M_n(\mathcal{A})) = M_{kn}(\mathcal{A})$. i_n is clearly an isometric Banach algebra homomorphism, and induces a (non-unit-preserving) homomorphism $i_n^k : M_k(\mathcal{A}) \to M_{kn}(\mathcal{A})$ by defining $i_n^k(A)$ (A being a $k \times k$ matrix (a_{ij}) with entries in $\mathcal{A}$) to be a block matrix built from k^2 $n \times n$ matrices $i_n(a_{ij})$ (so that the block at place (i,j) is $i_n(a_{ij})$). This new homomorphism is clearly isometric, as

$$\|i_n^k(A)\| = \sum_{i,j=1}^{n} \|i_n(a_{ij})\| = \sum_{i,j=1}^{n} \|a_{ij}\| = \|A\|.$$

Also, i_n^k maps idempotents to idempotents, as $i_n^k(A)^2 = i_n^k(A^2) = i_n^k(A)$. Thus, i_n^k is a continuous map $\mathrm{Idem}M_k(\mathcal{A}) \to \mathrm{Idem}M_k(M_n(\mathcal{A}))$, and thus induces a map

$$(i_n^k)^* : \pi_0\mathrm{Idem}M_k(\mathcal{A}) \to \pi_0\mathrm{Idem}M_k(M_n(\mathcal{A}))$$

(using Exercise 1 from Exercise Set 2). Thus we obtain a diagram

$$\begin{array}{ccccccc}
\pi_0\mathrm{Idem}M_1(\mathcal{A}) & \longrightarrow & \pi_0\mathrm{Idem}M_2(\mathcal{A}) & \longrightarrow & \pi_0\mathrm{Idem}M_3(\mathcal{A}) & \longrightarrow & \cdots \\
\downarrow (i_n^1)^* & & \downarrow (i_n^2)^* & & \downarrow (i_n^3)^* & & \\
\pi_0\mathrm{Idem}M_1(M_n(\mathcal{A})) & \longrightarrow & \pi_0\mathrm{Idem}M_2(M_n(\mathcal{A})) & \longrightarrow & \pi_0\mathrm{Idem}M_3(M_n(\mathcal{A})) & \longrightarrow & \cdots
\end{array}$$

We now obtain maps

$$\pi_0\mathrm{Idem}M_j(\mathcal{A}) \xrightarrow{(i_n^j)^*} \pi_0\mathrm{Idem}M_j(M_n(\mathcal{A})) \longrightarrow V(M_n(\mathcal{A})).$$

By the universal property of the colimit, we obtain a map $i_n^\# : V(\mathcal{A}) \to V(M_n(\mathcal{A}))$; considering the composition

$$V(\mathcal{A}) \xrightarrow{i_n^\#} V(M_n(\mathcal{A})) \xrightarrow{\iota} K_0(M_n(\mathcal{A}))$$

with $\iota([A]) = [([A],0)]$, then for $x, y \in V(\mathcal{A})$, since x and y can represented by equivalence classes $[X] = x$ and $[Y] = y$ for some $X \in \mathrm{Idem}M_p(\mathcal{A})$ and $Y \in \mathrm{Idem}M_s(\mathcal{A})$, then $[X]+[Y] = [X \oplus Y]$ maps into

$$[i_n^{p+s}(X \oplus Y)] = [i_n^p(X) \oplus i_n^s(Y)] = [i_n^p(X)] + [i_n^s(Y)] = (i_n^p)^*([X]) + (i_n^p)^*([Y]) = i_n^\#([X]) + i_n^\#([Y]),$$

making $i_n^\#$ a monoid map. By the universal property of the Grothendieck construction, the map $\iota \circ i_n^\#$ yields a homomorphism $i_n^* : K_0(\mathcal{A}) \to K_0(M_n(\mathcal{A}))$ of abelian groups.

We will now construct an inverse to i_n^*. Since elements in

$$V(M_n(\mathcal{A})) \cong \pi_0 \mathrm{colim}_j \mathrm{Idem} M_j(M_n(\mathcal{A}))$$

can be represented by equivalence classes $[A] \in \pi_0 \mathrm{Idem} M_j(M_n(\mathcal{A})) = \pi_0 \mathrm{Idem} M_{jn}(\mathcal{A})$ for some $j \in \mathbb{N}$, we can define a map $\alpha_n' : V(M_n(\mathcal{A})) \to V(\mathcal{A})$ by

$$\alpha_n'([A]) = [A],$$

that is obviously a monoid map.

In the same manner as above, then by the universal property of the Grothendieck construction, we obtain a map $\alpha_n : K_0(M_n(\mathcal{A})) \to K_0(\mathcal{A})$ of abelian groups with the property that

$$\alpha_n([(x,0)]) = [(x,0)], \quad x \in V(\mathcal{A}).$$

It now suffices to prove that the compositions of α_n' and $i_n^{\#}$ are the identities on the two monoids: by how the K_0 group homomorphisms are constructed from monoid maps, it then follows that α_n and i_n^* are each other's inverses. We have

$$\alpha_n'(i_n^{\#}([A])) = [i_n^{\#}(A)].$$

We will prove that there exists a path from $A \oplus 0$ to $i_n^k(A) \oplus 0$ for all $k \in \mathbb{N}$; then it follows that $[i_n^{\#}(A)] = [A]$ in $V(\mathcal{A})$ since $[B] = [B \oplus 0]$ for any $B \in \mathrm{Idem} M_k(\mathbb{C})$ (since the maps $\pi_0 \mathrm{Idem} M_k(\mathbb{C}) \to \pi_0 \mathrm{Idem} M_{k+1}(\mathbb{C})$ are the inclusions given by $B \mapsto B \oplus 0$).

Let $\{e_1, \ldots, e_{kn}\}$ be the standard orthonormal basis for $\mathbb{C}^{kn}$ and define a matrix $M \in M_{nk}(\mathbb{C})$ such that $Me_i = e_{n(i-1)+1}$ for $i = 1, \ldots, k$ and M is unitary (this can be done by making sure that there's a 1 in every column and row). Then it can be checked that $M^* i_n^k(A) M = A \oplus 0$. Let us do this for $n = k = 2$. If

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

then

$$\begin{aligned} M^* i_n^k(A) M &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_{11} & 0 & a_{12} & 0 \\ 0 & 0 & 0 & 0 \\ a_{21} & 0 & a_{22} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = A \oplus 0 \end{aligned}$$

so $A \oplus 0 \sim_s i_n^k(A)$ (Theorem 0.9 from Lecture Notes Section 2), and by the same theorem

$$A \oplus 0 \sim_h i_n^k(A) \oplus 0.$$

This proves that $\alpha_n \circ i_n^* = \mathrm{id}$. Proving $i_n^* \circ \alpha_n = \mathrm{id}$ amounts to proving $[i_n^{\#}(A)] = [A]$ in $V(M_n(\mathcal{A}))$ where A is an $(mn) \times (mn)$ matrix for some $m \in \mathbb{N}$, and this can be done similarly by for any $k \in \mathbb{N}$ taking the standard basis for $\mathbb{C}^{kmn}$ and a unitary matrix $M \in M_{kmn}(\mathbb{C})$ such that $Me_i = e_{mn(i-1)+1}$ for $i = 1, \ldots, k$. Hence i_n^* is a bijection and i_n induces an isomorphism $K_0(\mathcal{A}) \to K_0(M_n(\mathcal{A}))$.

(b)

Let $\phi : \mathbb{C} \to M_n(\mathbb{C})$ be the map

$$\phi(x) = \begin{pmatrix} x & 0 & \cdots & 0 \\ 0 & x & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x \end{pmatrix}.$$

Calculate the map $\phi_* : K_0(\mathbb{C}) \to K_0(M_n(\mathbb{C}))$.

For all $k \in \mathbb{N}$, ϕ induces an algebra homomorphism $\phi_k : M_k(\mathbb{C}) \to M_k(M_n(\mathbb{C}))$ in the same way as described in (a). Furthermore, note that for $A = (\lambda_{ij}) \in M_k(\mathbb{C})$, we have

$$\|\phi_k(A)\| = \sum_{i,j=1}^{k} \|\phi(\lambda_{ij})\| = \sum_{i,j=1}^{k} n|\lambda_{ij}| = n\|A\|.$$

Thus ϕ_k is continuous, and thus induces continuous maps $(\phi_k)^* : \pi_0 \mathrm{Idem} M_k(\mathbb{C}) \to \pi_0 \mathrm{Idem} M_k(M_n(\mathbb{C}))$ given by $(\phi_k)^*([p]) = [\phi_k(p)]$.

For all $k \in \mathbb{N}$, we know that $\pi_0 \mathrm{Idem} M_k(\mathbb{C}) \overset{\sim}{\to} \{0, 1, \ldots, k\}$ with a bijection given by $T_k([p]) = \mathrm{Tr}(p)$; it is well-defined, as proved in the lecture notes. Since

$$T_{kn}(\phi_k)^*([p]) = T_{kn}([\phi_k(p)]) = \mathrm{Tr}(\phi_k(p)) = n\mathrm{Tr}(p) = nT_k([p])$$

for all $p \in \mathrm{Idem} M_k(\mathbb{C})$. Since $\mathrm{colim}_k \{0, 1, \ldots, k\} = \mathbb{N}_0$, we obtain a unique map $T : V(\mathbb{C}) \to \mathbb{N}_0$ by the universal property of the colimit such that $T_k([p]) = T([p])$ for all $k \in \mathbb{N}$ and $p \in \mathrm{Idem} M_k(\mathbb{C})$; this is actually a bijection, since all T_k were bijections and the arrows $\pi_0 \mathrm{Idem} M_k(\mathbb{C}) \to \pi_0 \mathrm{Idem} M_{k+1}(\mathbb{C})$ are inclusions. Similarly, we obtain a bijection $T' : V(M_n(\mathbb{C})) \to \mathbb{N}_0$ such that $T_{kn}([p]) = T'([p])$ for all $k \in \mathbb{N}$ and $p \in \mathrm{Idem} M_{kn}(\mathbb{C})$.

$(\phi_k)^*$ now finally induces a unique map $\phi^\# : V(\mathbb{C}) \to V(M_n(\mathbb{C}))$ such that $(\phi_k)^*([p]) = \phi^\#([p])$ for all $k \in \mathbb{N}$ and $p \in \mathrm{Idem} M_k(\mathbb{C})$. Thus if $p \in \mathrm{Idem} M_k(\mathbb{C})$,

$$T'(\phi^\#([p])) = T'(\phi_k)^*([p]) = T_{kn}(\phi_k)^*([p]) = nT_k([p]) = nT([p]).$$

By identifying $V(\mathbb{C}) \cong V(M_n(\mathbb{C})) \cong \mathbb{N}_0$, we see that $\phi^\#$ is actually just multiplication by n. $\phi^\#$ is a monoid map by the considerations in (a), and thus, it induces a group homomorphism $\phi_* : K_0(\mathbb{C}) \to K_0(M_n(\mathbb{C}))$ given by

$$\phi_*([(l, m)]) = [(\phi^\#(l), \phi^\#(m))] = [(nl, nm)] = n[(l, m)].$$

Thus the induced homomorphism from ϕ on the K_0 groups is multiplication by n (note that both groups are isomorphic to $\mathbb{Z}$).