# Exam 2009

## Probability Theory 1 and Measure and Integration Theory

# Assignment 1

## Formalities

This is the first of the four compulsory assignments for the two courses *Probability Theory 1* and *Measure and Integration Theory*.

The assignment is divided into 3 problems with a total of 10 questions.

This is an exam, and the solution must be written and handed in individually **in two copies**. The solution must be equipped with the standard frontpage, which is available form the course webpage. You are only allowed to write on one side of the paper, the solution must be stapled in the upper left corner and no plastic-covers please.

The deadline for handing in the **two copies** of the solution is Monday, September 14 at **the beginning of the lecture** 13.15.

## Problem 1

We consider here the following two pavings $\mathbb{D} = \{(-\infty, x] \mid x \in \mathbb{R}\}$ and $\mathbb{H} = \{(x, \infty) \mid x \in \mathbb{R}\}$ on $\mathbb{R}$. From Problem 1.6(d) in the book it is known that $\mathbb{B} = \sigma(\mathbb{D})$.

**Question 1.1.** Show that $\mathbb{D} \subset \mathbb{H}^{\diamond}$ and that $\mathbb{B} = \sigma(\mathbb{H})$.

**Question 1.2.** Show that for any $x < y$, $x, y \in \mathbb{R}$ we have $(x, y] \in \mathbb{H}^{\diamond 3}$ and $(x, y) \in \mathbb{H}^{\diamond 4}$.

## Problem 2

In Problem 1 it was shown that the pavings $\mathbb{D}$ and $\mathbb{H}$ both generate the Borel-algebra $\mathbb{B}$. Let $\mu$ be a measure on $(\mathbb{R}, \mathbb{B})$ fulfilling that

$$\mu((-\infty, x]) = \frac{e^x}{1 + e^{-x}}. \tag{2.1}$$

Such a measure can be shown to exist based on the existence of the Lebesgue measure. For the questions below you may take the existence for granted.

**Question 2.1.** Argue that the paving

$$\mathbb{D} = \{(-\infty, x] \mid x \in \mathbb{R}\}$$

is stable under intersections and show that (2.1) uniquely specifies the measure $\mu$.

**Question 2.2.** Show first that $\mu(\mathbb{R}) = \infty$ and then that $\mu((x, \infty)] = \infty$ for all $x \in \mathbb{R}$. Show that the paving

$$\mathbb{H} = \{(x, \infty) \mid x \in \mathbb{R}\}$$

is stable under intersections and argue that $\mu$ is **not** uniquely specified by its values on $\mathbb{H}$.

**Question 2.3.** Compute $\mu((x, x + \varepsilon])$ for $\varepsilon > 0$ and $x \in \mathbb{R}$ and show that for $\varepsilon \leq 1$

$$\mu([x, x + \varepsilon]) \leq K\varepsilon(e^x + 1)$$

for some constant $K > 0$.

**Hint:** You can without further arguments use that for some $K > 0$

$$e^{\varepsilon} - e^{-\varepsilon} \leq K\varepsilon$$

for $\varepsilon \in [0, 1]$.

**Question 2.4.** Define

$$A = \bigcup_{n=1}^{\infty} (n, n + e^{-2n}].$$

Argue that $A \in \mathbb{B}$. Decide if $\mu(A) = \infty$ or $\mu(A) < \infty$.

## Problem 3

Consider the measure space $(\mathbb{R}^2, \mathbb{B}_2, m_2)$. Define

$$A = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0,\ 0 < y \leq 1 - x^2\}.$$

**Question 3.1.** Draw a sketch of $A$. Show that with

$$A_n = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0,\ n^{-1} \leq y \leq 1 - x^2\}$$

we have $A = \cup_{n=1}^{\infty} A_n$ and show that $A \in \mathbb{B}_2$.

**Question 3.2.** Define $B_1 = [0, 1] \times [0, 1]$. Argue that $m_2(B_1) = 1$ and show that

$$m_2(A) \leq 1.$$

Define for $n \in \mathbb{N}$ and $k = 0, \ldots, n-1$ the sets

$$B_{n,k} = \left[\frac{k}{n}, \frac{k+1}{n}\right] \times \left[0, 1 - \frac{k^2}{n^2}\right]$$

and define

$$B_n = \bigcup_{k=0}^{n-1} B_{n,k}$$

In the following you can use the formula

$$\sum_{k=0}^{n-1} k^2 = \frac{(n-1)n(2n-1)}{6}$$

without proof.

**Question 3.3.** Argue that $B_n \in \mathbb{B}_2$ and show that

$$m_2(B_n) \leq \frac{2}{3} + \frac{1}{2n} - \frac{1}{6n^2}$$

**Question 3.4.** Show that $A \subset B_n$ for all $n$ and argue that $m_2(A) \leq \frac{2}{3}$.

**Question 3.5.** Show that $m_2(A) = \frac{2}{3}$.

**Hint:** The $B_n$-sets above form outer approximations. It may be useful to consider the open sets

$$C_{n,k} = \left(\frac{k}{n}, \frac{k+1}{n}\right) \times \left(0, 1 - \frac{(k+1)^2}{n^2}\right)$$

to form inner approximations.

**Question 3.6.** Let $\overline{A}$ denote the closure of $A$ and $A^\circ$ the interior of $A$. Find $m_2(\overline{A})$ and $m_2(A^\circ)$.