Exam 2009

Probability Theory 1 and Measure and Integration Theory

Assignment 3

Formalities

This is the third of the four compulsory assignments for the two courses *Probability Theory 1* and *Measure and Integration Theory*.

The assignment is divided into 3 problems with a total of 10 questions.

This is an exam, and the solution must be written and handed in individually **in two copies**. The solution must be equipped with the standard frontpage, which is available from the course webpage. You are only allowed to write on one side of the paper, the solution must be stapled in the upper left corner and no plastic-covers please.

The deadline for handing in the **two copies** of the solution is Monday, October 12 at **the beginning of the lecture** 13.15.

NOTE: In several of the questions you are required to compute integrals using standard tools from classical Riemann integration theory. You are **not** required **in this assignment** to carefully explain all details to justify the actual computations in the framework of this course, where you formally use monotone convergence or integration by parts or combinations.

Problem 1

Question 1.1. Show that the function

$$(x,y) \mapsto \frac{y}{1+x^2y^2}$$

from $\mathbb{R}^2$ to $\mathbb{R}$ is integrable over $A = \mathbb{R} \times (-1,1)$ w.r.t. m_2 and compute

$$\int_A \frac{y}{1+x^2y^2} \mathrm{d}m_2(x,y).$$

Problem 2

Let μ be the measure on $(\mathbb{R}^2, \mathbb{B}_2)$ with density

$$f(x_1,x_2) = 1_{[1,\infty)\times[1,\infty)}(x_1,x_2)c\, e^{-x_1x_2}$$

w.r.t. to m_2 where $c > 0$ is an arbitrary constant. In other words, $\mu = f \cdot m_2$.

Question 2.1. Show that for $A \in \mathbb{B}$

$$\mu(A \times \mathbb{R}) = \int_{A\cap[1,\infty)} \frac{c}{x_1} e^{-x_1} \mathrm{d}x_1$$

and show that $\mu(\mathbb{R}^2) < \infty$.

In the remaining questions it is assumed that $c > 0$ is chosen such that μ becomes a probability measure, which is possible because we have shown in general that $\mu(\mathbb{R}^2) < \infty$. The value, $c \simeq 4.56$, can be found by numerical integration. Let, furthermore, $X = (X_1, X_2)$ denote a stochastic variable defined on $(\Omega, \mathbb{F}, P)$ with values in $\mathbb{R}^2$ and with distribution μ. That is, $X(P) = \mu$.

Question 2.2. Show that the distribution of X_1 has density

$$g(z) = 1_{[1,\infty)}(z)\frac{c}{z}e^{-z}$$

w.r.t. m. Argue that X_2 has the same distribution as X_1.

Question 2.3. Argue that X_1 has finite second moment and show that

$$EX_1 = ce^{-1} \quad \text{and} \quad EX_1^2 = 2ce^{-1}.$$

Question 2.4. Show that X_1X_2 has finite first moment and that

$$E(X_1X_2) = ce^{-1} + 1.$$

For the final question you can without further arguments use that since X_2 has the same distribution as X_1 it holds that also $EX_2 = ce^{-1}$ and $EX_2^2 = 2ce^{-1}$.

Question 2.5. Show that $X_1 + X_2$ has finite second moment and compute $V(X_1 + X_2)$. Decide if X_1 and X_2 are independent.

Problem 3

Consider the two transformations $t : \mathbb{R}^k \to \mathbb{R}^*$ and $r : \mathbb{R}^k \to \mathbb{R}$ with $t(x) = \log ||x||$ and $r(x) = ||x||$ with $||x||$ denoting the usual euclidean norm on $\mathbb{R}^k$ given by

$$||x|| = \sqrt{x_1^2 + \dots x_k^2}.$$

Clearly $r(x) = s \circ t(x)$ where $s(t) = e^t$.

With $f(x) = \frac{1}{||x||^k}$ (allowing for $x = 0$ with $f(0) = \infty$) we introduce the measure

$$\mu = f \cdot m_k.$$

In the following it can without further comments be used that t and s are measurable and that f is $\mathcal{M}^+$.

We will, for $w \in \mathbb{R}$, also need the translation $\tau_w : \mathbb{R} \to \mathbb{R}$ defined by

$$\tau_w(t) = t + w$$

and the $k \times k$ matrix

$$A_w = \begin{pmatrix} e^w & 0 & \dots & 0 \\ 0 & e^w & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e^w \end{pmatrix},$$

which we also regard as a (linear) map $A_w : \mathbb{R}^k \to \mathbb{R}^k$.

Question 3.1. Show that $\tau_w \circ t = t \circ A_w$ and then that $t(\mu)$ is translation invariant.

Remark: Formally $t(\mu)$ is a measure on $\mathbb{B}^*$ – the σ-algebra on the extended real line. Since $t(x) \in \mathbb{R}$ unless $x = 0$, and $\{0\}$ has m_k-measure 0, $t(\mu)$ gives measure 0 to $\{\pm\infty\}$ and you can safely from hereon identify $t(\mu)$ with a translation invariant measure defined on the Borel algebra $\mathbb{B}$, which satisfies that $r(\mu) = s \circ t(\mu)$.

Question 3.2. Show that $t(\mu) = c_k m$ where m is the Lebesgue measure and $c_k > 0$ is a constant. Then show that $r(\mu)$ has density $c_k g$ w.r.t. the Lebesgue measure m where

$$g(r) = \begin{cases} r^{-1} & \text{if } r > 0 \\ 0 & \text{otherwise} \end{cases}$$

Question 3.3. Show that if $h \in \mathcal{M}^+(\mathbb{R}, \mathbb{B})$ then

$$\int h \circ r \, \mathrm{d}m_k = \int h(||x||) \mathrm{d}m_k(x) = c_k \int_0^\infty r^{k-1} h(r) \mathrm{d}r.$$

Use this to show that

$$c_k = k m_k(B_k)$$

where $B_k = \{x \in \mathbb{R}^k \mid ||x|| \leq 1\}$ denotes the closed unit ball in $\mathbb{R}^k$.

Remark: For $k = 2$ we know that $m_2(B_2) = \pi$, hence $c_2 = 2\pi$. Taking $h(r) = e^{-r^2/2}$ we obtain, by computations as in Example 12.17, that

$$\left(\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \mathrm{d}x\right)^2 = c_2 = 2\pi.$$

This provides us with a alternative route to the computation of the normalization constant for the normal distribution than by using integration in polar coordinates. Integration using polar coordinates is an important technique, but the formal justification in Example 12.16 hinges on Lemma 12.11, which is not proved in the course and whose proof is not straight forward either. The derivations above do only rely on results that are completely proved in the course, and we obtain a notable integration formula in Question 3.3. More is even true. From the knowledge we just derived it follows that

$$\int h \circ r \,\mathrm{d}m_k = \left(\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \mathrm{d}x\right)^k = (2\pi)^{k/2}$$

but we also know from Question 3.3 that $\int h \circ r \,\mathrm{d}m_k = c_k \int_0^\infty r^{k-1} e^{-r^2/2} \mathrm{d}r$. Derivations as in Example 16.24 and knowledge of the Γ-function can then be used to derive the following formula

$$c_k = \frac{2\pi^{k/2}}{\Gamma(k/2)},$$

which in turn can be used to derive a formula for the k-dimensional volumne, $m_k(B_k)$, of the k-dimensional unit ball. As a digression, the number c_k is in fact the "surface area" – whatever that means – of the k-dimensional unit sphere.

Question 3.4. Compute

$$\int \frac{1}{(||x|| + \lambda)^{1+\lambda}} \mathrm{d}m_k(x)$$

for $\lambda > 0$.